Jet and Rocket Propulsion

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Lecture - 7

So, in the previous lecture we had discussed how to estimate the burn out velocity of a rocket vehicle as well as the burnout height, using a phenomenological approach where we balance the forces and we consider the trust as a force being supplied as a reaction force. And then in the previous lecture we started a different approach of the launch vehicle dynamic estimation, where we considered the exhaust as the factor producing the trust. So, we will continue from the same discussion.

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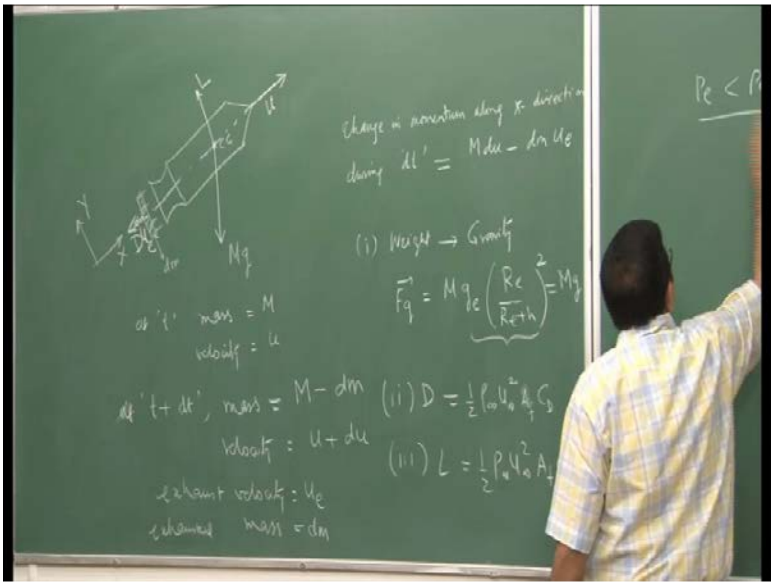

So, let me first draw the rocket that we were discussing last time. So, this is the rocket that we have discussing, this rocket is flying with a velocity u and this is the positive x direction, this is positive y direction. The forces acting on this rocket, where is the drag force acting in the negative x direction against the flight path, opposite to the flight path. Then the weight of the vehicle acting on with C G is given by M g, the lift is acting normal to the flight path given by L and we had considered that at time t mass of the vehicle was equal to M, the velocity of the vehicle was equal to u.

Then what we have said is that the rocket is flying at a particular condition, we look back at this rocket after time t plus delta t. That is after a small time interval delta t, what are the changes that has happened in this rocket that is what we want to see. So, we say that during this small time interval there is exhaust which comes out of this rocket. Let us assume that the mass of that exhaust which is coming out during the small time delta t was equal to d m and the velocity of this exhaust is equal to u e. So, then the mass at time t plus delta t the instantaneous mass is equal to the initial mass minus the mass that has gone out. And the velocity of the vehicle has increased by a small among delta u.

So, therefore, the new velocity is u plus delta u. And we have considered the exhaust velocity to be equal to u e and let us exhaust mass exhausted mass rather; let me write it exhausted mass is equal to d m. So, this is the problem we have been discussing in the previous class and we have shown that the change in the momentum of this vehicle, total momentum of this vehicle along the x direction during this small interval of time d t is equal to M d u minus d m u e.

This is what we had derived in the previous class. Where M is the mass of the vehicle at time t, d u is the increment in the vehicle velocity during this small time d t, d m is the mass exhausted from the vehicle during this small time interval and u is the u e is the exhaust velocity. So, this is the expression that we had derived in the last class. Now, let us continue with the discussion coming back to this diagram and now let us just revisit the different forces that we had talked about. Previously we had identified four forces, where thrust was a reaction force appearing separately. And then we consider only the vehicle not the exhaust therefore, the rate of change of momentum was just M d v d t. Now, we are considering the exhaust also therefore, we do not consider thrust as a separate force so then the forces acting on the vehicle or only this three lift drag and the weight. So, the weight of the vehicle is due to the gravity right. So, this is equal to force because of gravity we have is the same expression we continue.

