Aircraft Dynamic Stability & Design of Stability Augmentation System
Professor A.K. Ghosh
Department of Aerospace Engineering
Indian Institute of Technology Kanpur
Module 6
Lecture No 36
Numericals

Hello friends. My name is Prashant Kumar. I Am TA for This Course. Till now in this course you have learnt about Phugoid mode, longitudinal dynamics, lateral dynamics, short period mode, spiral, Dutch roll and roll modes. Today I will be emphasising more on the numerical part of that. So before continuing on to numericals, let us see what were the dynamics equations which we derived for longitudinal as well as for lateral mode.

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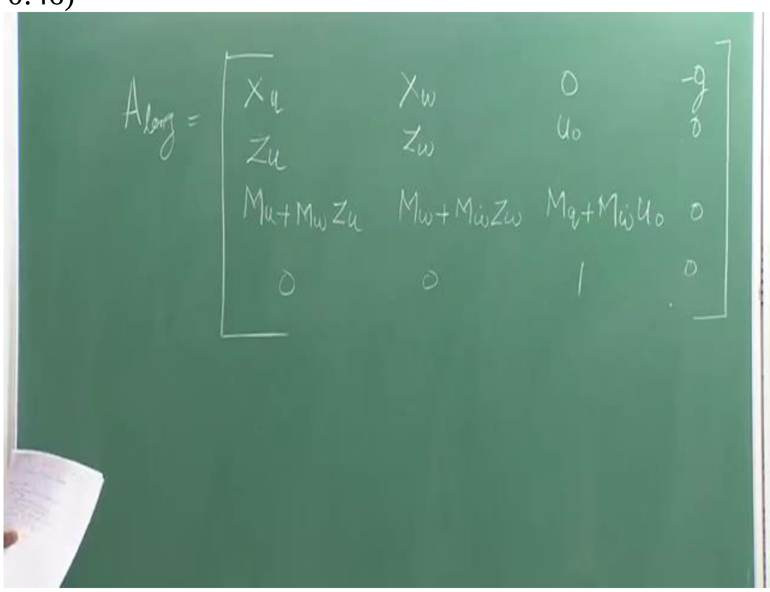

In previous lectures, you might have seen the longitudinal matrix was given by. This is the matrix of longitudinal dynamic mode. Now, in longitudinal mode, there are 2 modes, short period mode and Phugoid mode.

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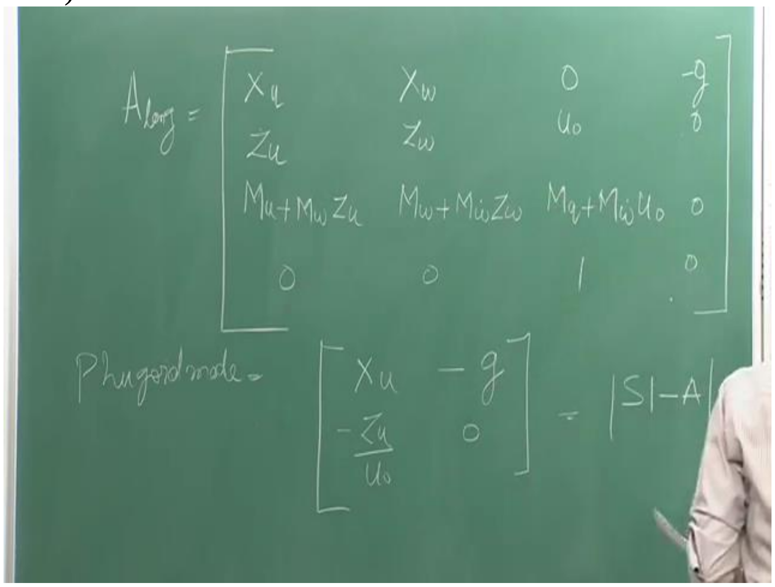

And the approximation matrix for those are, for Phugoid mode, the matrix will be given by XU, -G, ZU by U not, 0. So this is the matrix for Phugoid mode. So characteristic equation for this Phugoid mode will be Si - A equal to 0.

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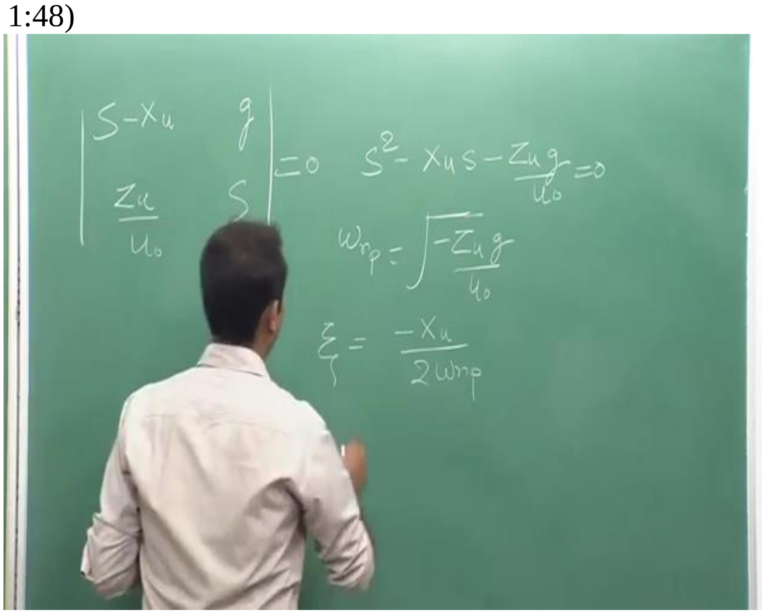

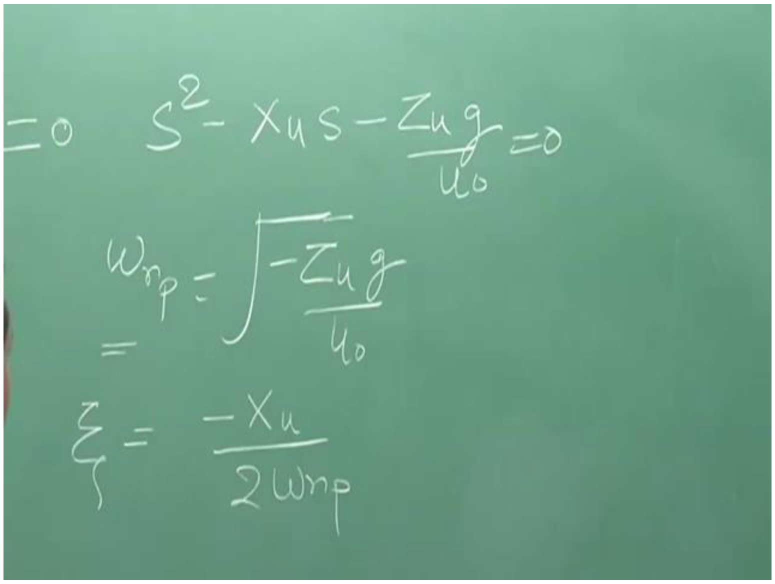

Equating this matrix, we will get that S - XU, G, ZU upon U not, S equal to 0. So the characteristic equation will be S square - XU of S - ZUG upon U not equal to 0. From this characteristic equation, we can see that the natural frequency for Phugoid mode will be at the root of - ZUG upon U not. Similarly we can derive the value of Zeta from this equation.

That is, Zeta will be - XU divided by 2 Omega NP. Now this was the natural frequency given for Phugoid mode.

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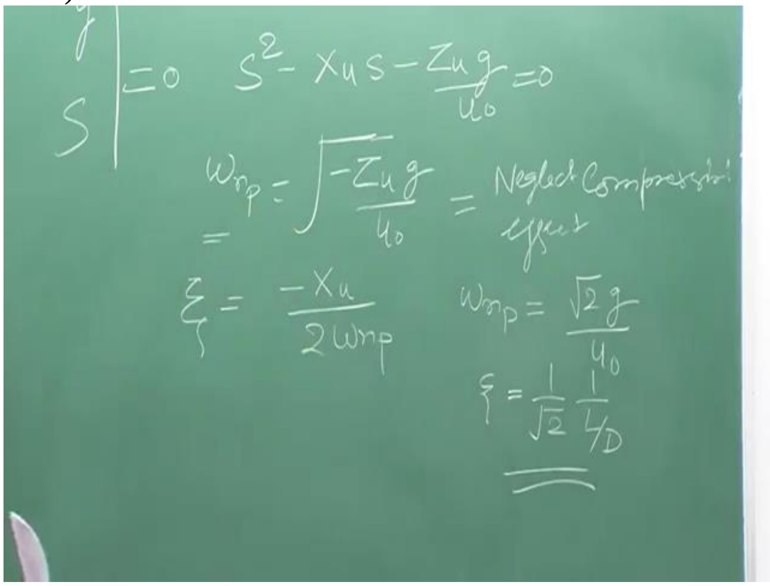

If we neglect the compressibility effect in Phugoid mode, then natural frequency formula reduces to Omega N can be written as root 2 G upon U not. And the value of Zeta will become 1 by root

2 1 upon L upon D. These all characters and is, you have seen in your previous lectures. I am just repeating what was taught in the previous lectures. So this is for Phugoid mode. Now we will see for short period mode.

