Jet and Rocket Propulsion
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Lecture - 17

Good morning, welcome to this lecture. So, in the previous lecture we have been discussing the Orbital Mechanics. We have derived the equation for the particular orbit based on the conic sections, based on force balance. And towards the end of the last lecture we had derived the expression for the orbit in terms of its energy. So, let us continue from that point and take it further.

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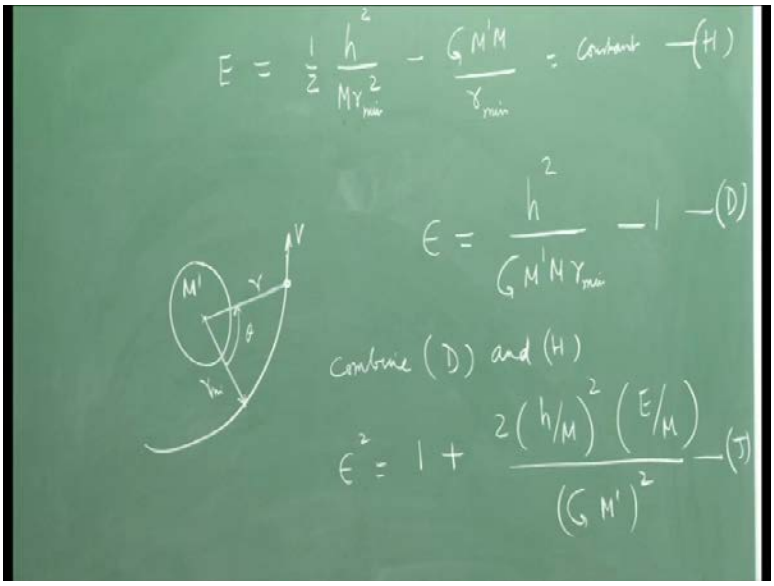

So, what we have seen in the last lecture is the orbital energy is given as half h square upon M r min square minus G M dash M upon r min equal to a constant, and we call this equation as H. In this equation h is the angular momentum of the vehicle in the orbit; which is a constant because there is no angular acceleration, theta dot is constant. So, therefore angular momentum is constant. M is the mass of the vehicle in that orbit and r min is the minimum distance of the vehicle from the surface of the from the center of the heavenly body. This term came from the kinetic energy of the vehicle; this term came from the potential energy of the vehicle, where g is the universal constant; M dash is the mass of the heavenly body. And as I have just said M is the mass of the vehicle and since in the orbit.

So far in our discussion in Orbital Mechanics we are considering that the thrust has been given impulsively. And after that there is no thrust; when we are talking about the vehicle dynamics. It is moving purely because of the gravitational force. So, if the thrust is 0. Then when it is in the orbit, there is no net energy being added to it. And if the energy is not being added, then the total energy remains constant. Therefore, this is the energy of the vehicle at its minimum point, which is constant. So, at everywhere in the orbit we will have the same energy. So, this is what we have been talking about. M dash, this is my minimum point r min, the vehicle is somewhere here moving with a velocity v, this is r and this is theta. Now with this expression, first for theta equal to 0; when we say at any value of theta, we will have the same condition. So, this is what we have been discussing.

Now, earlier we had derived an expression for the conic sections. And we have shown that the eccentricity of a conic section epsilon is given as h square upon G M dash r upon r min minus 1. And we had called this equation D. This we had derived in the previous lectures. Now, if I take this expression for D and combine it with H; if I combine these 2 equations, then I can get an expression for epsilon in terms of other parameters. So, after combining I will get epsilon square is equal to 1 plus 2 times h by M square upon E by M divided by G M dash square. Let me call this as equation J. So, here what we have is an expression for epsilon which is the eccentricity of the conic section in terms of the angular momentum, the mass and the energy. This is the expression for the eccentricity.

