Aircraft Dynamic Stability & Design of Stability Augmentation System
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Module 5
Lecture No 30
Routh-Hurwitz Stability Criterion

Welcome friends. By now, you are familiar with the dynamics of the aircraft and the dynamic stability. In today's lecture, we will study the Routh stability criteria and the Routh array. Before we proceed for the Routh stability criterion, let us understand why this method was developed and how this is useful for us.

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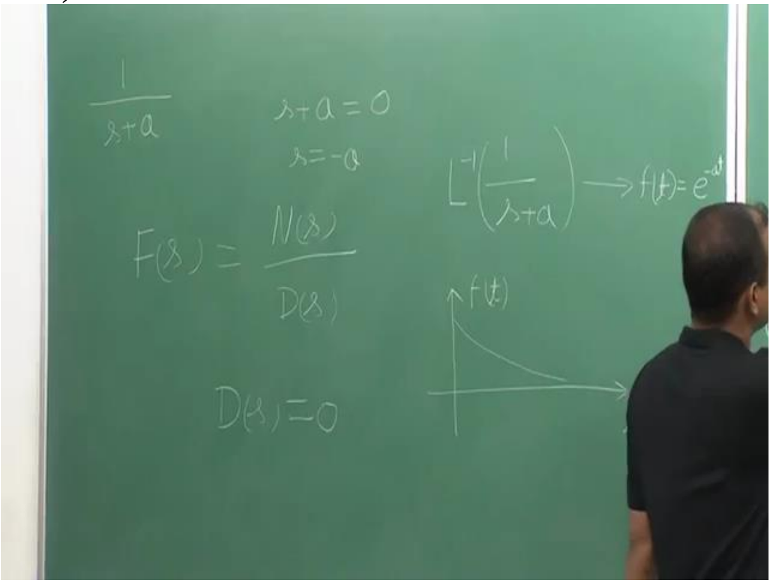

You know, if we have a system with the transfer function F of S is equal to N of S by D of S and if I equate this D of S with a 0, this I call as the characteristic equation of the transfer function. And the roots of this equation are known as the poles of the transfer function. For a system to be stable, the poles of this transfer function must lie in the left half of the S plane. Suppose I have a transfer function having a first order characteristic equation 1 by S + A.

In this case, my characteristic equation will be S + A is equal to 0. And the pole will be equal to S is equal to - A. If I take the Laplace inverse of this transfer function, suppose this is my F of S then the F of T will be E to the power - AT . And if I plot it it will be something like this. This is T and this is my F of T.

As the time tends to infinity, the response will die out and that is why our response is stable. Let us see more examples. Let me write on the board.

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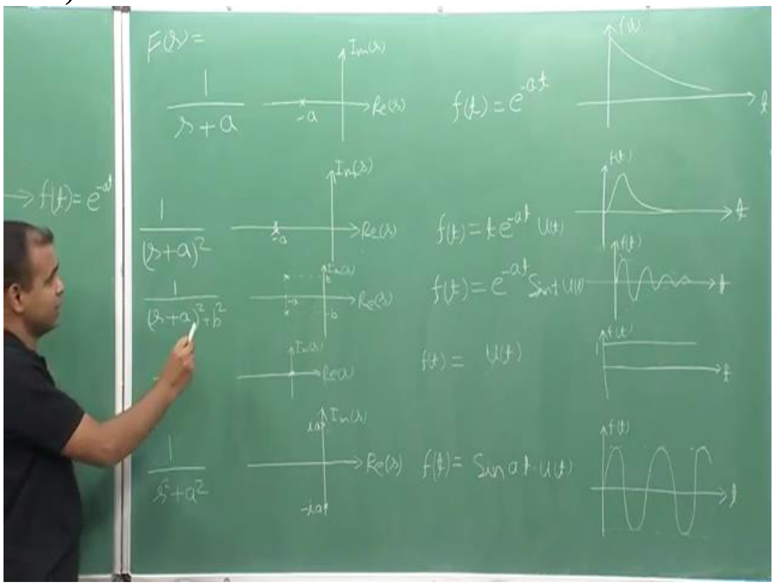

So if my pole is at - A, in time domain the F of T will be E to the power - AT and the response will be like this. If my transfer function is 1 by S + A to the power 2, in this case I will have 2 poles at S is equal to - A and its inverse Laplace will give us F of T TE to the power of - ATUT and it will look like this. In this case also, the system is stable.

If my system is 1 by S + A whole square + B square, in that case I have 2 poles in the left half, 2 complex poles, they are complex conjugate and in this case, my time domain equation will be E to the power - AT Sin UT and I will get the response like this. For 1 by S, I have one pole at the origin. The response is U of T and it will look like this.

In this case, the system is marginally stable. Because it is not tending to 0 and it is not divergent also, that is why it is marginally stable. If I have the transfer function 1 by S square + A square, in this case I have 2 complex conjugate poles and they are on the imaginary axis. In time domain, it will be sin ATUT and it will oscillate between 2 finite values and we call the system is marginally stable.