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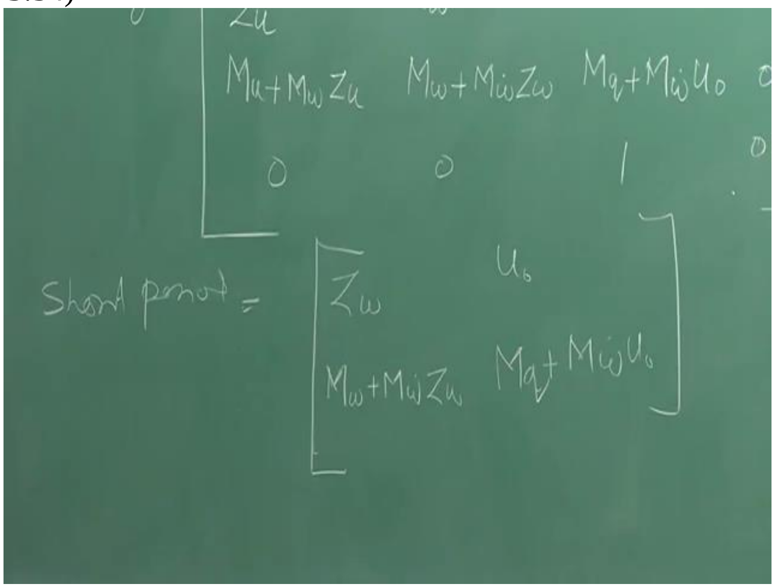

In short period mode, the matrix was given by…

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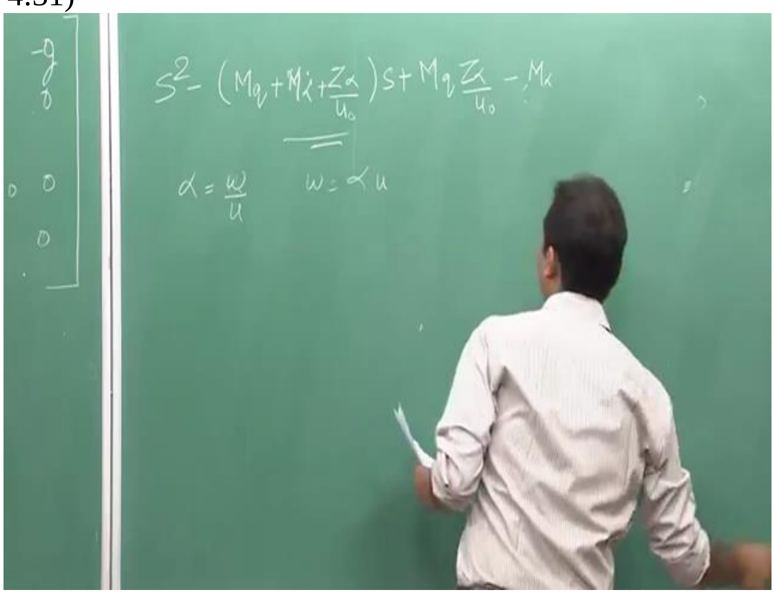

Similarly, the characteristic equation for this short period mode will be S square - MQ + alpha dot + Z alpha upon U not S + MQ Z alpha upon U not - of M alpha. Now you might have noticed, I have written M alpha dot and M alpha and straight off MW. That you can easily derive since you know that alpha is given by W upon U. So that is why, you can replace W by alpha into U. So when making this substitution in equation, you will come to this characteristic equation.

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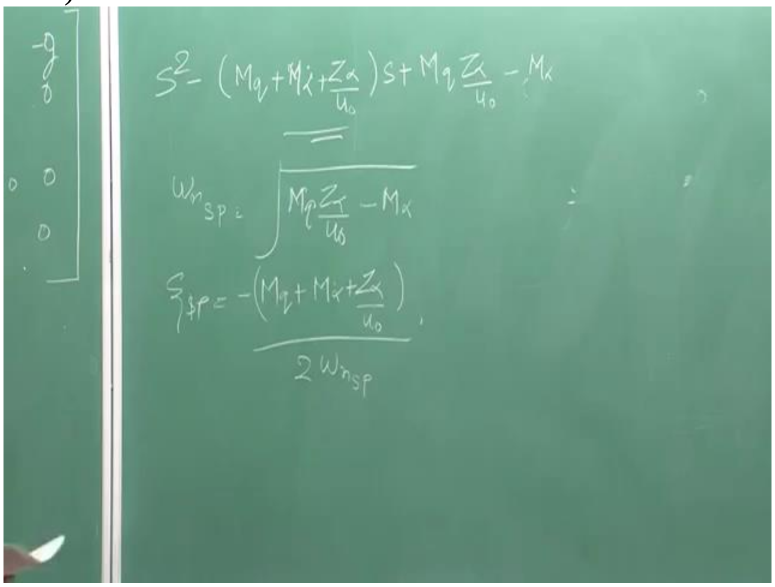

So from this characteristic equation, we can see the natural frequency for short period mode will be under root of MQZ alpha upon U not - of M alpha. And Zeta will be - of MQ + M alpha dot + Z alpha upon U not divided by 2 Omega N short period. So up to here, this all was taught in the class.

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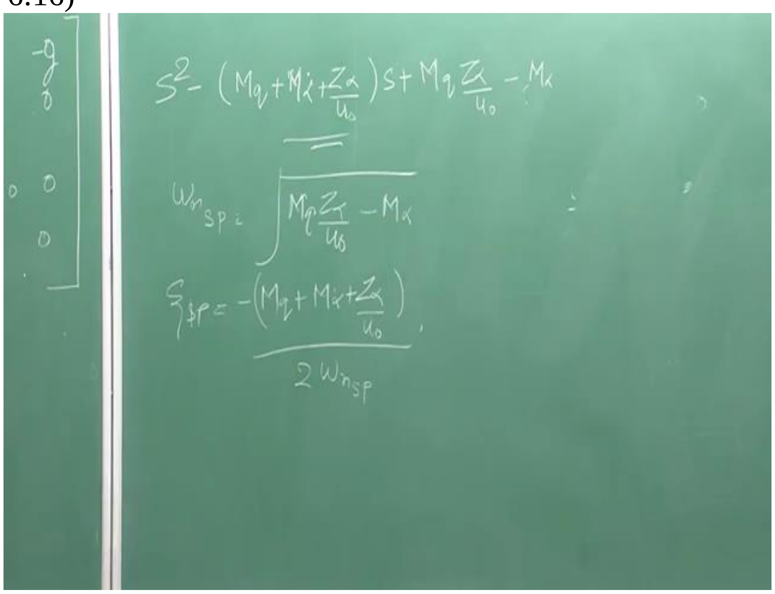

So this will be the formulas will be using in numerical which I am going to write out now.

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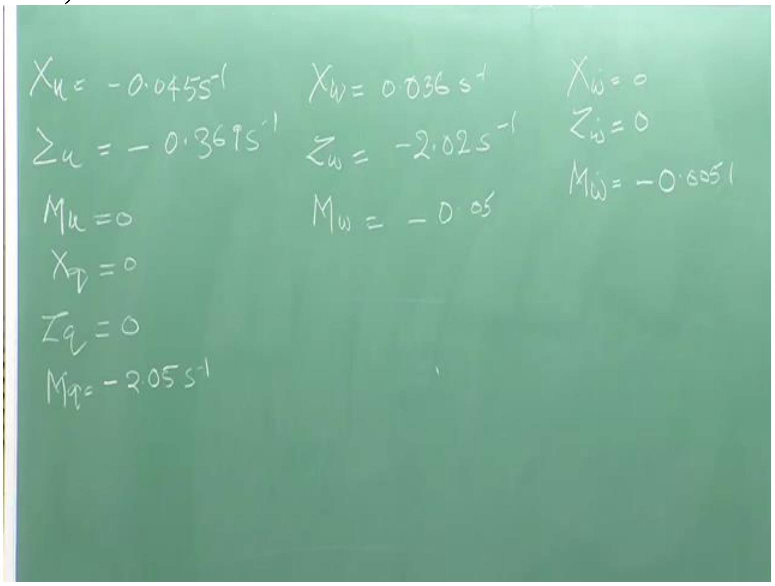

So let us take a numerical on longitudinal mode. Using this longitudinal derivative, you can directly substitute these variables in this equation and you will reach this longitudinal matrix. Characteristic equation of this longitudinal matrix, will be derived from the same formula that we did for short period and Phugoid mode.