Where M is the instantaneous mass, g is the acceleration due to gravity at the sea level, R e is the radius of earth and h is the height of the vehicle at particular instant t. Now, this to together is the local acceleration due to gravity. So, we can write this as M times g. So, that is the gravitational force or the weight. Then the drag again we have seen this before, the drag is equal to half row infinity u infinity square S C D or S we can also write as A f right, where C D is the drag coefficient, A f is the frontal area or weighted area, u e is the free stream velocity of the incoming air, row infinity is the density of air at that altitude or density of the medium may not be here. What we can see from here is that, if we were outside the atmosphere then row infinity is 0 therefore, the drag is zero, as we go up high up in the altitude then density reduces therefore, the drag reduces. So, initially at the launch pad we have substantial drag, but as we go up the drag continuously keeps on reducing. So, this is the expression for drag. The lift is given by half row infinity u infinity square A f C L. So, once again here C L is the lift coefficient, A f is the frontal area, u infinity is the free stream velocity, row infinity is the density.

This lift and drag or essentially the aero dynamic parameter depends on the aero dynamic design as well as the operating altitude and operating speed. So, these are nothing to do with the rocket designer person. This is something aero dynamic should provide to the rocket designer. So, these are the two values that we will be considering. However, one more thing that will come into picture in this case, which did not appear in the previous analysis, which is the forces exerted by the exhaust pressure, the pressure forces, because there will be pressure differential may be pressure differential here.

So, we have discuss that, there are various type of expansion possible. We can have an under expanded nozzle, we can have a ideally expanded nozzle, we can have a over expanded nozzle. We have also discussed in the previous lecture that the trust produce is maximum for ideal expansion, because at that condition P e is equal to P a that is the exit pressure is equal to the atmospheric pressure.

So, therefore, there is no contribution from the pressure forces and we have also shown that for either the over expansion or for the under expansion the thrust is going to be less than the ideal expansion we have already discussed those things. We have also discussed that typically what we like to have is over expansion not under expansion, because of the fact that when the vehicle is launched from the ground the pressure is low as it goes up in the altitude the pressure continuously decreases and as the pressure is decreasing, if we start with a different type of expansion there is a monotonic decrease in the expansion.

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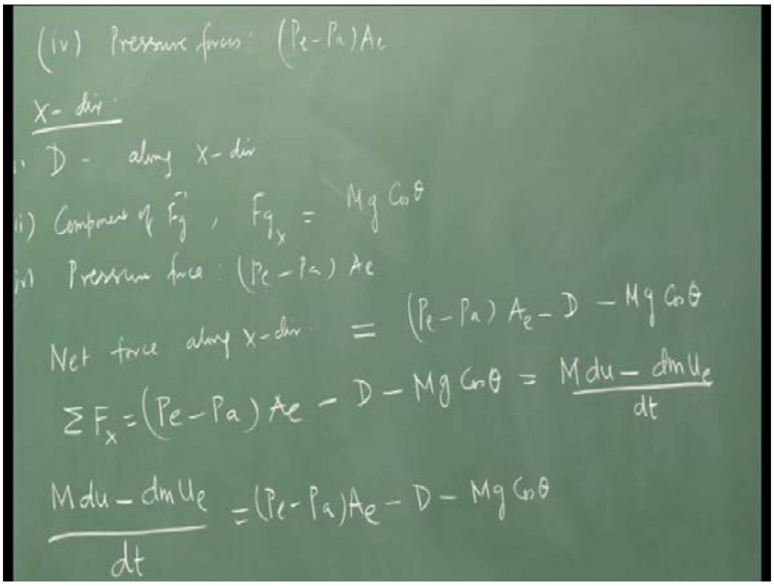

So, when, but if we start with a point where, let us say P e is less than P a, this is under sorry, when we start with under expansion, P e is less than P a is under expansion. We start with this condition P e is less than P a, as it goes up P a falls, the atmospheric pressure falls. And then at one point of time, it will become equal to P a beyond that P e will be greater than P a. So, we have a thrust variation something like this whereas, if we start with this condition and then this P a is falling. So, we have a continuous drop in thrust right.

So, therefore, this is a preferred condition, P e is less than P a then at sudden instance we will reach this condition where P e is equal to P a. So, this is a digression form what we have been discussing. Now, let us this is over expansion is more, yeah this the over expansion condition. So, that is why we usually consider over expansion, but ideal case will be this, where we get the maximum trust. So, now, since we are considering the exhaust separately there should be a contribution from this pressure force, if the expansion is not ideal. So, that also not needs to be accounted for in the momentum equation. So, the forth force in this case is equal to the pressure forces.