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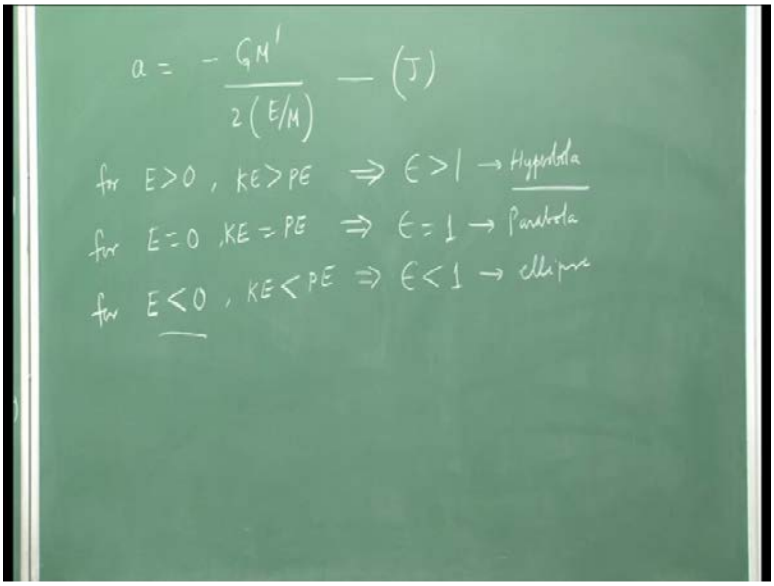

And, we had also obtained an expression for ‘a’, which appears in the conic section expression in terms of epsilon. So, if I look at that expression for ‘a’ and put it back here, I will get an expression for ‘a’ in terms of the orbital parameters as minus G M dash divided by 2 upon E by M. So, once again let us point out here that earlier we have got expressions for ‘a’, epsilon, etcetera in terms of r min and other parameters. Now, I am just combining so that I am bringing the energy content also. Previously, we had not talked about the energy. Now I am combining these equations to bring in the energy. So, let me call this equation as equation J.

Now, previously we have discussed how the eccentricity dictates what will be the vehicular path. Right. We have discussed that if epsilon is greater than 1, we have a hyperbolic path. And we have shown that hyperbolic path is not possible because it is going to come down. We have shown that if epsilon is equal to 1, then we are going to have a parabolic path; which will be a fly-by. We have also shown that if epsilon is equal to 0, we have a circular path with the radius is equal to the minimum distance that we have talked about. We have also shown that the possible orbit; circular orbit is the possible orbit. Another possible orbit is ellipse; where epsilon is in between 0 and 1. We have discussed these things in detail.

Now, let us see that how does the path depends on the energy; particularly, the relative importance of the kinetic energy and potential energy. For that, let us first consider a case for E greater than 0. E greater than 0 means if I come back to this equation, the kinetic energy is greater than the potential energy. Right. That is what E greater than 0 means. So, if kinetic energy is greater than potential energy, E is greater than 0; which essentially means kinetic energy is greater than potential energy. Then if I come to this equation, this is greater than 0. So, this is greater than and all these terms here are greater than 0. Therefore, this entire thing is greater than zero. So, 1 plus this is greater than 0. Therefore, epsilon is greater than 1. Sorry, 1 plus this is greater than one. So, therefore this means epsilon is greater than 1; which essentially means that we have a hyperbolic path. And we have shown that hyperbolic path is not possible. Therefore, if my kinetic energy is greater than the potential energy, we do not get at a proper trajectory. ok

Next case, let us look at for E equal to 0; which means the orbital energy is 0, which means my kinetic energy is equal to the potential energy. So, the two energy components are balanced. So, there is no net energy in the orbit. In that case, this term is going to be

0. If this is 0, this entire thing is 0. Therefore, epsilon is equal to one. So, we get to epsilon is equal to 1, which means what we have is a parabola. And we have shown that parabola is also fly-by. It would not be captured in the orbit. So, E is equal to 0; that means the kinetic energy is equal to the potential energy. The vehicle will just fly-by. It would not be captured by the gravitational force of the bigger body. It will just fly-by ((Refer Time: 09:39)).