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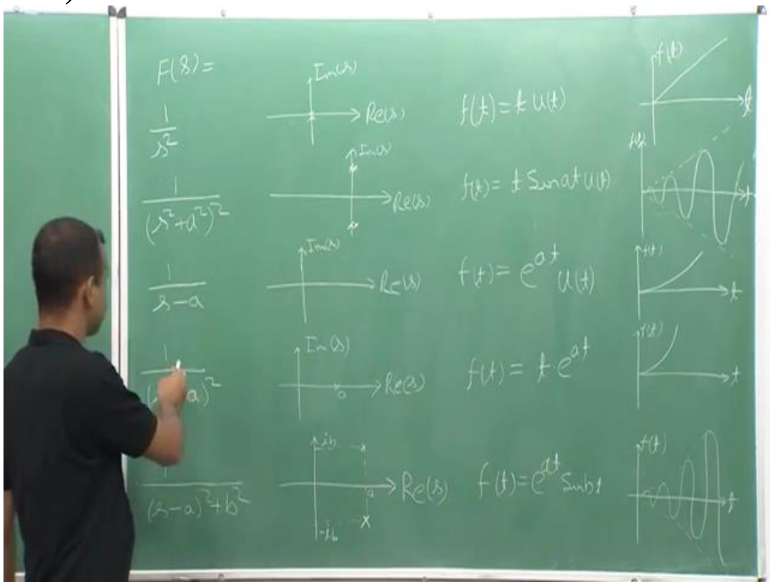

If my transfer function is 1 by S Square, in this case, there will be a double pole at the origin and my response will be TUT. It will be diverging and the system is unstable. For 1 by S square + A square whole square, in this case, there are 2 double poles on the imaginary axis and a response will be T sin ATUT and it will be a sinusoidal wave which is increasing its magnitude and this will also be unstable.

If I have a transfer function, 1 by S - A, in this case, my pole will be at S equal to A and in this case it will be E to the power ATUT and this will diverge with the time and the system is unstable. For 1 - S - A whole square, in this case I have 2 poles at S is equal to A and it will diverge faster than the previous one and this will also be unstable.

For 1 - S - A whole square + B square, there will be 2 poles in the right half in complex conjugate. And it will be E to the power AT Sin BT and it will be unstable. So one message is clear. If pole is in the right half, the system will be unstable and if there is double pole on the imaginary axis, in that case also the system is unstable. In this example and for this function also, the T sin AT and the system is unstable.

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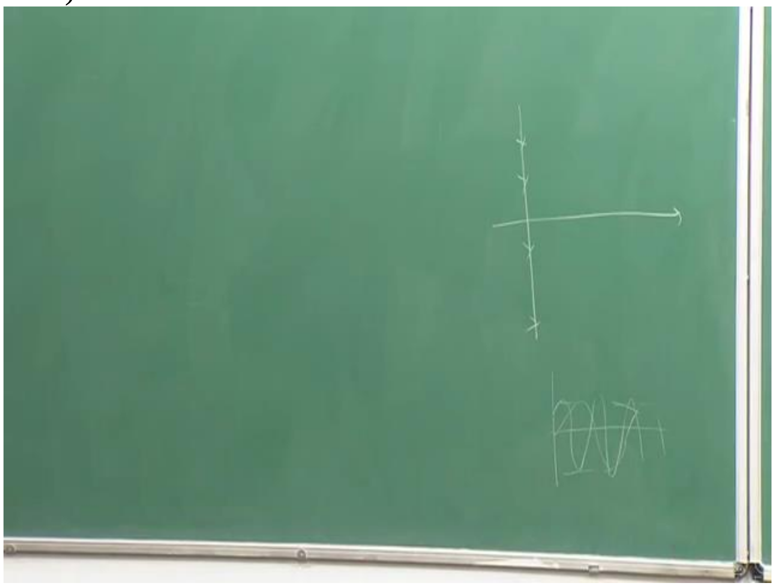

But if I have 2 poles on the imaginary axis at different locations in complex conjugate, in this case I will get, for individual pair I will get 2 sinusoidal waves of different magnitude and the resultant will be the superposition of both. And the system will be marginally stable. So if there is pole in the right half, the system is unstable even if I have a bigger transfer function.

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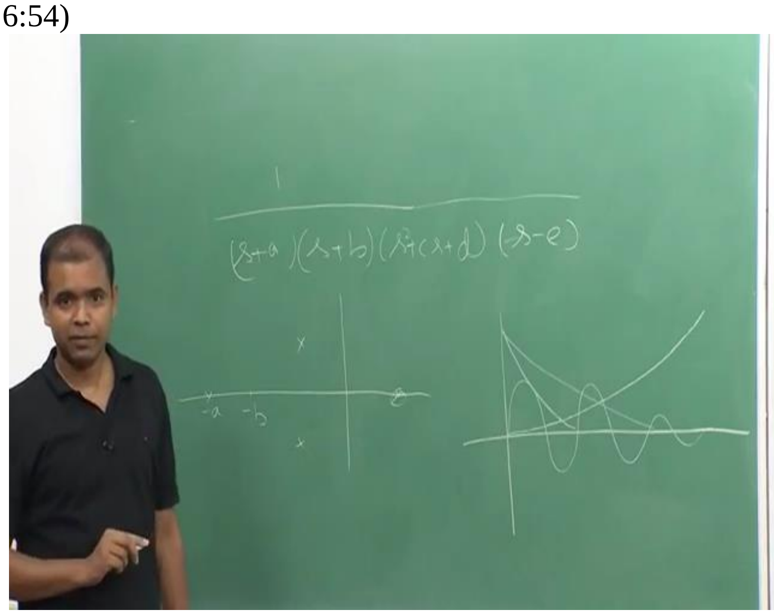

Suppose I have a transfer function something like this, 1 by S + A into S + B into S square + CS + D. Something like this and S - E. In this case, I will have poles like at - A, - B and suppose in this case, I will get one complex conjugate and one pole in the right half. If I see its response for this poles, S is equal to - A and S is equal to - B, I will get this decaying exponential.

For that also, I will get one decaying exponential for second one. For this complex conjugate, I will get some decaying sinusoid. You can see, the poles which are on the left side, the responses due to them are decaying with time and as T tends to infinity, they will die out. But the response due to this pole, I will get an exponential which will be building with time and as time tends to infinity, this will also become infinity.