And that is equal to P e minus P a times A e, this will be applicable if it is not ideally expanded, if it is ideally expanded P e will be equal to P a. Now, let us look at what are this forces doing, the drag force is along the, along x direction or is along the direction of flight. We consider x as the direction of flight. The gravitational force, so first let us look at the forces acting along the x direction. One is the drag force, second is the a component of F g, a component of gravitational force will also be acting along the x direction and that component will be equal to let we say F g x which is equal to M g cos theta, according to this diagram where this angle is theta.

Then the net and the special force is also acting in the x direction, which is equal to P e minus P a and A e. Now, the net force then along x direction is equal to the pressure force P e minus P a times A, which will be acting in this direction right upward in the positive x direction minus the drag force. So, this is minus D and this term the gravitational component is also acting in the minus x direction so that we will also be minus. So, this is M g cos theta. So, therefore, this is the net total force external force acting on the rocket vehicle. Now, from Newton second law of motion this is equal to the rate of change of momentum. So, now, let us use Newton second law of motion. So, then first of all total sigma f along x direction is equal to and the total changing momentum in the x direction is equal to this. So, this can be written M d u minus d m u e.

So, this is the application of Newton second law of motion on this rocket vehicle. Now, let us simplify it little bit, what we can first what we will do is we will divide both sides by d t. So, sorry just one more thing, this is the net force acting which is equal to rate of change of momentum, this is the change in momentum, the rate of change is momentum will be d t divided by d t. So, therefore, let me rewrite it again d m d u minus d m u e by d t is the rate of change of momentum is equal to the sum of external forces, this is the governing equation that we have.

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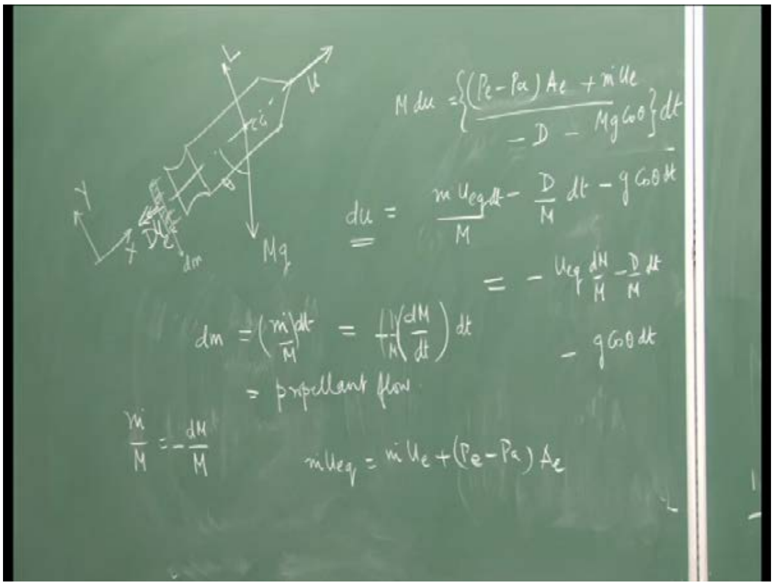

Now, let us look at this equation little will more closely. So, first of all let us look at this term d m change in mass, d m is nothing but equal to m dot d t, where m dot is rate of change of mass. So, this is m dot d t, where d t is the time interval. Now, this is equal to the initial mass was M, so this is d m d t times d t and the mass of the vehicle is decreasing. So, there will be minus sign here. So, d m is minus d M d t d t. So, this is the expression for d m.

Where once again like to rewritten it that the change is occurring because of the propellant flow. So, this is nothing but propellant flow, d m is a propellant flow that brings about this change in mass. So, then what we can do is, if I look at this expression we will have M d u. So now, going back to this equation M d u is equal to P e minus P a times A e plus this term d m d t, d m by d t times u e is m dot u e it had a negative sign when we take it to the right hand side this becomes positive and then after that we. So, this term here will be equal to d M d t is equal to m dot d t.