Then the other scenario is for E less than 0. E less than 0 means my kinetic energy is less than potential energy because E less than 0 means this term is negative. And this is going to be negative, when this term is less than this. Therefore, my kinetic energy is less than my potential energy. So, therefore when kinetic energy is less than potential energy, the overall energy is less than 0. Then this term is negative; everything else here is positive. So, therefore if I look at this equation, this term is negative. I have 1 plus a negative term, which will make it less than 1. Then we take the square root. Epsilon is less than one. So, for this case when kinetic energy is less than potential energy, my epsilon is less than 1. And that is an ellipse. So, we have an elliptic trajectory. And this is something that is possible.

So, what we have shown from this discussion? That, in order for the energy captured into an elliptic orbit, my kinetic energy should be less than the potential energy. Typically, what we say is that kinetic energy should be equal so that there is no net energy. But, in order to capture it in the orbit, what we have shown is the kinetic energy should be less than the potential energy. Only then, it will be captured in the orbit; in an elliptic orbit.

So, this discussion then shows that we have a case where we can capture it by putting E less than 0. Now if we are talking about lesser, some vehicle, which has to escape from the gravitational pull, then we are talking about escape velocity. Out of these cases, which one will give us the escape case? Parabola fly-by. Right. So, if we have to escape from the gravitational pull of the heavenly body or say earth, we have to take a parabolic path. Right. It should correspond to parabolic path, which essentially means that E equal to 0. There should not be any energy. Only then, it will escape. So, therefore from here now we can get an expression for the escape velocity. So, let me try to get it from here as you can notice now the importance of this equation H.

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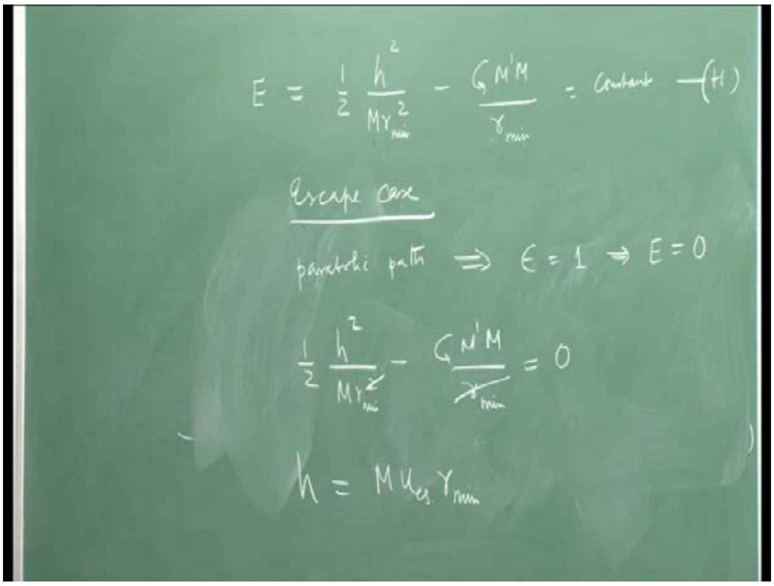

So, next we talk about escape case. For it, it has to be a parabolic path. As we have discussed that, that is what is going to be the fly-by case. So, we should have a parabolic path. And this implies epsilon is equal to 1; which implies that the total energy is equal to 0. Now, if I put this condition back into this equation, what I have is h square upon M r min square minus G M dash M by r min is equal to 0. Now, let me first cancel this r min. And h is my angular momentum, which is equal to M. Now, let say that the velocity I am talking about is escape velocity. So, we will write it as u escape; and r. right. And as we have said that everywhere it is constant, so we will take the minimum radius only. So, we will put it as r min. right. So, this is my h value.