That is why, the system will be unstable. So it does not matter how many poles are in the left half. Even if a single pole is in the right half, the system will be unstable. If I have characteristic equation up to second or third order or maximum up to fourth order, it is easy to find the roots of the characteristic equation or the poles.

But it becomes a tedious task to determine the poles of the transfer function if I have a characteristic equation of 5^{th} order, order 5 or 6 or 7. And in this case, the Routh stability criteria is used and this converts this tedious task into a simple task.

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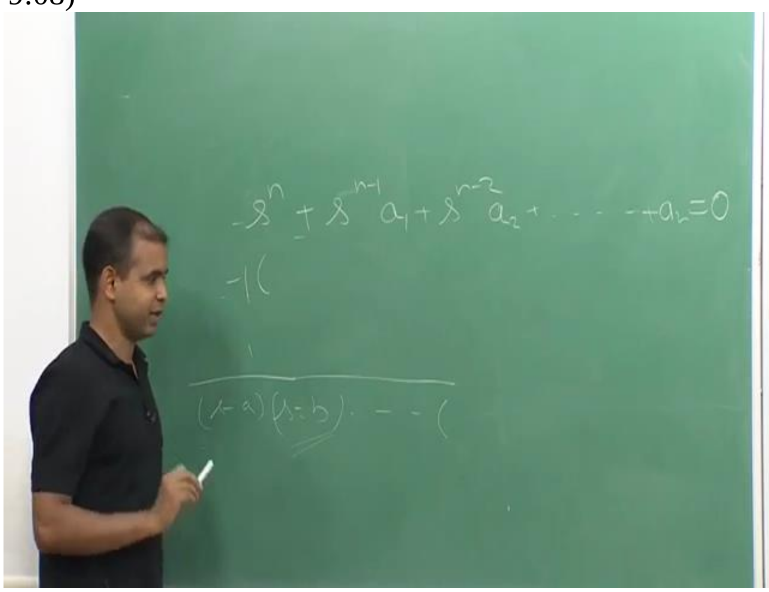

Suppose my characteristic equation is S to the power N + S to the power N - 1 A1 + S to the power N - 2 A2 + something + AN is equal to 0. So all coefficients of S must be positive or must have similar signs. Even they are all negative. In that case, we can take - 1 negative and all will come positive. So it does not matter, in the case of stability, it does not matter whether they are positive or negative.

All must have the same sign. If there is any sign change in the characteristic equation, the system will be unstable. And it is quite obvious because if I have S - A S - B or something like that, if there is any pole having negative sign, it will reflect a negative sign in the characteristic equation. This means, there is one pole, at least there is one pole in the right half.

So the first criteria of the Routh stability is, all coefficients of the characteristic equation must have similar sign. The second criteria given my all coefficients are positive, there may be both possibilities. The pole may lie in the left half or may lie in the right half also. And to check this, we use the Routh array.

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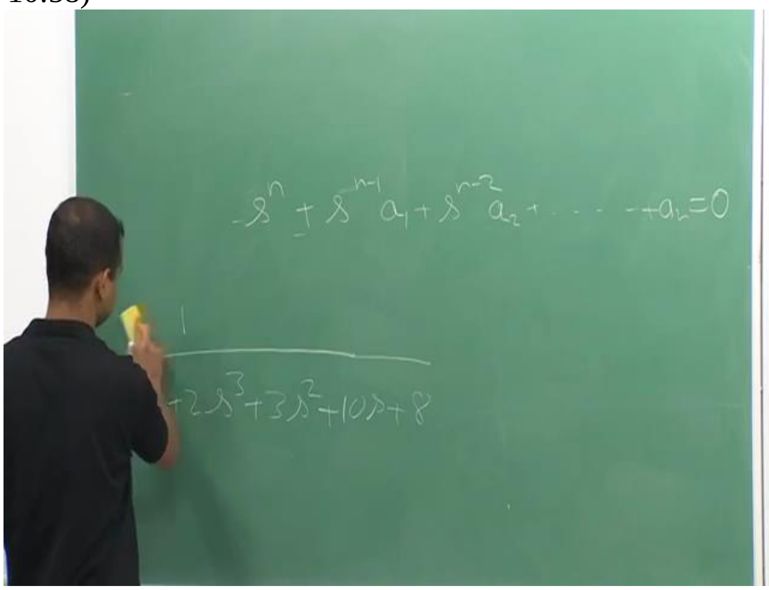

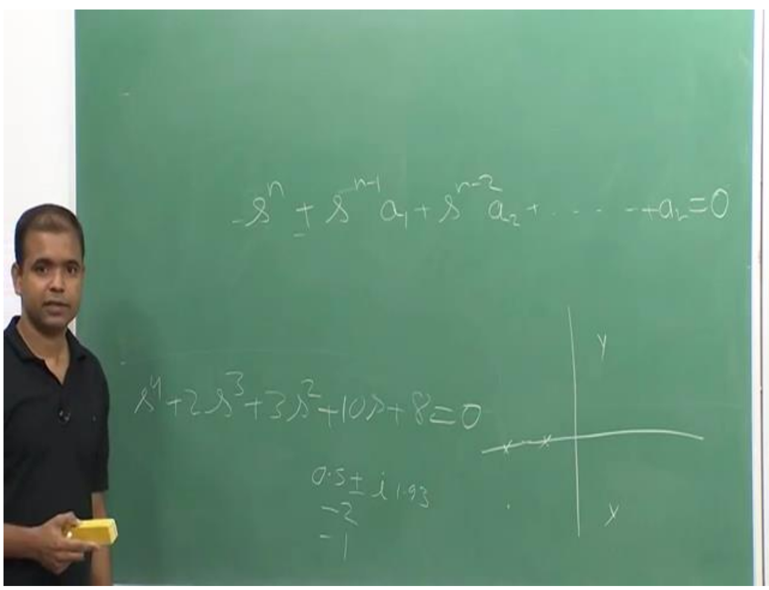

See this example. If I have a transfer function like 1 by S to the power 4 + 2S cube + 3S square + 10 S + 8. In this case, the roots of this characteristic equation, if I write this as a characteristic equation, the roots are 0.5 + - I 0.93. - 2 and - 1. Here, my all coefficients are positive. Still I have this pair which is in the right half of the S plane.