So, this term here d M u e by d t is equal to m dot u e it has the negative sign when you take it to right hand side it becomes positive. So, this is equal to m dot u e minus drag minus the gravitational term and on this times d t first this d t have taken to the right hand side. So, this is the expression for M d u. We can further simplify it and write an expression for d u alone the changing velocity is equal to, first of all let me look at this two term together what is this? We know that m dot u equivalent is equal to m dot u e plus P e minus P a times A e right. Therefore, this two term together is m dot u e.

So, write it here as, m dot u equivalent minus D by M d t, where D is the drag, M is the instantaneous mass minus g cos theta d t that is the acceleration due to gravity. Now, let us look at m dot term, m dot is, this will have divided by M right, because we are taking this M to the right hand side. Now, m dot by M from this expression, we write m dot by M is equal to d M by M right, if I divide both of this by M right. So, m dot there will be a d t here, first of all, because this d t is also coming in here.

So, m dot d t by M is equal to minus d M by M. So, now, we take this and put it back into this equation, this simplifies to minus u equivalent d M by M minus D by M d t minus g cos theta d t. So now, we have got an expression for d u as we can see that this in includes the equivalent velocity, it is not in terms of the trust directly it is in terms of the equivalent velocity and the mass change is all already been in corporate in this equation. So now, we wrote with this equation and now let us go back to the same similar cases that we had discussed for the other approach in the previous lecture and try to get the expression for velocity increment as well as the height that can be achieved.

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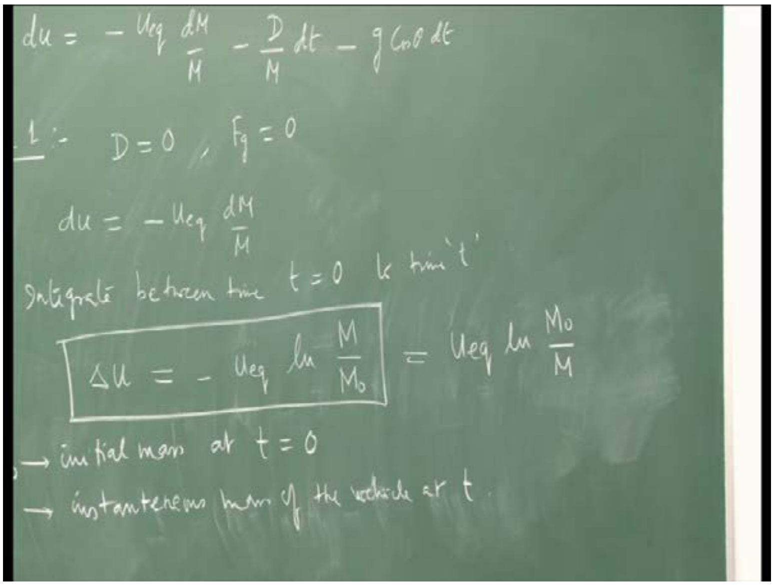

So, let me first rewrite this equation d u is equal to now minus u equivalent d M by M minus capital D by M d t minus g cos theta d t. This is the general expression, like to point out here one thing that, this is the expression in the positive x direction, velocity increment in the positive x direction. We have lift is not appearing here because lift is in the y direction. So, we do not consider the lift at present. So, this is the generic equation, which can be applicable to any rocket vehicle. Now, let us do the case studies. Case 1, first let us consider that the drag is 0 and we also consider that the gravitational force is 0 or g is 0, acceleration due to gravity is 0. So, for this case then, from this equation this term goes to 0 this term goes to 0. So, what we have left with a simple equation d u equal to minus u equivalent d M by M, as we can see this expression is must simpler than what we have done before.

And now we can just integrate it from time t equal to 0 to sometime t and we get the velocity change. So, when we integrate this between time t equal to 0 to sometime t then we get this is equal to delta u minus u equivalent d M by M integration is just lock. So, this will be lock, M by M naught right, where M naught is the initial mass of the vehicle at time t equal to 0 and M like in our discussion here is instantaneous mass of the vehicle at time t . So therefore, this is not the expression that we have obtained.