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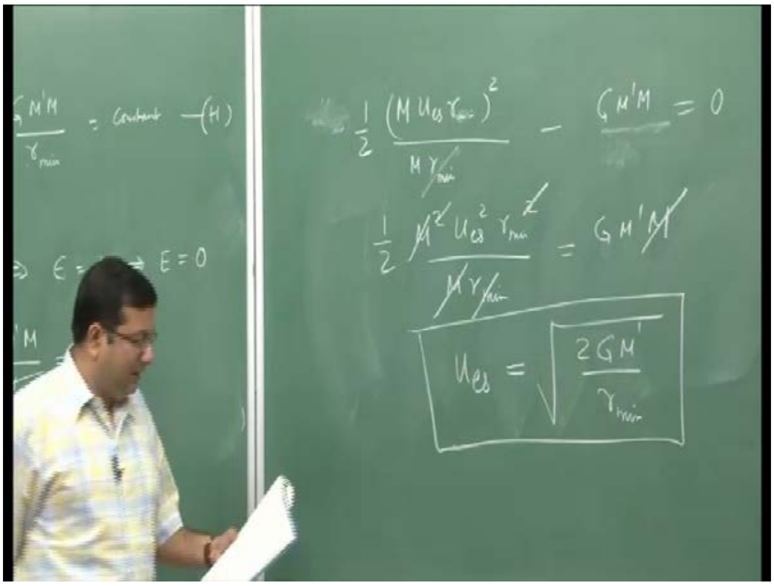

If I take it and put it back into this equation, then what I will get is that u escape. Let me just first put it back here. Half M u escape r min square or just as I say r square. Okay, put it r min only. r min square divided by M r min minus G M dash M upon G M dash M is equal to 0.

So this, I will simplify this as half M square u escape square r min square divided by M r min equal to G M dash M. I cancel this. M and M will cancel off; this will cancel off. So, what I have is escape velocity is equal to square root of 2 G M dash by r min. So, this gives me an expression for the escape velocity in terms of the universal gravitational constant, the mass of the planet and the distance of the location of that vehicle from the center of the planet. Now, we can take it further and find out that if we launch a vehicle from earth surface, then r min is equal to radius of earth. right. So, this will be then replaced by radius of earth. We can find out the escape velocity from earth. right. So, this is the velocity with which it has to be launched so that it can escape the gravitational force.

So, this is first thing; now we are getting somewhere. We started this discussion to get the velocity increments that we are looking for. Now, we see that this is what is going to be the escape velocity. Next, let us take it little further and try to find out the period or the time it will take for the vehicle to move around in an orbit. For that, I am not going into the detail. I will give you as homework again.

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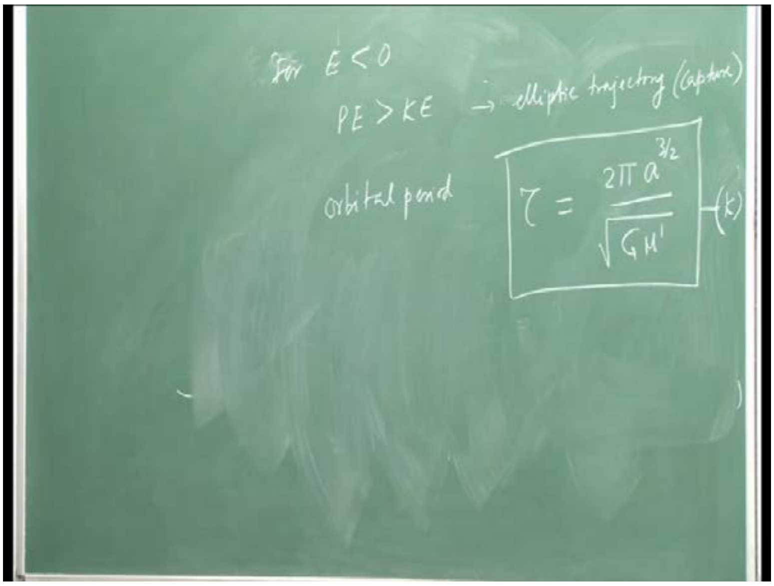

For epsilon less than 0, we know that potential energy is greater than kinetic energy. Right. And that will lead to an elliptic trajectory. We have shown this before; which essentially means capture. That is, the vehicle is captured by the gravitational force of acting ((Refer Time: 17:13)) from the bigger planet. right. So, this is the orbit capturing. For that, this is the condition.