If I plot it, I will get this pair here, somewhere here and one pole here and one pole here. So even when my all coefficients are positive, there may or there may not be roots in the right half of the S plane. For this, we use the Routh array to determine whether any pole is lying in the right half or not. Let us understand this Routh array with examples.

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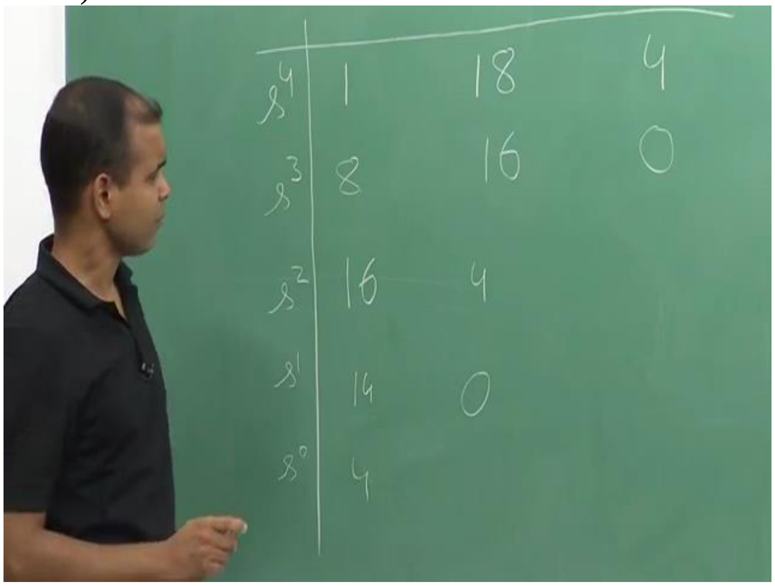

In our first example, suppose my characteristic equation is S to the power 4 + 8S cube + 18S square + 16 S + 4 is equal to 0. first of all, the all coefficients must be positive or all coefficients must be negative. They must have the same sign and the first criteria is satisfied. So to construct the Routh array, we write the highest power, S to the power 4.

And then, if this is even then we write all the coefficients having even power of the S in descending order. So, one coefficient of this. Then 18. Or S to the power 0 that is 4 the constant term. Then you write, in descending order, all the powers of S. Here, the coefficient of S cube is 8 and then S to the power 1, that is 16.

Now we will put 0 because there is not any term. Here, we multiply these 2 terms and subtract the multiplication of these 2 and divide by this. So in this case we will get, 18 into 8 - 16 and divide by 8. This will be 16. At this place, 8 with 4 and 1 with 0 and divide by 8. So this will be 4. You can directly write 4 if here it is 0 because this has to become 0 and we are ultimately dividing by this term only. So we can directly copy it here. Here, we have to multiply these 2 terms, subtract this and divide by this. So 16 into 16 which is equal to 256 - 8 into 4 divide by 16. And this will be 14.

And this term, you can write directly here because this multiplied by this - 0 divide by 16. This will be 0. Here, this multiplied by this. So this will be 4. So the second Routh stability criteria is there must not be any sign change in this column. As many sign changes are here, the same number of poles are lying in the right half of the S plane. In this case, there is no any sign change, in this column. So there is no pole in the right half and the system will be stable. Let us solve another example with a different case.

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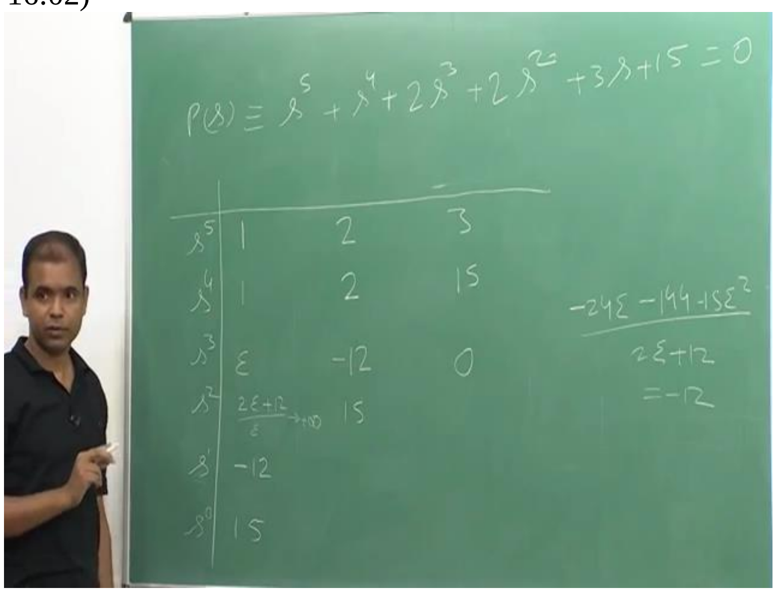

The characteristic equation, P of S is given by S to the powers 5 + S to the power 4 + 2S cube + 2S square + 3S + 15 is equal to 0. So if you construct the Routh array, the highest power is S to the Power 5. This is odd number. So I will write all the coefficients having order power of S. So this is 1. And the coefficient of S to the power 3 is 2 and 3 here.