So, starting from the initial we can now get expression for anytime, this negative sign is quite odd. So, what we will do is we just be over and we can write it as u equivalent l n M naught by M. So, as you can see this is a much more simplified expression that. The difference between this expression and what we had obtained earlier is the fact that there we had considered the trust at an angle little vector thrust, here we are not considering that we are saying that the exhaust is coming in the x direction negative x direction that is why those complexities do not arrive here. So, but we have discussed that why do we need to have that small variation that because of the stability.

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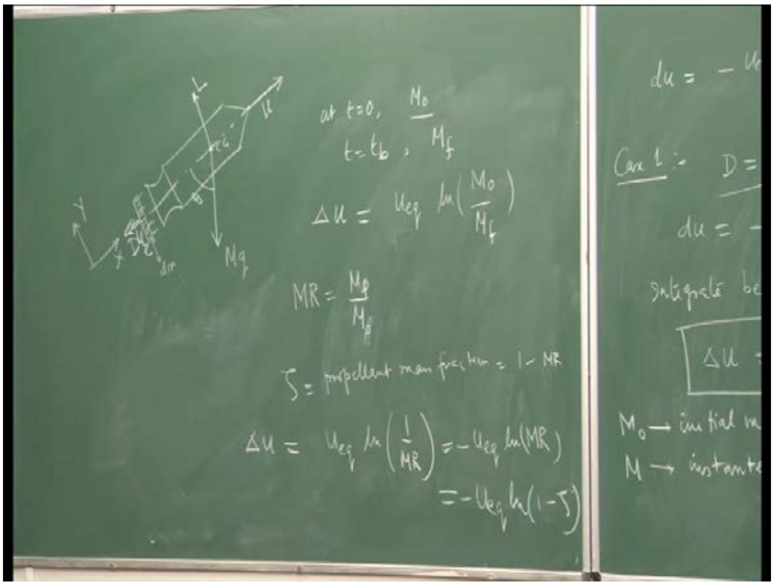

So, now let us continue with this discussion and see that what we get after this. So, let us consider a practical total burnt out at time t equal to 0, we had initial mass M naught. And then the rocket is launched till the complete burn out we want to find out how much velocity change is occurring. So, at time t equal to t b, which is the burnt out time the mass is equal to M f, which have been writing as M b bus, but here since we are talking about a different approach. So, just to differentiate it, I will write it as M f is the final mass.

So, the velocity increment during this entire operation then is equal to u equivalent l n M naught by M f. Now, if you recall in one of the previous classes I have defined this mass ratio and that mass ratio is given as M R, which we have defined. So, let me write it here M R is equal to M naught by M f, which we have defined and sorry, it is M f by M naught. And we had also defined the propellant mass fraction zeta, which is equal to 1 minus M R this we had defined before.

So now, coming back to this expression then we get delta u is equal to u equivalent l n 1 up on M R, according to this definition. We can also write it as equal to minus u equivalent l n M R, we can write it like this. Now, notice one fact here that M R is always less than 1, because fuel mass is equal less than the total vehicular mass. So, M of M R is always less than 1 therefore, this term is going to be negative. So, this becomes positive, delta u is positive and since zeta equal to 1 minus M R. So, therefore, M R is equal to 1 minus zeta. So, this can also be written as minus u equivalent l n 1 minus zeta. So now, we have got the single stage rocket dynamics the velocity change in terms of the mass ratio or the propellant mass fraction, I like to point out here that typically for any rocket these are the factors that are given to the designer ok.

So, you have to work with this factors only that is why it important to express everything in terms of this ratio. And particularly becomes important when we go to multistage because therefore, different stages will have different numbers for this ratio and we have to work with all them together. So, therefore, it is important to work with this ratios the mass ratio or propellant mass fraction or structural coefficient which will come later everything. So, this is the first case now we have discussed, when the drag is 0 no gravity then this is the exit velocity expression, which is expressed in terms of the mass ratio as well as the propellant mass fraction. Now, continuing from this then let us consider second case. So, this is our basic governing equation. So, I will just written it here and look at second scenario. In this case, we will consider the gravity, we still neglect that drag and consider the gravity.