We can find out the orbital period. That is, the time it will take to move around; to take one revolution around the planet. You can solve it yourselves. It is not very difficult to do. We will be given as tau equal to 2 pi a to the power 3 by 2 divided by square root of G M dash. This is my orbital period. Let me call this as equation K. So, now we have derived many equations. We will now look at how to use these equations. So far, now we have derived most of the equations required for getting the orbit, Orbital Mechanics, what will be the velocity etcetera.

Next what I will do is, first of all I am going to list all the equations that I have derived which are going to be helpful. After that, I will summarize how do we use these equations to estimate the required velocity. That is what our next course of action is. So, now I am in the summary mode. I am going to summarize what we have discussed so far.

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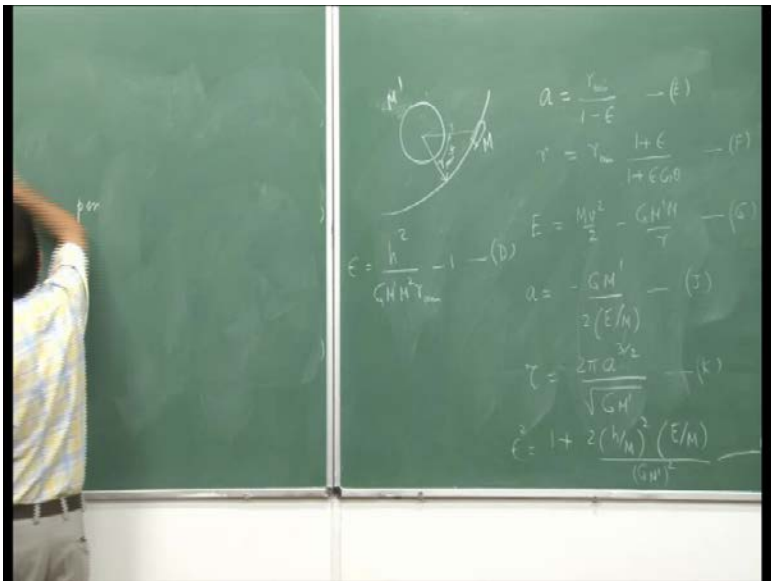

So, first let us look at what are we trying to do. We have a body of mass M dash and around which a vehicle or artificial vehicle is moving which has a mass M. It is following certain path. The minimum distance of this vehicle from the center of this heavenly body is r min. Now, let me just list all the derivations or expressions that I have derived. One of them is an expression for ‘a’. We have derived this; that ‘a’ is equal to r min upon 1 minus epsilon. And this we had called as equation E. Then we have derived an expression for r; where r is the intended path of the vehicle is equal to r min 1 plus epsilon upon 1 plus epsilon cos theta. We have called this as equation F; where this angle is theta and epsilon is the eccentricity of the path that is been followed, which comes from the conic sections. Then we have got an expression for the orbital energy given by E. So, which is equal to M v square by 2; which is the kinetic energy minus the potential energy. This was our equation G.

Then if I combine these equations, I get an expression; modified expression for ‘a’, which is equal to minus G M prime upon 2 upon E by M. It is the modified equation for ‘a’. And I have called this equation as J. And I have got an expression for the period of the vehicle in an elliptic path around the planet, which was given in terms of ‘a’ as 2 pi a to the power 3 by 2 divided by square root of G M dash. And I call this equation as K. And then what I have got is, in this equation, one thing that is so far not listed is the eccentricity. So, I have got an expression for the eccentricity also. Epsilon square is equal to 1 plus 2 upon h by M square times E by M divided by G M dash square. And I had called this equation; no, this I had not given any name. This was equation I.