In descending order, I will write, S to the power 4, S to the power 3, S Square, S and S to the power 0. The coefficient of S to the power 4 is 1. The coefficient of S Square is 2 and S to the power 0, that is the constant term is 15. Here, 1 into 2 - 2 divide by 1. So this will be 0. - 12. And the term here I will get is 0.

So if we are getting a 0 in the first column, we will replace it with a very small positive number and further we will proceed our calculation. So we will replace it with a small positive number that is Epsilon. Here I will get 2 Epsilon + 12 by Epsilon. And Epsilon is a very small positive number which is tending to 0.

Here I will get 15 into Epsilon divided by Epsilon. So it will be 15. If Epsilon is tending to 0, this term will tend to infinity. If limit Epsilon tends to 0, this will be infinity. So this will be tending to + infinity. The term that I will write here is the - 12 times 2 Epsilon + 12 by Epsilon - 15 Epsilon the whole divide by 2 Epsilon + 12 by Epsilon.

And if I solve it, this is - 24 Epsilon - 144 - 15 Epsilon square by 2 Epsilon + 12 and if Epsilon is tending to 0, this will be - 12. So I can write here directly, - 12. And here I will get 15. So in this case, the first column, this is a positive number and here is one sign change from + infinity it is becoming - 12 and then again from - 12 it is becoming 15.

There is 2 times sign change means the transfer function having characteristic equation, this PS, will have 2 poles in the right half of the S plane and if there are 2 poles in the right half, means the system is unstable. Let us solve another example.

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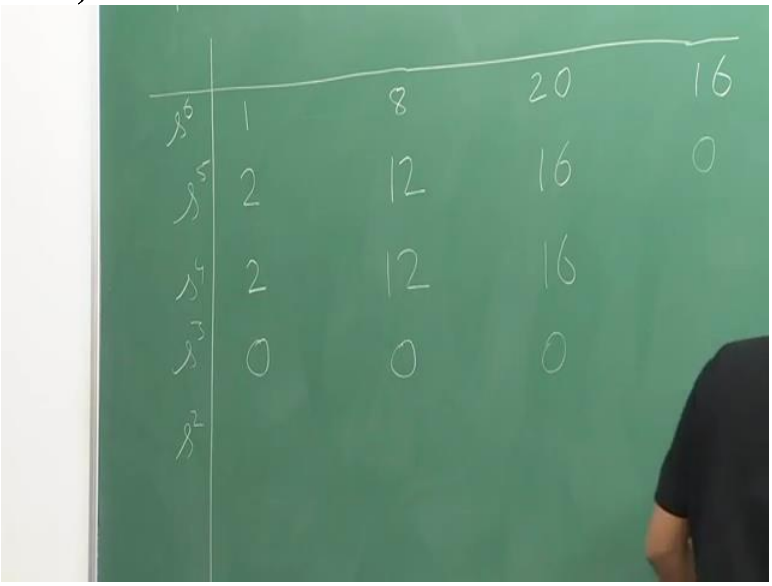

P of S is equal to S to the power 6 + 2S to the Power 5 + 8S to the power 4 + 12S cube + 20S square + 16S + 16 equal to 0. If I construct the Routh array, the highest power is S to the power 6 and this is even. So I will write all the coefficients having even power of S in descending order. So this will be 1, 8, 20 and 16.

Now the coefficient of S to the power 5 in this array. 2, 12, 16 and here I will write 0. For S to the power 4, I multiply 8 into 2 which is 16 - 1 into 12 divide by 2. So this will be 2. 2 into 20 which is 40 - 16 will comes 24 divide by 2. So this will be 12. And here I can write this 16 directly because here is 0.

Now, for S to the power 3, I will get these both numbers are same. So this will be 0. 24 - 24 will be 0. And similarly here also, 2 into 16 - 2 into 16 will be 0. And obviously here will be 0. Here, all the terms are 0. In this case, we form auxiliary equation and auxiliary equation will be the row in which we are getting all the coefficients 0, just before that row.

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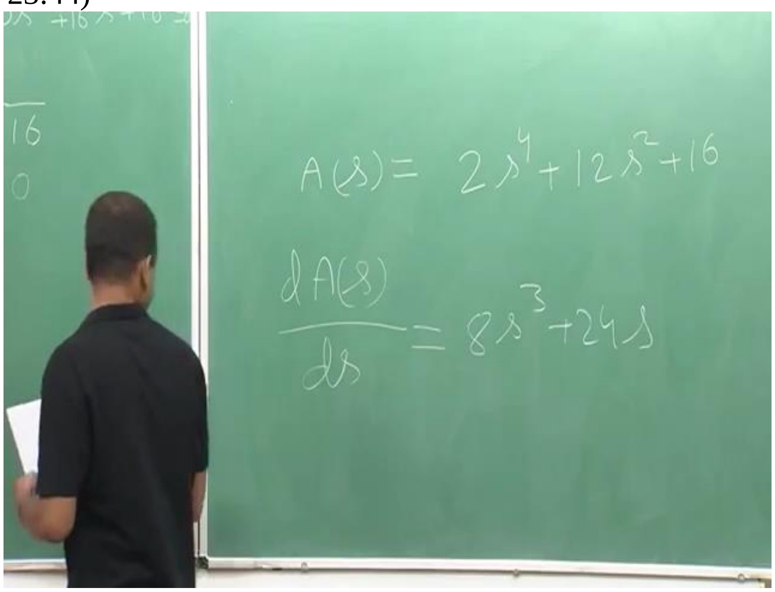

So in this case, our auxiliary equation will be A of S let me write it on another board. So auxiliary equation will be 2S to the power 4 + 12 square + 16. After differentiating, this auxiliary equation and whatever the coefficients I will get, I will replace this row containing all 0s with those new coefficients. And if I write DA of S by DS, I will get 8S cube + 24S.