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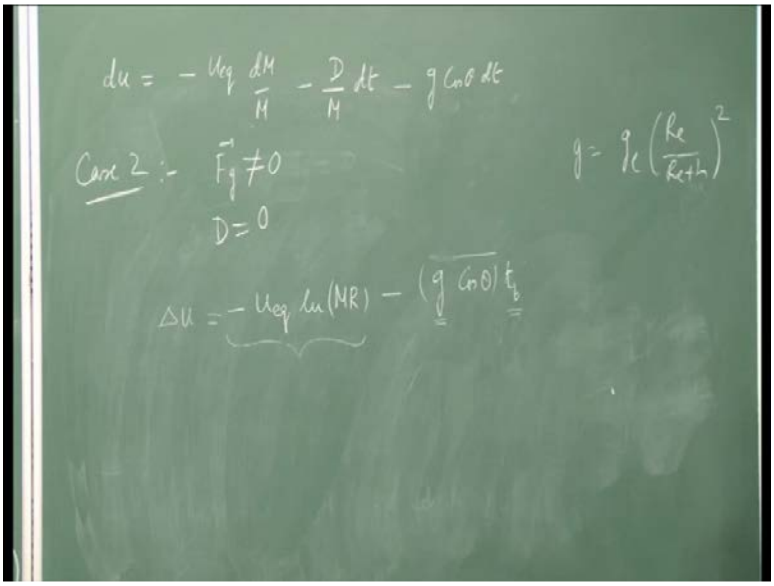

So case 2, case 2 is then F g is not equal to 0, D is still equal to 0. We still consider no drag, but now, we are considering the acceleration due to gravity. In that case coming back to this equation then, this term goes to 0, this term remains. Now, this term as we can see is g cos theta d t when we integrate this equation this term gives me this and integrating this between time t equal to 0 to d t to t gives us g cos theta t that is it, nothing else changes. So, we can bring it here and finally, we get delta u is equal to u equivalent, minus u equivalent l n M R, that is this term after integration is giving me this minus g cos theta t and if you are, but sorry you are burning up to t b right, up to the burning time t b. So, this gives me g cos theta d t b. One point I would like to mention here, that during a flight g may vary cos theta may vary right. So, if this two are varying then we have to account for that also, but when we are integrating this then we can take the average value since we are integrity. So, we can take the average value of g cos theta. So, average g cos theta times t b gives us then the expression of velocity increment with an incremental velocity with the x effect of gravity also. So, they live is the change in velocity.

So, here t b is our burning time which we have discuss before here, t b is our burning time and cos theta particularly this cos theta is the integrate at average value of cos theta that is that the angle that the slight partners with the vertical is the average value of that. So, this is actually not over the entire thing, but only the cos theta average cos theta. Now, this integrate this averaging is valid if we are talking about short trust intervals we are talking about large interval of course, then we cannot consider the average value then what we have to do it at every instant we have to consider the attitude and then integrated.

Now, gravitational acceleration h so for as long as we are close the earth gravitational field or close to the earth surface it does not very much, but once we start moving away from the earth then it starts to vary with h the altitude, because we have shown that g e equal, g equal to this we have already talked about this. So, as we go away from the earth surface h becomes important then g will start to drop. So, therefore, this value is g as long as we are close to the earth surface it will be a constant, but beyond the certain height cannot consider this to be a constant.

Then in this integration, we considered the variation of g as well and then accordingly we integrate. So, therefore, that can be considered. Now, one point I would like to emphasize here, we have mentioned that, this term is always going to be positive, because M R is less than one. So, this term is always positive, g is always positive it cannot be negative right, acceleration due to gravity is always positive. Theta is among between 0 and one so that is also positive, time is already increasing. So, this everything here is positive. So, therefore, this term is also positive. So, this is positive minus a positive quantity. Therefore, the effect of gravity is to actually decrease the acceleration due to the increasing velocity. So, we have to ensure that this term should be large enough to compensate it for this loss, because of gravity. So, therefore, in the design state that is why we are writing all in terms of this parameter that our u equivalent for the mass ratio should be chosen in such a way that we get a substantial velocity increment that is the M finally, to get a high velocity, which is not much effected by the variation in gravity.

So, that needs to be. So, the therefore, the thrust must be high enough to compensate for the gravity effect here this is the trust appear the term appearing here is from the trust. So, this is the second case that we have talked about. Let us now look at the another scenario the third case. Now, in the third case, what we will consider is that now we do not consider the drag is absent. Let us say it is the flight is inside the atmosphere. So, we consider the drag as well. So, in case 3, what we will consider is that the drag is also present. So, we have the gravitational effort we have the drag as well.