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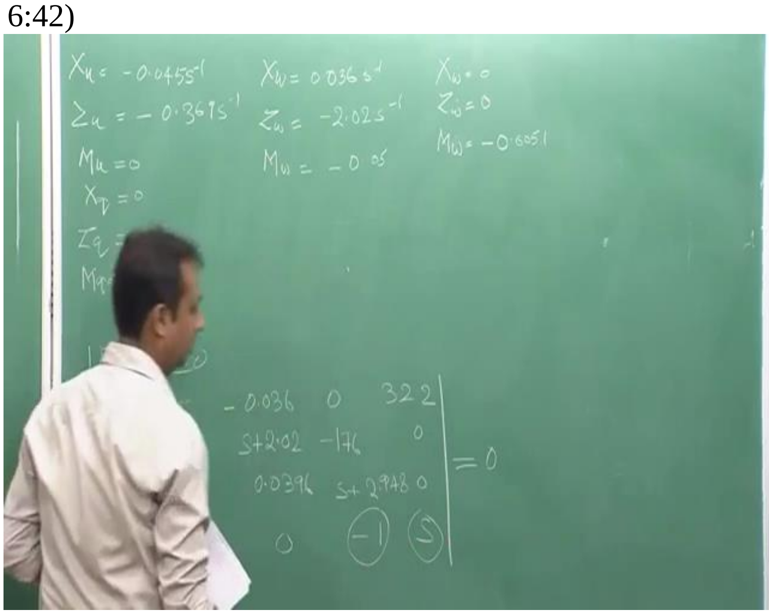

That is SI - A equal to 0 where A is the longitudinal matrix. Now since you know, this is a 4 cross 4 matrix, so the order of the characteristic equation will be of the fourth order. So let us derive the characteristic equation using this formula. The longitudinal mode characteristic equation will be determinant of …

So equating this determinant with 0, we will get the characteristic equation. So it will be a pretty long calculation. You have to bear with me. Since you know, this is a 4 cross 4 matrix, so what will be my best strategy is to go for a minimum calculation. So I will be choosing this element and this element because like that I have to only calculate determinant of 2 sub matrices.

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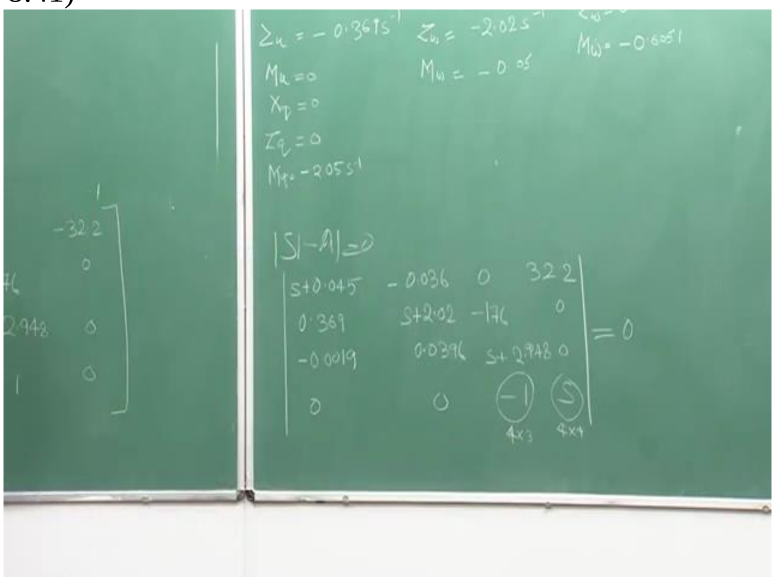

So I will be taking this and the general form of calculating determinant of any matrix will be that element - 1 to the power of what is the location of that matrix, that row and column, sum of that. Since this is a 4 cross 4 matrix, this is 4 cross 3 and this is 4 cross 4. So, sum of the (())(9:09) of rows and column.

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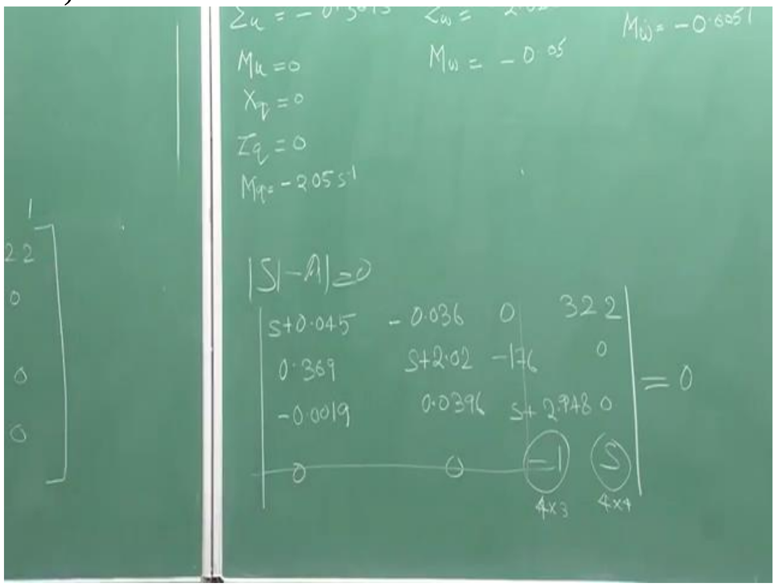

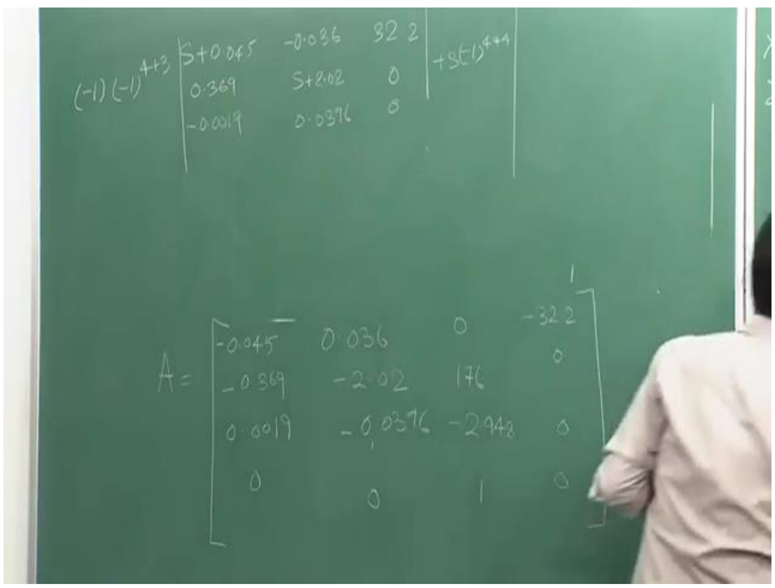

That is 4 + 3. Then determinant of the, matrix remaining by eliminating the rows in which that element is there and the column in which element is there. That is, determinant of this matrix I have to calculate. That is S + 0.045, - 0.036, 32.2, 0.69, S + 2.02, 0. Then - 0.0019, 0.0396, and 0. Then + S. And same using that formula - 1 + 4.

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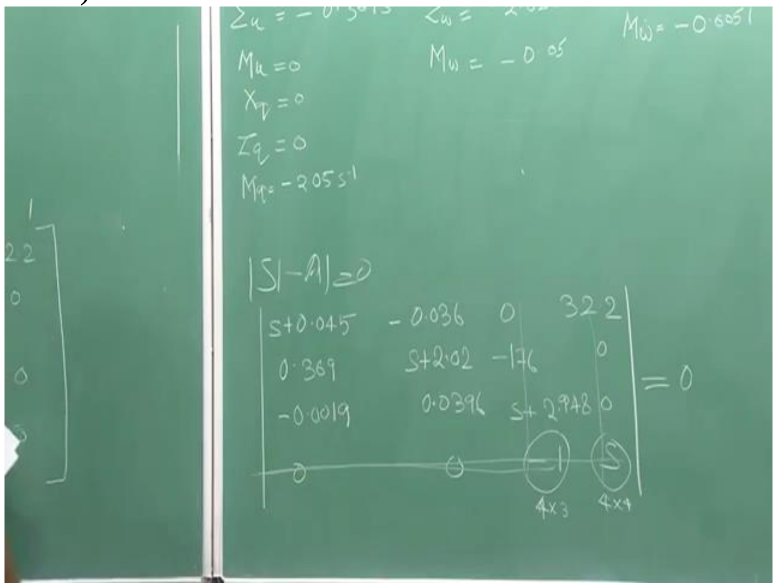

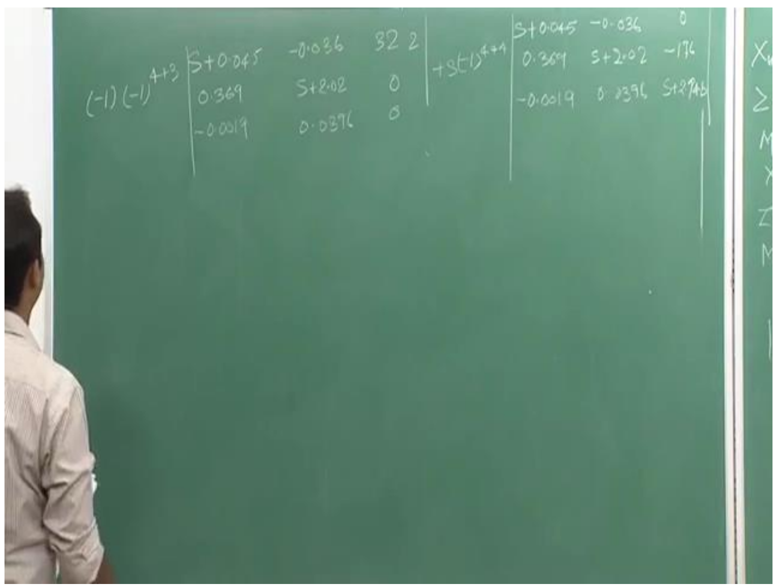

And determinant of the submatrix by eliminating the row and column in which the element lies. S + 0.045, 0.396, - 0.0019, - 0.036, S + 2.02, 0.0396, 0, - 176. Then, S + 2.94.