So the coefficients, I will write here, 8 and 24. And I will proceed my Routh array. The coefficient that I will get here will be 6 if I calculate. And here I will copy this 16 directly because here is 0. So this will be 16. S to the power 1.

So here I will get 24 into 6 - 16 into 8 divided by 6 and this will be 16 by 6. And here is obviously 0. For S to the power 0, I will get 16. So in this case, all the coefficients are positive but when we get all 0 in a row, this means we have few poles on the imaginary axis and in this case we have to check whether they are double pole or not. If they are double pole, our system will be unstable, we have seen earlier.

So in this case, we have auxiliary equation 2S to the power 4 + 12 square + and we will have to find all the roots of this equation whether they are double or not.

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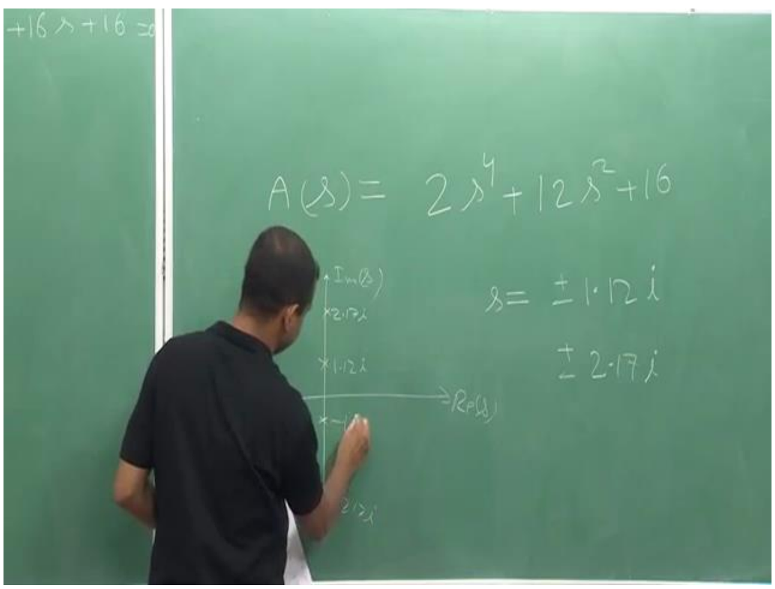

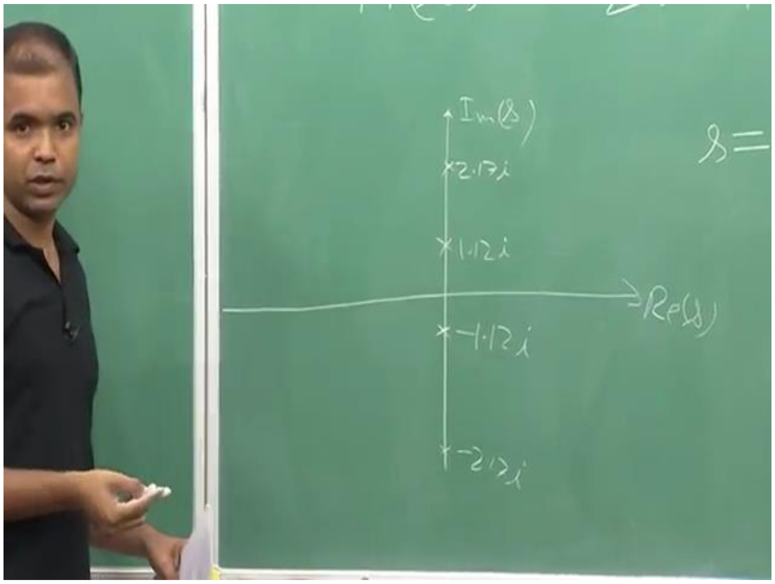

This is a fourth order equation but it is very easy to solve because there is no S cube term and if I replace S Square with T, it will become quadratic and if I solve it, I get S equal to + - 1.12I and + - 2.17I. If I plot it, it will be something like this, 2 complex conjugates and 1 pair somewhere here, 2.17 I, - 2.17I. - 1.12I.

So there is not any double pole on the imaginary axis. Means our system is marginally stable. It is not unstable. Let us see another example.

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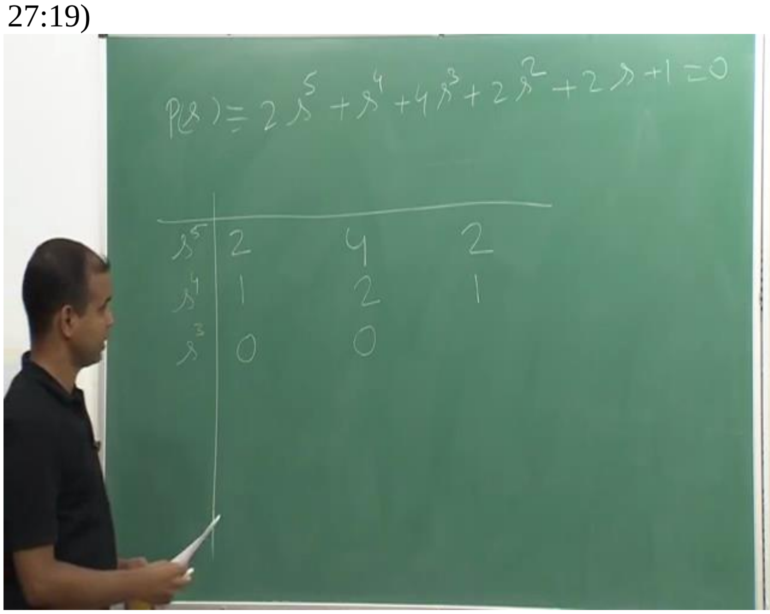

The characteristic equation is given by 2S to the power 5 + S to the power 4 + 4S cube + 2S square + 2S + 1 equal to 0. In this case, the Routh array if I form, starting from S to the power 5, 2, 4 and 2. S to the power 4, 1, 2 and 1. The coefficients in this row, S cube. 4 - 4, it will be 0. Here, 2 - 2, to be 0.