We had an alternative expression for 'a', epsilon and the eccentricity in terms of the angular momentum. So, that was eccentricity is equal to h square upon G M dash M square r min minus 1. This was our equation D. So, this side I have listed now; all the relevant equations which we have derived for the vehicle motion in orbit. Now, before we proceed further I would like to define some more parameters; particularly, for the elliptic orbit. In elliptic orbit, typically we do not talk about r min. We talk about apogee and perigee. So, let me talk about apogee and perigee, little bit about the elliptic orbit and then I will list how we solve these problems.

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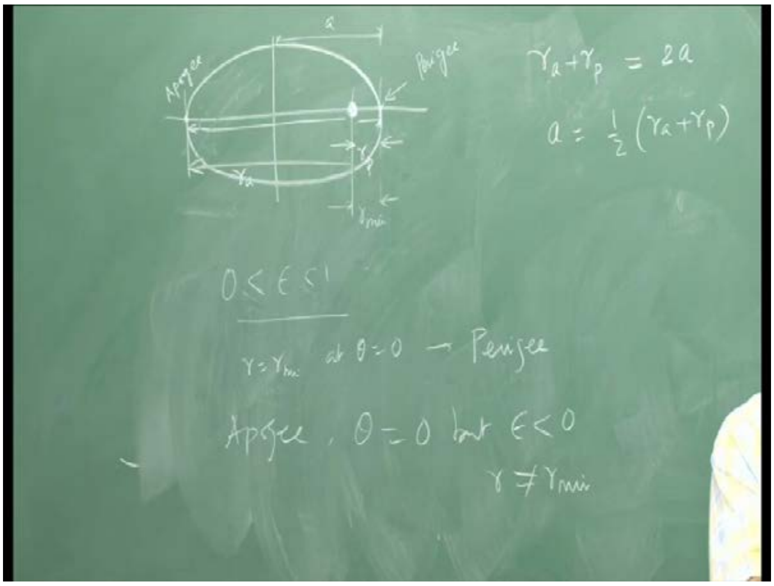

So, we are looking at an elliptic orbit. This is an elliptic orbit and this is where our planet is and the vehicle is moving in this orbit. So, the least distance of the vehicle from the planet; that is, this point is called the perigee. And the maximum distance of this vehicle from the planet is called the apogee. These are the defined properties. So, the minimum distance of the vehicle from the planet center is the perigee; maximum distance is apogee.

And according to our definition, then this is equal to r min. Right. And we have shown that the eccentricity for an ellipse is in between 0 and 1. And we have also shown that r equal to r min at theta equal to zero so that this point we have theta equal to 0. Now, we can show one thing here; is, if we are at this point, we have the minimum theta is equal to 0. But if we are at some other point, let us say here, right, there also theta is equal to 0. Right. But, does it correspond to this? No because at that case epsilon is less than 0. So, corresponding to apogee; so this is for perigee. Corresponding to apogee, we have theta equal to 0, but epsilon is less than 0. And it does not correspond to r equal to r min. ok. So, this is something that again comes from the geometry of ellipse.

So now what we have is, we call this distance r p. So, this is the distance of the perigee from the planet surface. We call this distance, r a. Distance of the apogee from the planet surface. Then what is r a plus r p? r a plus r p is 2 times the semi major axis ‘a’ because this was our ‘a’. right. So, therefore semi major axis ‘a’ is nothing but r a plus r p. This is something that is true for elliptic axis, sorry, elliptic trajectories. Now, we have defined all the require properties. Let me next now summarize our solution procedure.

So, what we are going to solve first of all? We are trying to find out the velocity increment required or the velocity of the vehicle in this orbit; that is, the vehicle is moving in a given orbit, what should be its velocity? That is what we want to find out because that has to be given to our vehicle designer so that they can design the rocket accordingly.