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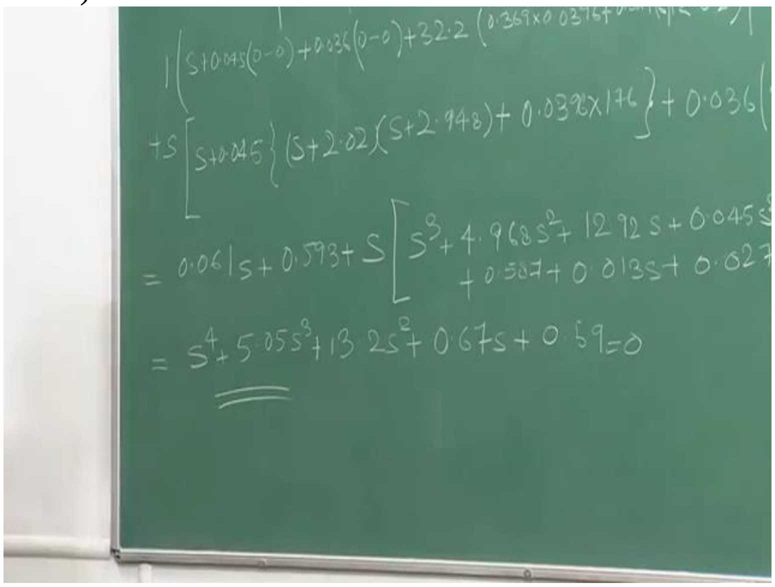

Now for solving this equation, we will get this will be - of - 1, so this will be 1. And determinant of this submatrix will be S + 0.045 into, this eliminating 0 into 0, 0 - 0. - of -, that will be + 0.036. So we will be left with only 32.2 and 0.369 into 0.0396 - of, - and - will be + 0.0019 into S + 2.02.

+ S and this will be the determinant of the submatrix which will be S + 0.045 into S + 2.0 to into S + 2.948 - 0.0396 - and - will be + into 176. - and - will be + 0.036 into 0.369 into S + 2.948 - 176 into 0.0019. So this will reduce to 0.0 61S + 0.593 + equating this part, we will get S into S cube + 4.968 S square + 12.92S + 0.0 45 S square + 0.223S + 0.587 + 0.013S + 0.027.

Which will be further reduced to S to the power 4 + 5.05 S cube + 13.2 S square + 0.67S + 0.59 equal to 0. So this is your characteristic equation for that longitudinal matrix. So I will be showing you how to calculate the roots of this characteristic equation or higher degree polynomial. But before that, we already derived the formula for short period mode and long period mode. Let us see what are the roots for short period and long period mode.

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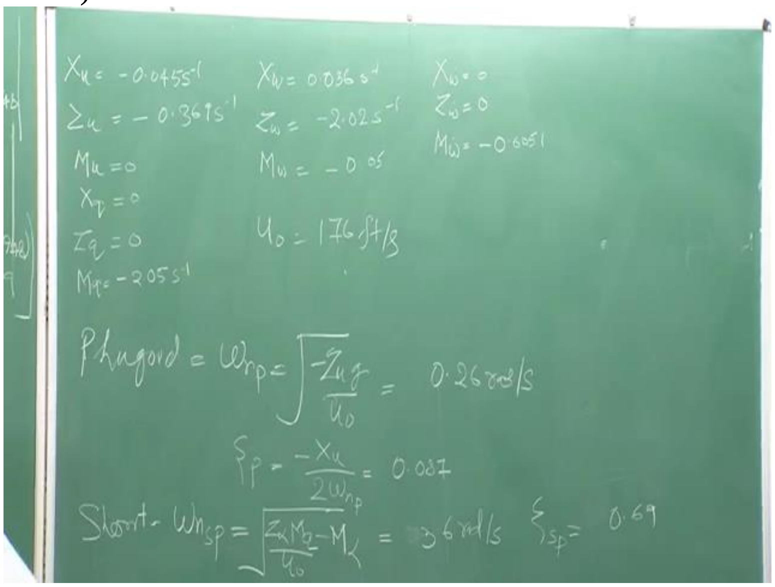

For Phugoid mode, we already wrote that Omega N Phugoid will be root over - ZUG upon U not. U not is 176 ft/s. So it will come around 0.26 radian per second. Similarly, value of Zeta will be - XU upon 2 Omega N Phugoid which will be 0.087.

Similarly for short period, Omega N short period will be root over Z alpha MQ upon U not M alpha mom which will give 3.6 rad per second. And similarly Zeta for short period mode will be 0.69. I am not doing the calculations. Just you have to substitute these values in the formula and you will get that value.

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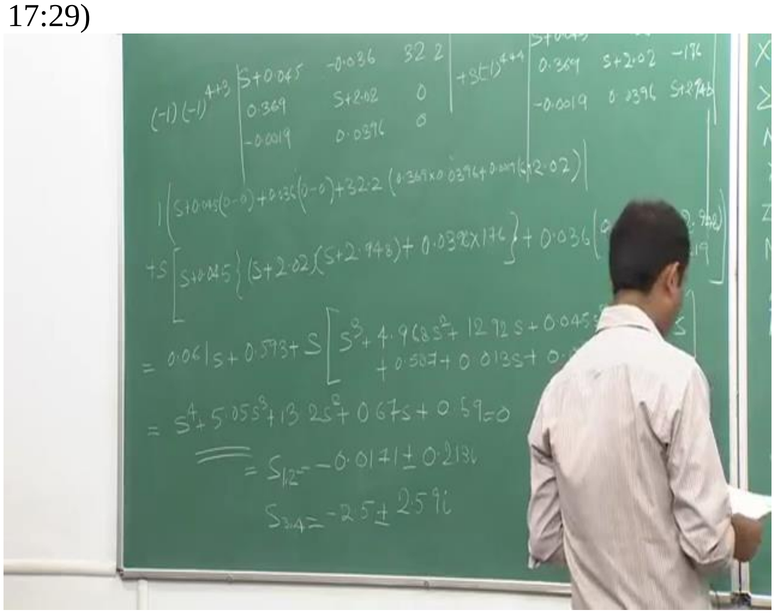

And the characteristic root for this equation will be, I will show you how to derive this afterwards. That characteristic root will be equal to since this is a fourth order equation, so we will get 4 roots. That too, complex conjugate pole. S12 equal to - 0.0178 + - 0.213I. S34 will be - 2.5 + - 2.595.

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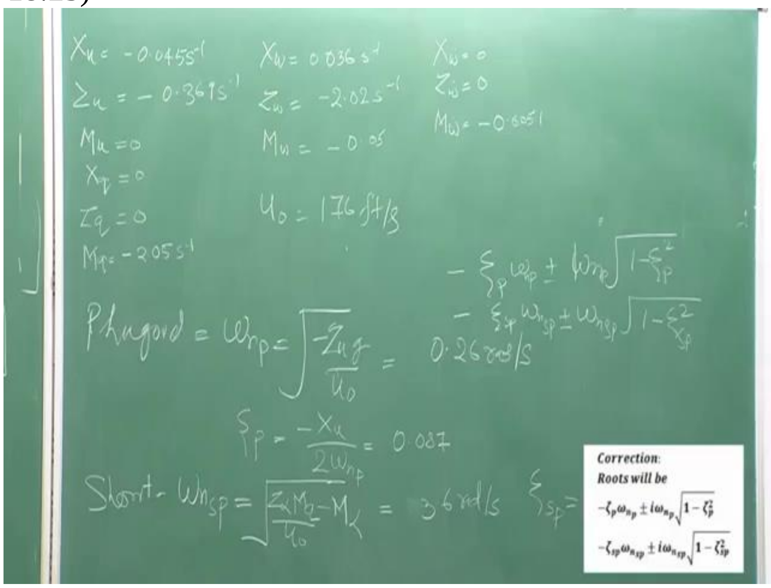

So these were the roots derived for the characteristic equation of the fourth order from approximation. Roots will be given by - Zeta Phugoid Omega Phugoid + - Omega N root over 1 - Zeta square. Similarly for short period approximation, the roots will be given by - Zeta short period Omega N short period + - Omega N short period root over 1 - Zeta short period S square.