Again we are getting all the terms this row 0. So we will have to find the auxiliary equation and then we will have to find the new coefficients.

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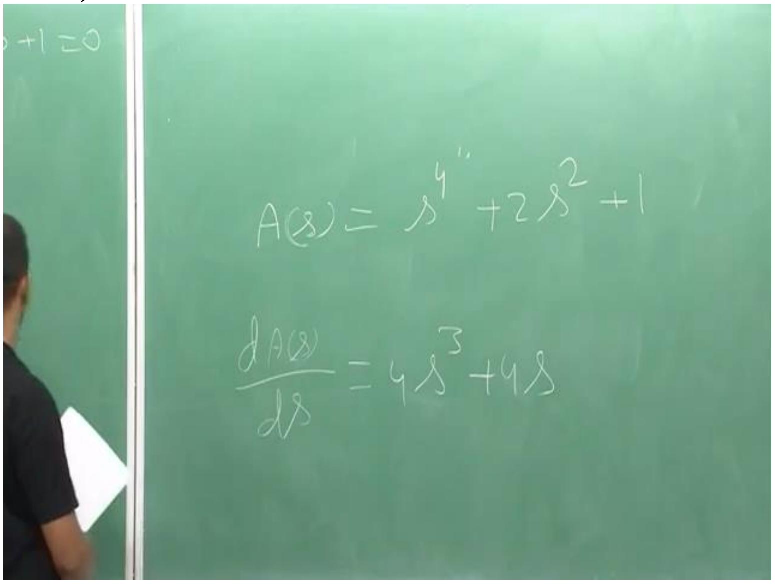

Suppose I write this auxiliary equation A of S, this is S4 + 2S Square + 1. And DA of S by DS will be 4S cube + 4S.

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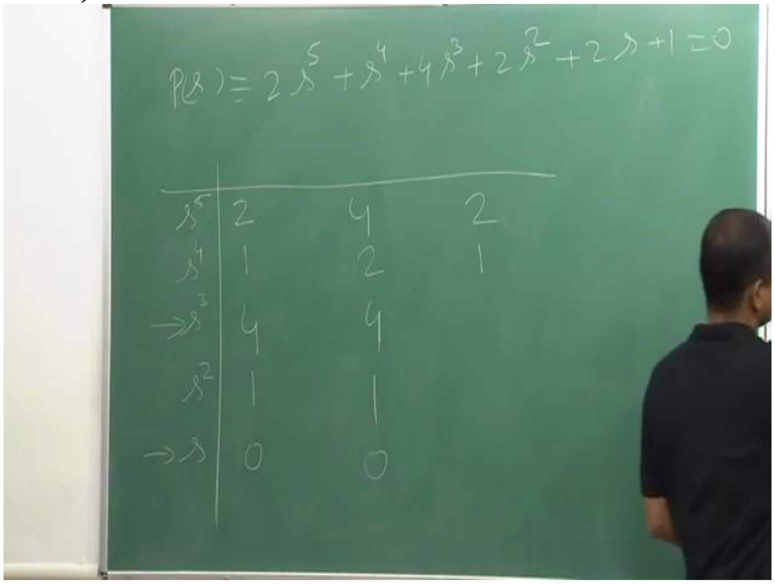

So I will have to replace this 0. In this row, we are getting all the 0s. Now for S square we will get 8 - 4 that is 4 divide by 4, that is 1. For S, I will get 4 - 4, this is 0. And this is already 0 here. So again we are getting all 0s in this equation. So I will have to form one more auxiliary equation.

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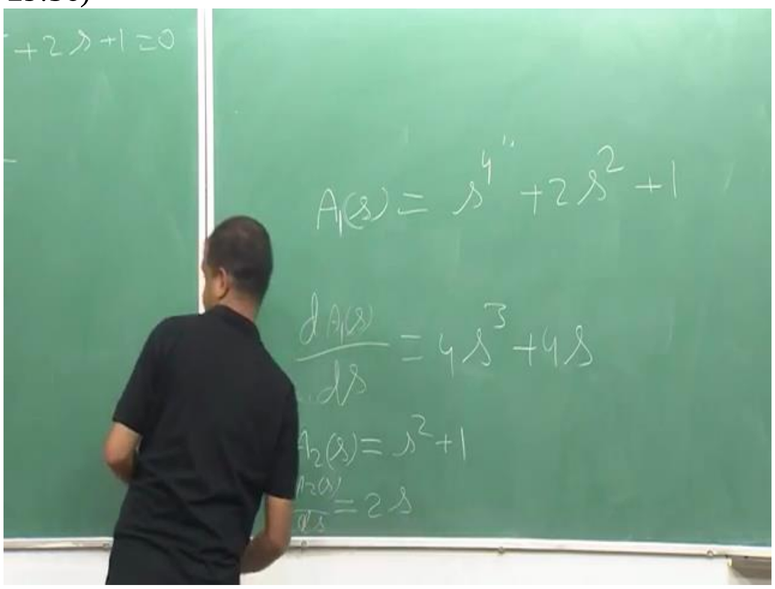

So if I write this A1 of S, now I will get second auxiliary equation, A2 of S and this is S square + 1. And if I differentiate it, DA2 of S by DS, I will get 2S.