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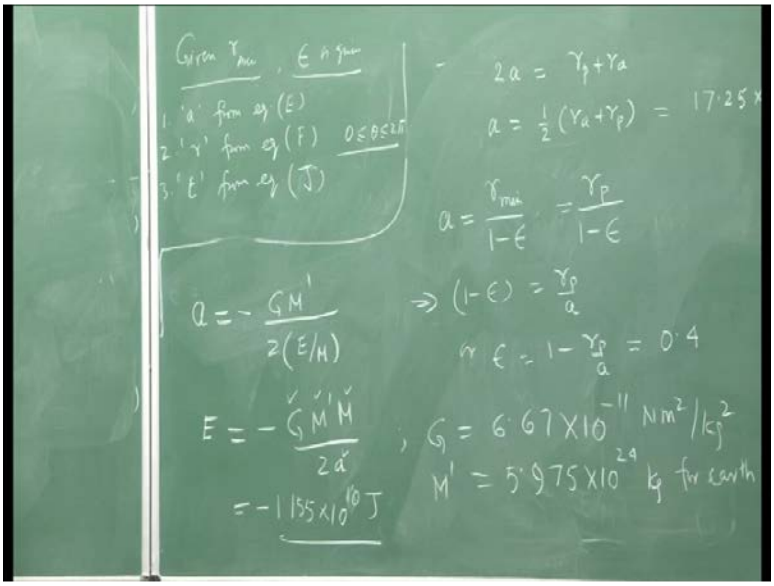

So, let me now define the problem that we are solving. We have given r min. So, the minimum distance of the vehicle from the planet surface is given. And we have a given eccentricity epsilon. We see that if these two parameters are given; r min and epsilon are given, then our path is specified. The orbit is specified. So, we have a given orbit now. Now for this given orbit, what are the things that we can find so that finally we can get the velocity? So for this, first thing we can do is my r min and epsilon are given. So, looking at equation E, I can get ‘a’. So first thing is, ‘a’ from equation E. right. I can directly get ‘a’. So, I am now getting more and more orbit parameters so that semi major axis is now obtained.

Next, once we have ‘a’ for different values of theta, if I look at this expression equation F, r min is given and epsilon is given for different theta; theta varying from 0 to 2 pi. right. The entire period we get r. So, second thing is we get ‘r’ from equation F; when theta varies from 0 to 2 pi. So, now what I see is that if my r min and epsilon are given, my path or orbit is completely defined. My ‘a’ is obtained; my ‘r’ is obtained at every theta location so that the orbit is completely defined. Now, the next step is if the orbit is defined, what should be the vehicle velocity in that orbit so that the vehicle stays there.

So, the next thing is third. For that, first we estimate the energy of the vehicle in this orbit E. So, let us get ‘E’ from equation G. So, let me just draw this portion here and then I will write here. So, from equation three from equation G, I can get that total energy of the vehicle in this given orbit now. Now, my orbit is given. Then once I have that I go to…, sorry, no, ‘E’ I should first calculate from here; equation J. If I look at this equation, ‘a’ I have already calculated. Right. I have already calculated a, G is known, M dash is known. Only unknown here is E. So, from solving this equation, equation J, I can get the value of E. So from equation J, I get E.

Next step is once I know ‘E’, I come to this equation. In this equation, ‘r’, I can put r equal to r min or yes r equal to r min. E is known. We know that energy is constant everywhere. So, we can give this value which we have obtained from here. only unknown now is my v square or v. So, I can calculate v from equation G. So, I know the velocity now that is required. Once I have calculated v and the period is known, then orbit is known, then getting time period is very trivial.

So, fifth is calculating tau from equation K; where, this gives me the time period. So, this is my procedure. Using this procedure, we can calculate each and every parameter required to estimate the state of the rocket in its particular orbit or state of the satellite or vehicle in its orbit. So, I have summarized all the processes.