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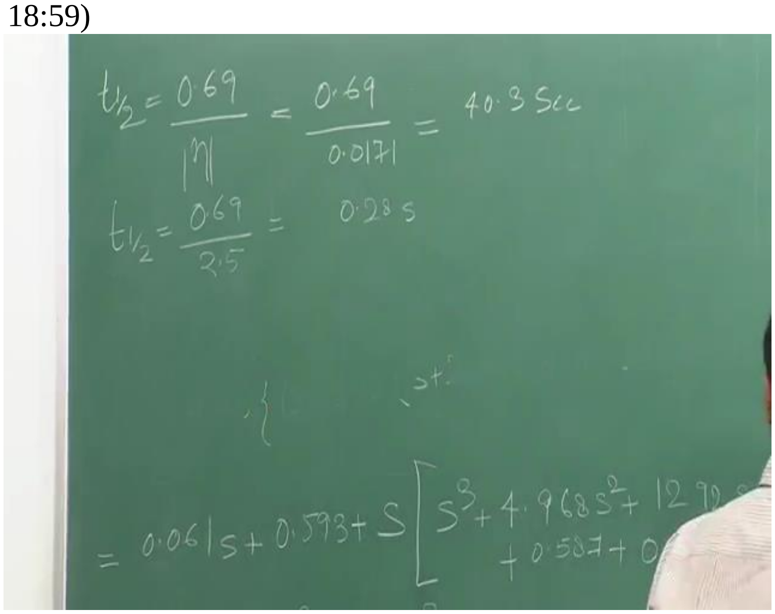

Now for Phugoid mode, if we calculate the value of T half, it is given by 0.69 by Eeta where Eeta is the real part of that root, will calculate to 6.09 divided by 0.0171. This is for the characteristic equation, the roots which we got for the characteristic equation. What will be the of T half? It will be, this is mode. So this will be equal to 40.3 seconds.

Similarly, T half for short period mode will be 0.69 divided by 2.5 which will give me 0.28 seconds. This was the T half we got for, roots we got for characteristic equation.

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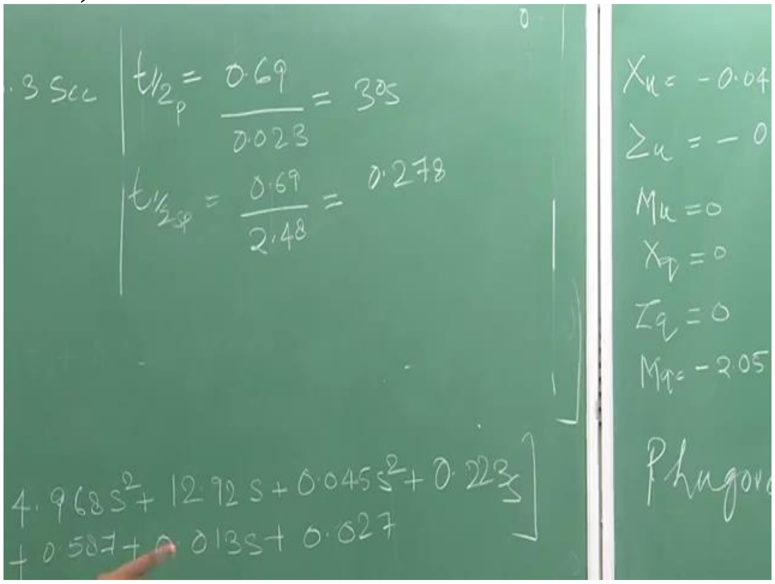

Now the roots which we got while approximating Phugoid and short period mode, from that, T half will be, T half Phugoid mode request to 0.69 divided by 0.023 equals to 30 seconds. Similarly, T half for short period will be 0.69 divided by 2.48 equals to 0.278. You can check that by calculating the value of short period and Phugoid mode approximation. Just substituting the values of Zeta and Omega N, you will get the value of roots what will be for Phugoid and short period mode.

Now, as you can see the exact root gave me a T half of 40.3 seconds. While doing approximation for Phugoid mode, I got T half as 30 seconds. So as you can see, the T half which I got from the exact root analysis was 40.3 seconds and the T half which I got from Phugoid mode during approximation is 30 seconds. This gives an error of about 25%. While for short period mode, the error is very minimal.

Or you can say, the approximation was very accurate. So now coming back to how to derive the exact root of this fourth order polynomial or higher order polynomial? For that we will be using a numerical method known as Newton Rapson for calculating roots of higher-order equation.

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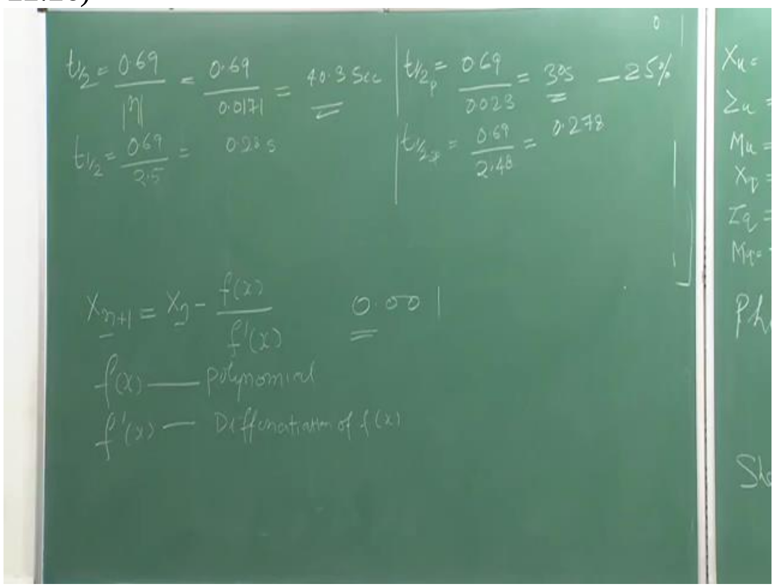

Newton Rapson method numerical calculation works on the formula given by XN + 1 equals to XN - F of X divided by F dash X where FX is your polynomial for which we have to derive characteristic roots or we have to derive roots. F dash as the word is clear, this will be the derivative of this polynomial. Now this and is your previous value and what will be the value you get from substituting this value in the function? You will get next value.

And you substitute till you reach, your solution converges or you reach a tolerance value. For instance, I have given my tolerance value as 0.001. So when this tolerance value is reached, my difference between the previous value and the next value is within the limit of 001. My solution will convulse there. So I will show you how to the find roots of characteristic equation that is fourth order longitudinal matrix.

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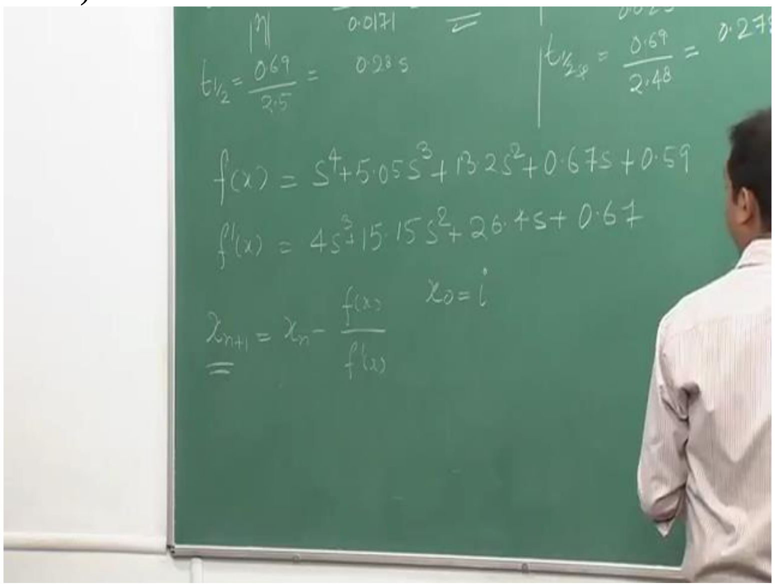

So my function, FX will be given by S4 + 5.05 S cube + 13.2 S square + 0.67S + 0.59. No differentiation of this F dash X will be 4 S cube 15.15 S square + 26.4S + 0.67. Now using Newton Rapson method, that is XN + 1 equals to XN - F of X upon F dash X. Now there is one limitation to Newton Rapson role. If you know the roots are in complex pair, you have to give initial value as a complex number.

Otherwise, you should give it a real number. The solution would converge if your characteristic equation contains only complex roots. So solution will not converge and it will continue to run for an infinite loop. So that is why, if you know that your solution contains a complex root, so better give the value of XN as a complex number.

Since we know, this is a fourth order polynomial which consists of 2 roots, both complex and conjugate, so suppose this initial value of M I will give, X not equals to I. Let us see what will be the value of this Newton Rapson where it converges for the value of X not equals to I. So for X not equals to I, substitute this value of I in my equation, FX and F dash X to get the value of this numerator and denominator which will be subtracted from X equals to Y.