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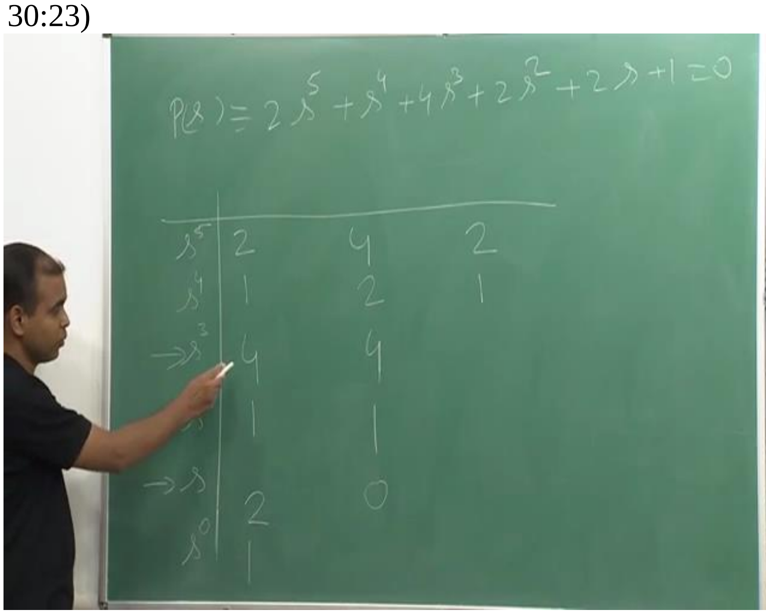

So I will have to replace this 0 with 2. And now for S to the power 0, I will get one. So all the coefficients are positive but we have to check whether any, there is any double pole on the imaginary axis or not. For this we will have to find. So we have 2 auxiliary equations. The roots for this equation we get, that is already the root of this equation. So we have to find the root of this equation only, of A1 of S.

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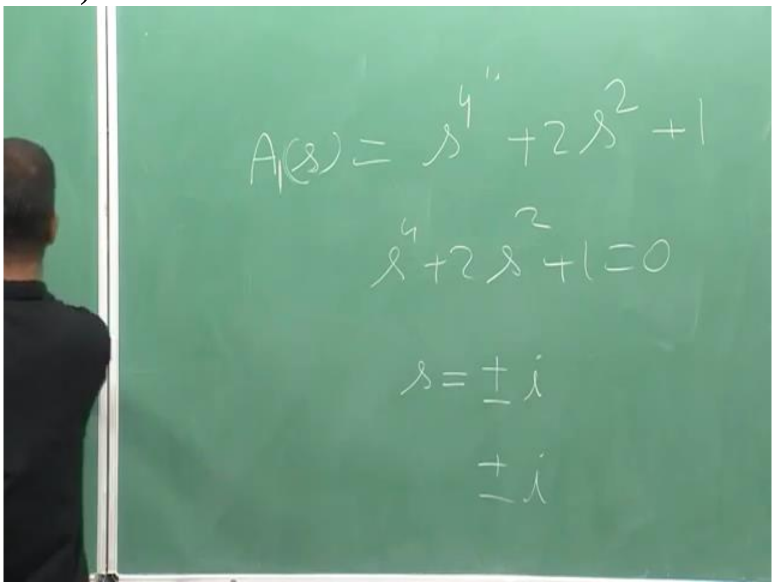

If I write S to the power + 2 square + 1 equal to 0, if I replace this S square with a new coefficient and I solve it, I get 2 roots as S equal to + - I and the second root is also + - I.

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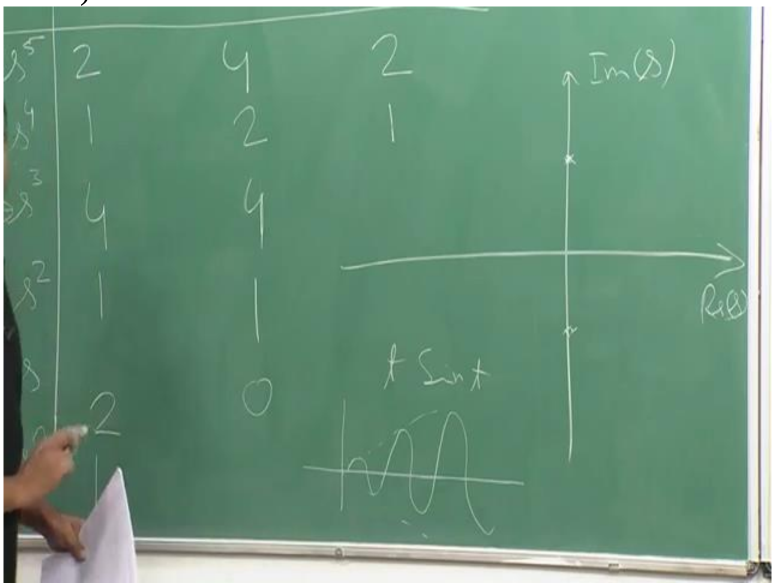

So if I write it complex plane, real part of S, imaginary part of S. So I have 2 roots here, 2 roots here. So we have double pole on the imaginary axis and my system will be unstable. If you remember, its response was T sin T. It will look something like, this is my T. Its response will be something like this and system is unstable. Thank you very much.