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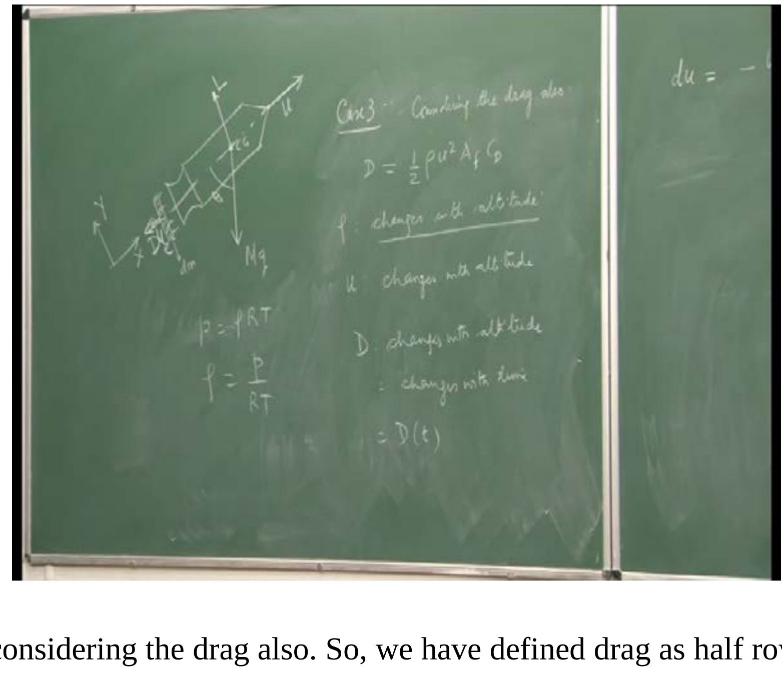

So, case 3 is considering the drag also. So, we have defined drag as half row u square A f C D, this is how we have define the drag. Now, let us see what happening here, density is changes with altitude, because we consider the atmosphere to be if the perfect gas considering a perfect gas. Then we know that for perfect gas equation of state pressure equal to row r t therefore, row which is the density is equal to pressure by r t. Now, temperature as we go up in the atmosphere initially decreases remains constant that increases again then remains constant decreases like that at different levels of the atmosphere temperatures varies. The pressure monotonically varies from the earth surface up to the edge of the atmosphere the pressure continuously decreases. The net effect is the density continuously decreases as we go up high in the altitude.

So, that there is a change in the altitude the density also changes. Therefore, considering everything same the drag will be reducing as we go up and up in the altitude. Second point here is that u as we have just discussed, del u is always positive right. So, therefore, as del u is positive, u is also changing and increasing so u changes with altitude. Now, I would like to point out here one thing I said that del u is positive for that is only during the power flight, if we cut of the engine this term goes to 0 then this term will take over and reduced del u and will come to 0 velocity. And that will, we will come back again when we talk about the maximum height. So, during the power flight del u is positive therefore, u also changes with altitude.

So, once again what we have saying is the density is decreasing with altitude, but velocity is increasing. Considering the design is same C D is constant let us say, area is constant then the drag will be increasing, because the dependence is u square and the variation in velocity is much higher than the variation in density. So, therefore, the drag continuously changes with altitude so D changes with altitude. Now, here altitude represents essentially the passage of time, because the variation in altitude is because you are increasing increases with time. So, therefore, in other words we can say that it changes with time ok.

So now, if we come back to this expression then, this drag here is a function of time. Now, we have to integrate this equation so D equal to D t. So, coming back to this equation and this dependence is not a linear dependence unfortunately as you can see that everything is coupled.