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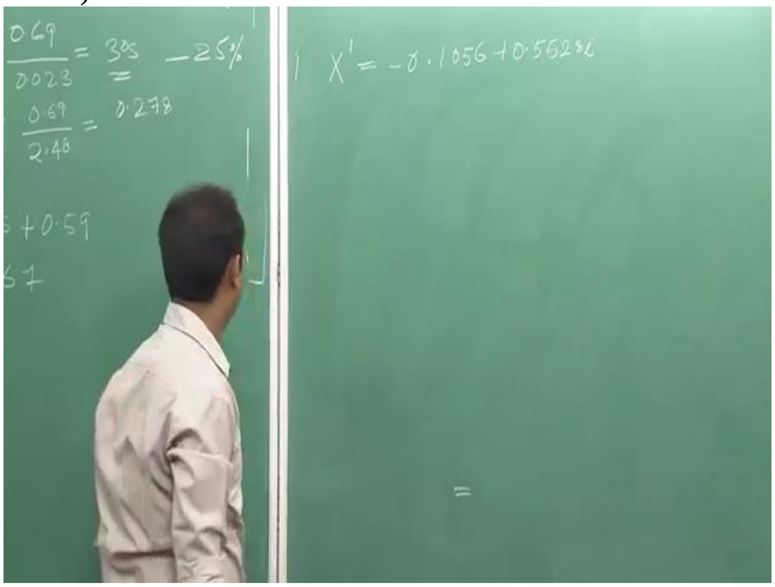

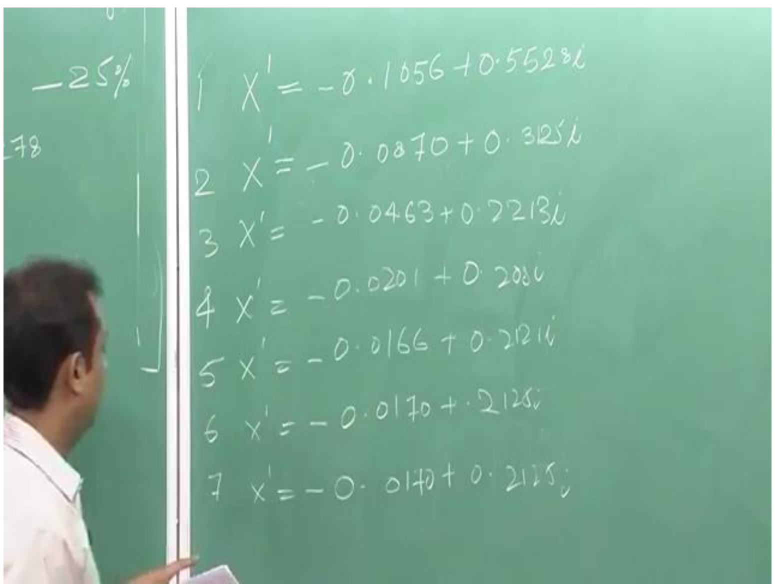

This gives me, for first iteration, X dash will be equal to - 0.1056 + 0.55289. Now substitute this value, the new value of X in the equation of Newton Rapson. So now XN will become the value which is obtained after iteration 1. And the new value of XN + 1 will be after second iteration, it will be, again repeating the same process, this will be…

Now as you can see, after 6^{th} iteration, the solution of this equation will have the same value, this value will be repeated. So we can say, the root of this characteristic equation or we can say one root of this characteristic equation will be 0.0170 + 0.2125I. Now since we know, these are fourth order equation, it will have 4 roots and all the roots will be conjugate of each other.

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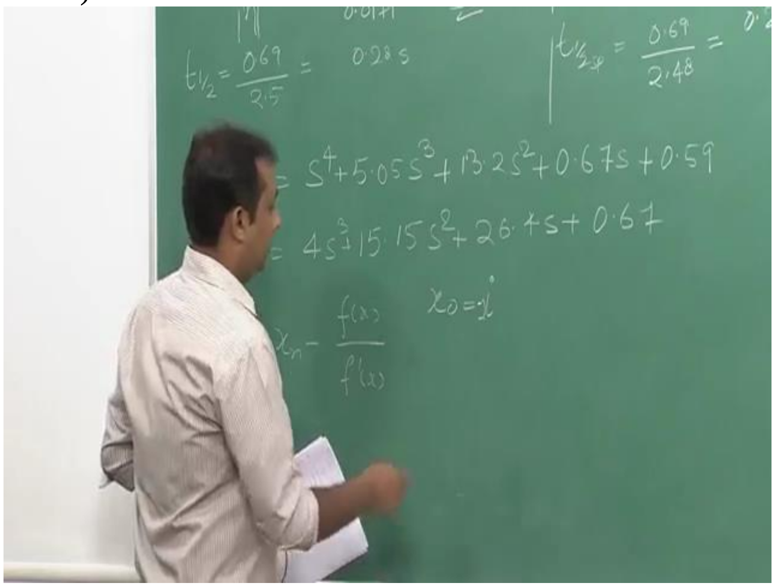

So they will be same roots with a mirror image of this root, that will be - 0.0170 - of 0.2125 I. You can verify this by putting the initial value of X not as - I. The root will converge at that particular point. Now this was, for first root means or you can say pair of root. Now if you want to derive the second root or the second conjugate root, you have to change the initial value of X not to some higher value.

Because the Newton Rapson roots will converge to the nearest roots. So if I give - I or - 2I, it will continue to converge. That particular point , you will not be able to get second root.

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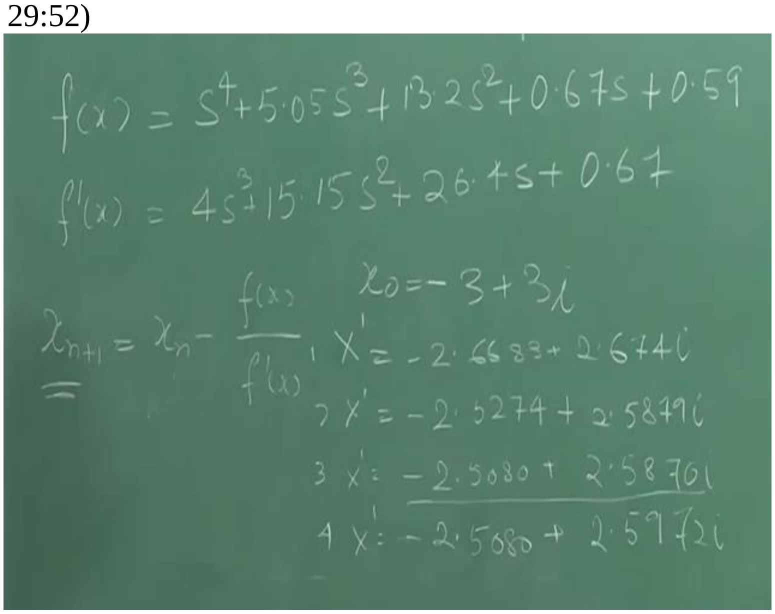

To get second root, let us take the value of X not as - 3 + 3I. Now if you give initial value as - 3 + 3I, your Newton Rapson will give solution as X dash for first iteration will be - 2.6683 + 2.674I. This is for second iteration, we will give - 2.5274 + 2.5879 I. third iteration will give X dash equals to - 2.5080 + 2.5870.

fourth will give X dash equals to - 2.5080 + 2.5972I. So you can see, as the roots are repeating, so we can say, this is another root of the equation. So this is one root. And we have a mirror image of those. So another root will be same with a - sign. So we can say, these are the roots or you can say these are the exact roots of the characteristic equation given by this polynomial.

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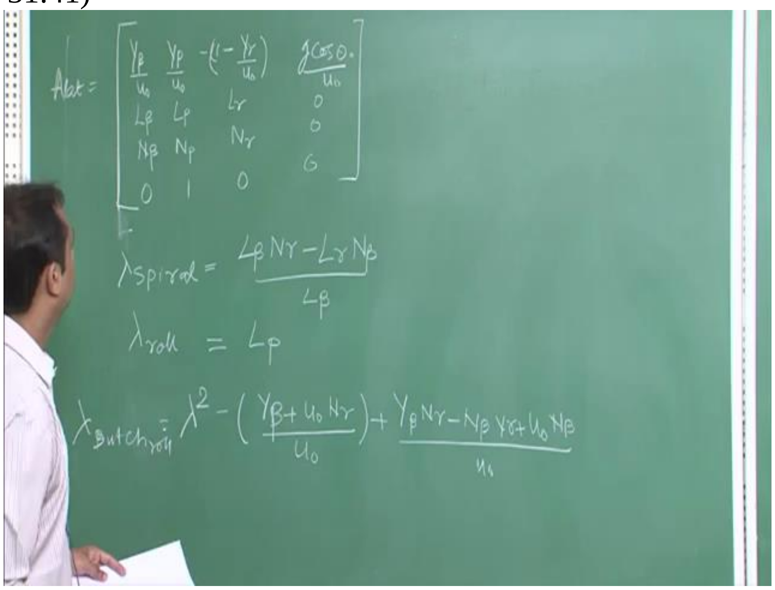

Previously we saw the characteristic equation of the longitudinal mode. So now we are doing a problem on lateral mode. Here also you know, the lateral matrix is given by this formula of matrix. The roots of lateral directional characteristic equation is our spiral roll and Dutch roll. The roots for different modes are, for spiral mode, is given by L beta NR - LR N beta divided by L beta.

Similarly for roll, it is LP. For Dutch roll, you can get the roots by solving the characteristic equation. Lambda square - Y of beta + U not N of R by U not + Y by beta NR - N beta YR + U not N beta the whole by U not. This is for Dutch. These are the roots for various modes approximation which will be the roots given for spiral roll and Dutch roll modes. So based on this, let us see a numerical.

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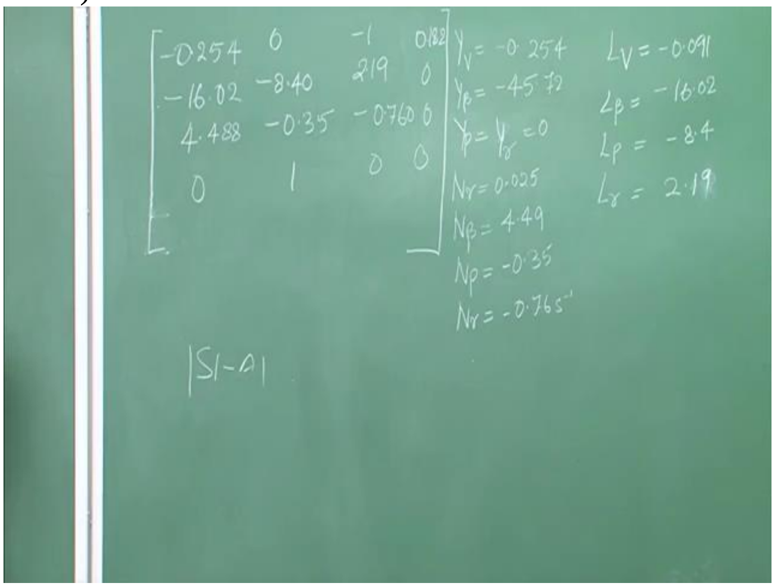

The matrix for lateral dynamics is given by this equation and various lateral derivatives are mentioned here. So first of all, what we will be doing? We will be finding the characteristic equation for this matrix and as we know the characteristic equation for any matrix is given by SI - A, determinant of this equals to 0.

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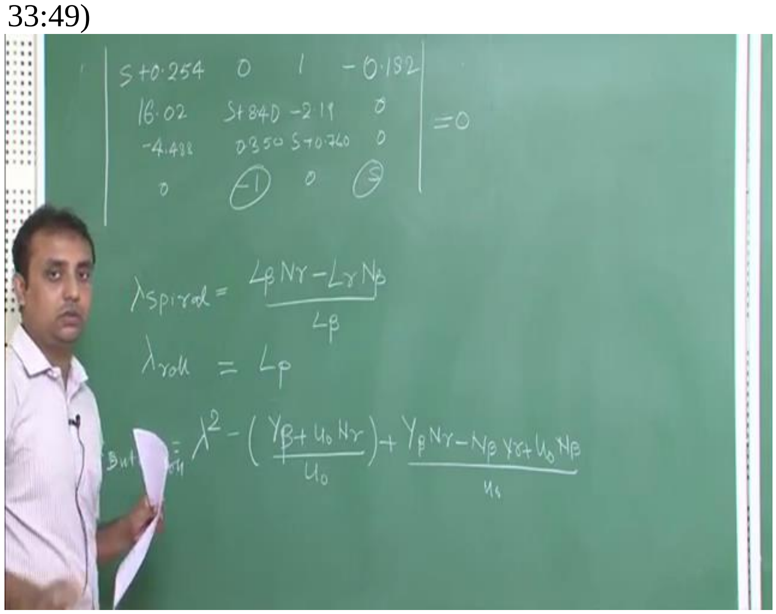

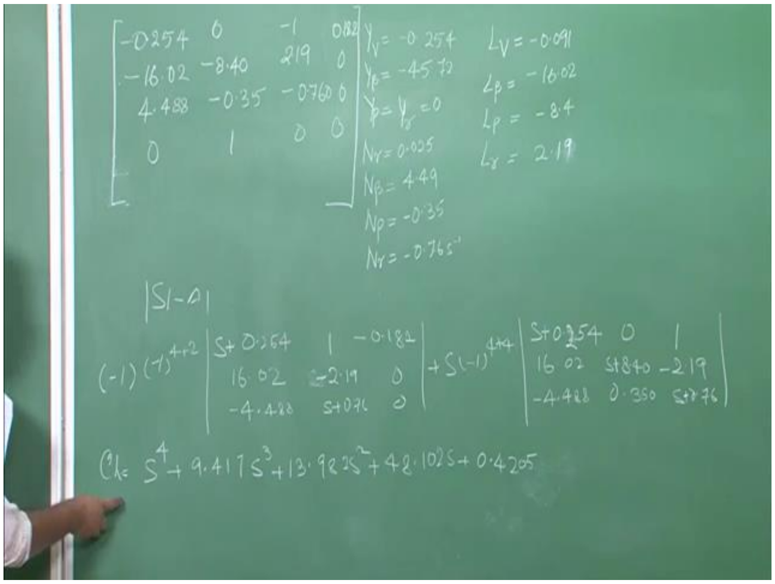

Characteristic equation we will get, S + 0.254, 0, 1, - 0.182, 16.02, S + 8.40, - 2.19, 0, - 4.488, 0.350 S + 0.760, 0, 0, - 1, 0. Determinant of this equating to 0. So for my convenience or you can say the calculation will be easy if I choose this element and this element because I have to deal with only 2 submatrices.

So we will get characteristic equation as 0.254, 1, - 0.182, 16.02, - 2.19, 0, - 4.488, S + 0.76, 0. + S - 1 to the power of 4 + 4 determinant S + 0.5254. This is 2. 0, 1. 16.02. S + 8.40, - 2.19, - 4.488, 0.350, and S + 0.76. Now I will not be doing the full calculation.

Just do the calculation and the final matrix which you will get for this will be known as the characteristic equation for lateral directional matrix is, S4 + 9.417 S cube + 13.982 S square + 48.02S + 0.4205. So this is our final characteristic equation for the given matrix. Now, if you want to calculate the roots of this characteristic equation or the exact roots, we will be using the Newton Rapson method which you have already seen.

Now as approximate solution or approximate root for this matrix will be given by these approximate roots for different modes.

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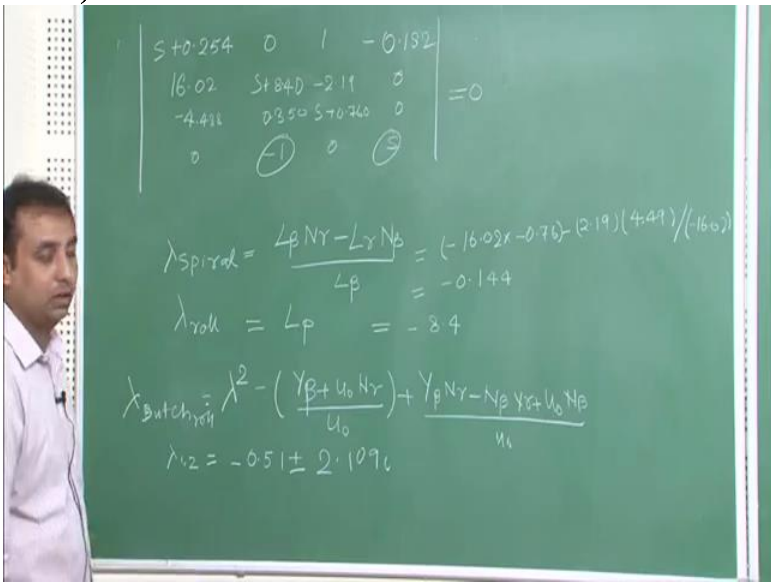

So equating these using derivatives given for these questions, the roots for spiral mode will be - 16.02 into - 0.76 - 2.19 into 4.49 divided by - 16.02 which will be equal to - 0.144.

Similarly for roll mode, and will be simply LP and NLP is given as - 8.4. Similarly for Dutch mode, you can substitute the value of Y beta with derivative from the equation and the roots for this equation will be Lambda 12 equals to - 0.51 + - 2.109I. Now using this equation, you can calculate the natural frequency and the value of zeta. Now what am I interested in?

What will be the values I will use so that this convulse very quickly. So the trick for that part is, the approximations of these modes are very easy.

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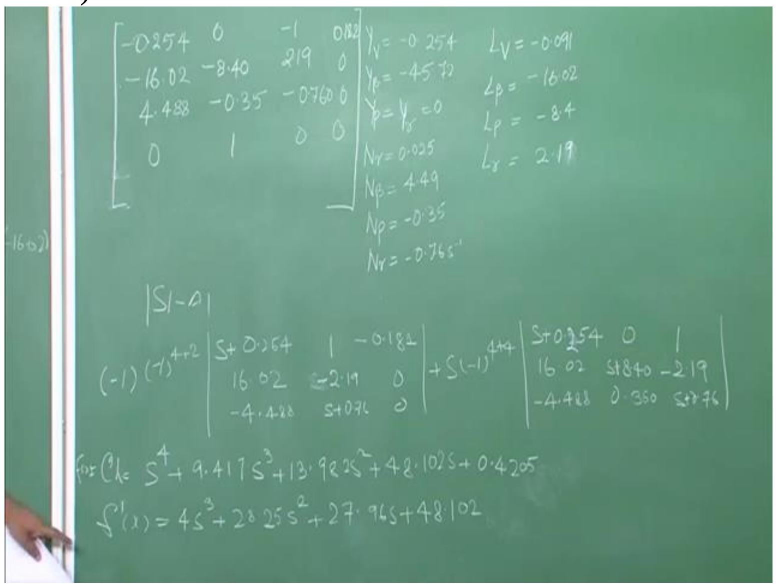

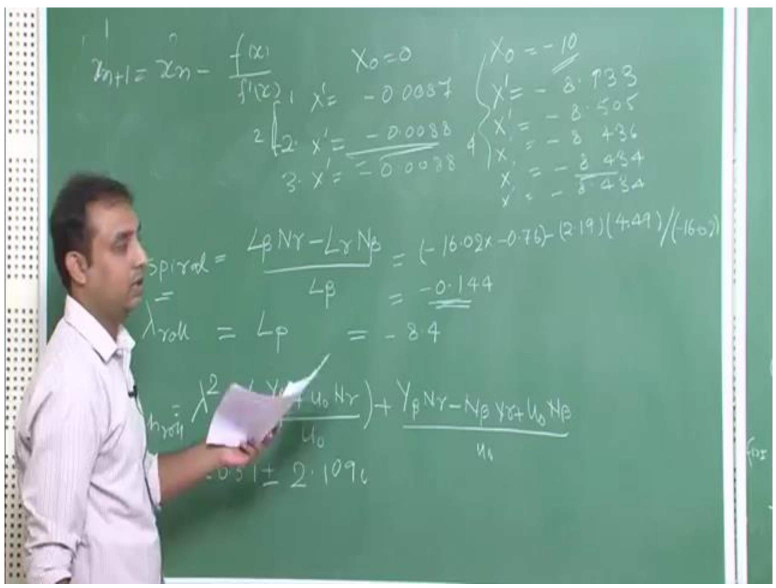

So keep the initial values and the Newton Rapson formula which we already mentioned was XN + 1 equals XN - F of X by F dash X where FX is our characteristic polynomial for which we have to determine the roots and F dash will be simply derivative of that. Let me write the F dash value of this also. This will be F of X.