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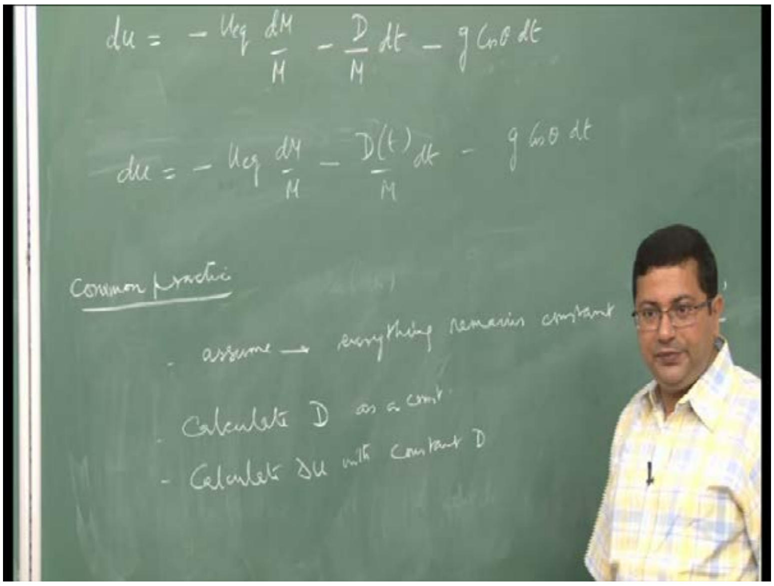

So, this is our governing equation now, where minus D t by M d t minus g cos theta d t this is what we have to integrate now, which is not very easy to integrate, because we do not have a well defined close form term for D as a function of t, we do not know how drag is changing with time, it is not very easy to figure out. So, typically what is done is a roundabout approach and solving this. So, the common practice is this, let us assume that everything remains constant during a small implement of time d t. So, we assume that this theta remains constant of d t, gravitational acceleration remains constant in this small interval, the D remains constant, the mass remains constant everything is constant u equivalent is also constant during its time. Then now, we can very easily integrate this we can get a value of this entire right hand side at a small time then slowly we keep on changing it.

So, numerical integration that is what we can do. So, what we can do is then we calculate this value of drag calculate D as a constant, because during this small interval our density is constant, u is constant. So, we can calculate this value of D during this small interval of time. Once we have done that now, we put this value back into this equation now this is appearing as a constant right. So, now, after that we integrate this and get a velocity increment D u. So, next we calculate D u with constant D.

We implement this D u with constant D, we calculate this value of D u now, what we do is in the next time step, we replace this u with u plus D u and the corresponding change in altitude also we estimate replace the density with that and recalculate it for the next time step. So, again we put that value here and integrate this that is why, that is how, we keep on integrating in time a marching in time till we get the final value. And also one more point here is that, when we were doing this we also keep on changing this theta whatever change in theta is required that we keep on doing, because that comes from the component balance. So, we keep on upgrading this term and do a time marching and estimate the final value. That is all we estimate the final velocity increment when we consider the drag and gravity as well.

So, this gives us the most complex scenario, which can also be done, but has to be done numerically. I would like to point out here 1 more thing, that in all this analysis we have not talked about y direction. If we look at y direction then unless we provide a trust vectoring there will be a non 0 lift. So, there will be at tendency of the vehicle to move in this direction. So, in order to get 0 lift, we have to give a vector, a small vector to the trust which we have discussed in the previous lectures.

So, therefore, in order to provide in order to have 0 lift, the trust must be given at an angle to the u direction or the x direction as we have already discuss. This balance the normal component of gravitational force, otherwise the gravitational force will not be balanced and it will start to wear of this intended path. So, now, let us there is no lift and we do not this give vector then there is a unbalanced force acting like this and that will take it downward. So, it will come down. So, therefore, in order to compensate for this gravitational pool.

We have to give a slight vectoring to the thrust. So, we put thrust at a particular angle here. And now, coming to the other way then, we can actually achieve various type of stable flight by providing thrust vectoring. If you do not provide thrust vectoring there will be a tendency to go like this. We will provide thrust vectoring it will go like this let say, after a word up to a point we want to bring it down we were different thrust vectoring it comes like this right.

So, the cruise missile where we want to change the orientation or attitude of or the flight path we can do that flight suspect ring very easily. So, therefore, thrust vectoring is an important method in the flight of rocket vehicles. So, with this we come to an end of our discussion of this method. Now, what we will do is in the next lecture, we will go back to the single stage rockets and look at different performance parameters like the maximum height etcetera. So, here we have a estimated the velocity change, next we look at the burnout time and burnout height etcetera for single stage rocket which are discussing so far.

Thank you.