F dash X will be given as 4 S cube + 28.25 S square + 27.96 S + 48.102. So this is FX, this is F dash X. So I was telling you, the trick part or you can say, what should be the value of this XN so that my series converges very easily. So for that, as you know, you can, the approximation for modes are very simple. So just use these values as initial value stop. So your series will converge with minimum number of iterations.

So let me show what will be the value for different, if we substitute these values in the initial value of this Newton Rapson formula, how many iteration it will require to reach a convergence. So for instance, since we have 3 roots and as you know that for a single root, just change the sign of this in the initial value, you will get both the roots.

Let me show you how many minimum iterations are required for a series to get converged. So let me put, X not equals to 0. So see, since my first root is very close to 0, so substituting this value in this function or you can say FX and F dash X, my value of F dash will be - 0.0087. This is first iteration.

After second iteration, now substitute this value in place of XN and you will get X dash equal to - 0.0088. third iteration, - 0.0088. So you can see, despite putting the value of 0, my iteration has reduced. Just after 2 iteration, I get the root of the equation. So this will be the root, exact root for your spiral mode.

So as you can see, the difference between, calculated by using exact formula and exact root which we got from the Newton Rapson method. That is why the approximation for spiral mode is not very accurate. So as you can see, my putting the value of X not, my series converges in 2 iterations itself. For second root, now let me put X not equals to - 10.

Simply you can put - 9 or - 8, it will even converge very fast. But let me put - 10. My first iteration will give me - 8.933, second iteration will give - 8.505, third iteration will give - 8.436, fourth iteration will give - 8.434, fifth iteration will give - 8.434. So you can see, after fourth iteration, I got the value of root, what will be the exact root of the characteristic equation.

So the more closer you put initial value to your approximate root, many a number of iterations will be required to reach the exact root. So my suggestion is that if you have already calculated the approximate mode, just keep that initial value close that value, you will get the exact root. And similarly since we have got these 2 roots, now that third root will be for this Dutch roll.

As I already mentioned that the live addition of this Newton Rapson method is that if you give a real number, it will always remain in real domain and your series will not converge if it has a complex route. So better give your roots in terms of complex numbers. Then your series will be giving you the exact root of that equation. Number of iterations required to reach third root.

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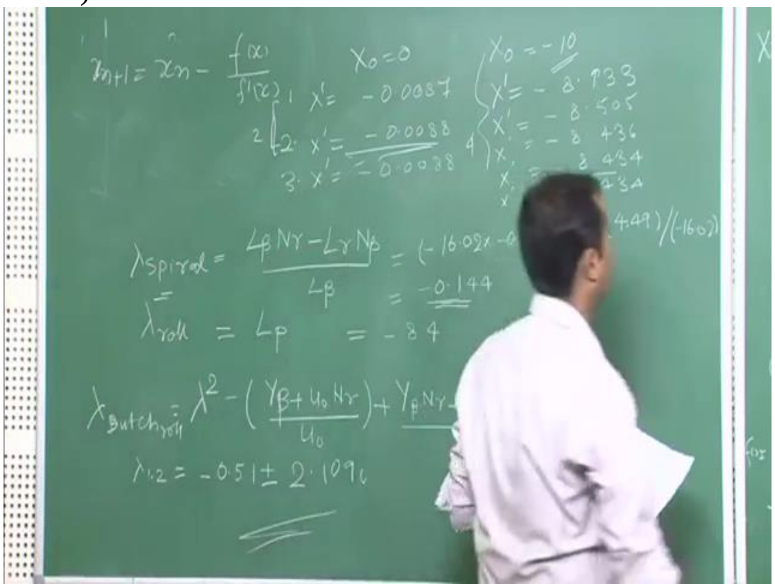

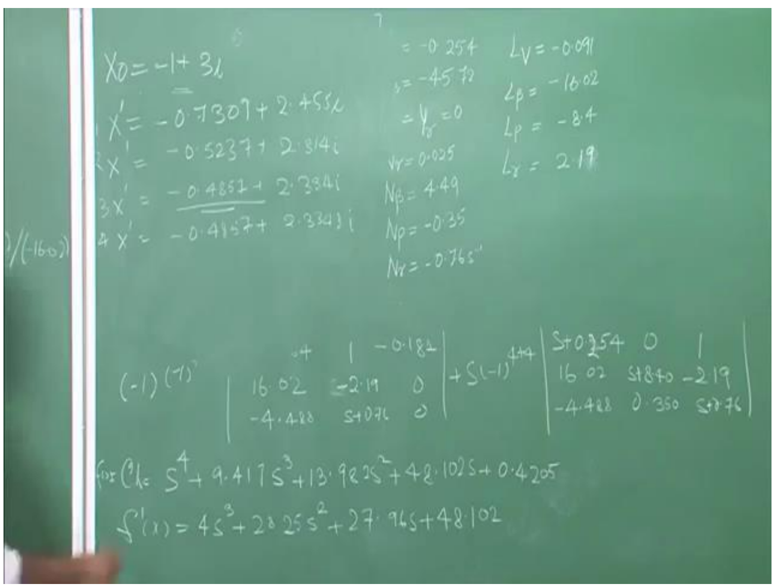

Let me put, X equals to - 1 + 3I which is very close to the root we calculated using approximate mode. So let me see how many iterations it requires. X dash after first iteration, value will be - 0.7309 + 2.4555. X dash after second iteration, it will give - 0.5237 + 2.214I. third iteration, -

0.4857 + 2.234I, fourth equation X dash equals to - 0.4857 + 2.3348. Now since the roots are repeating, you can see that this will be the root of the equation.

So you can see, the closer I will be with the approximate root or initial value is close to your approximate mode, number of iterations required will be less. So finally, this is the simplest method or you can say, this is the least troublesome method you can use to derive the root of order polynomial.

There are other methods. You can approximate this by using a, since you know, this is for lateral dynamics, you can use 2 first order equations and one second order equation. Then you can equal the coefficients. But it is very lengthy. + when you get the solution, that will also become a fourth order equation. So you will not reach everywhere. So this is the pattern, method to calculate the root of any Nth order polynomial.

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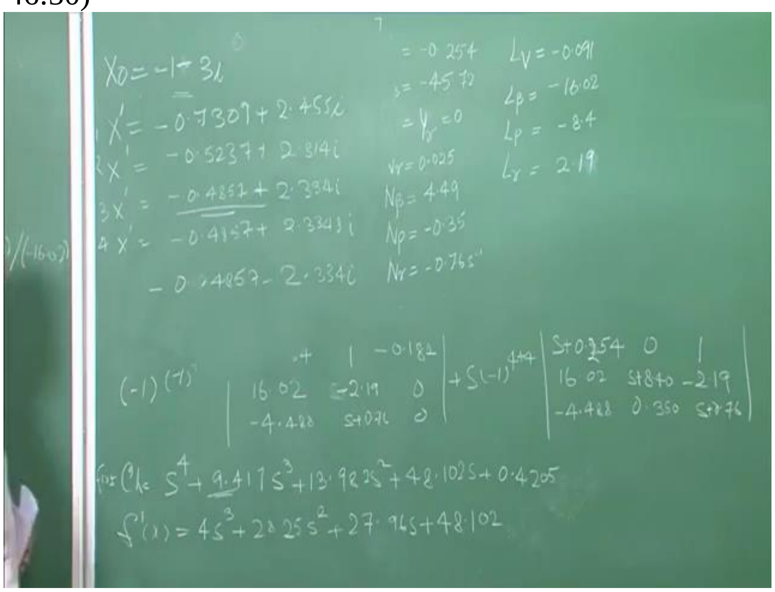

And one thing more, as I told you, since this is for 1 root, now as you know that complex roots are in pair which will be mirror images about X axis. So this will be somewhere about - 4. something here. Then another root will be just mirror image of this about X axis which we can get, just replace this initial value with - 1 - 3I. We will get another solution with same number of iterations. It will be - 0.04857 - 2.334. So this is the method to calculate characteristic equation of Nth order. And more the numerical you will do, it will be clear to you and that was my point of discussing this numerical to you. Thank you.