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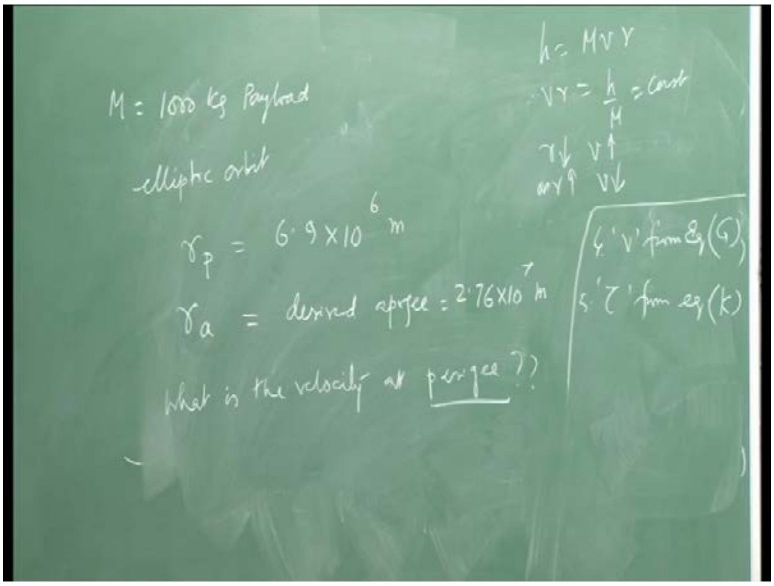

Now, let us solve a problem to make it little more clearer. So, let me first specify the problem. So, what is the problem statement is let us say we want to launch 1000 k g payload. We want to launch a 1000 k g payload. So, my M is equal to 1000 k g into an elliptical orbit. Now when we are saying elliptical orbit, we have to specify these two parameters. This always is needed. The perigee is always needed. So, let us say the perigee of the orbit is given as 6.9 into 10 to the power 6 meters. That is the perigee. The next thing we want to know is either the eccentricity or some other orbital parameters so that we can define the orbit.

So, in this case let us say what is given to us is the apogee height, r a. So, r a is the desired apogee. This is equal to; let us say given to be 2.76 into 10 to the power 7 meters. So now, the apogee and perigee are given. We have to find out what is the velocity at perigee. What is the vehicle velocity at the perigee? This is what we need to find out. Now, I would like to point out here one thing that my angular momentum is constant. right. If angular momentum is constant, an angular momentum we have seen is M v r. if M v r is … So, if angular momentum is constant means mass is constant. Therefore, h by M; which is equal to v r is constant.

Therefore, we do not have the same velocity everywhere in the orbit. When it is closer, velocity is higher; when it is further away, velocity is lower. right. So, since h is equal to M v r, therefore G r is equal to h by M; which is a constant. Therefore as r decreases, v increases; as r increases, v decreases; which essentially means that at the perigee, we have the maximum velocity. And at the apogee we have the minimum velocity. So therefore in this problem, then what we have been asked to do is to find out the maximum velocity. We could have placed the question like that. That, what is the maximum velocity of the vehicle in its orbit? Right. Then we have to say that the maximum velocity is going to be at the perigee, which comes from this discussion. ok.

So now coming back to this problem, what is given to me is the distance of the perigee, distance of the apogee and the payload. Nothing else is given. I want to find out what should be the velocity. So, for that let us start first from the definition of 2 a, which is the length of the major axis is equal to r p plus r a; r p is given, r a is given. So, I can get an expression for the semi major axis 'a', which is equal to half r a plus r p. Now, these values are given. So, I can calculate this. This comes out to be equal to 17.25 into 10 to the power 6 meters.

Now once we have got this, we go to our equations for 'a'; which says that 'a' equal to r min upon 1 minus epsilon. And r min is my distance at the perigee. So, this is equal to r p upon 1 minus epsilon. Therefore, I can get 1 minus epsilon equal to r p upon a or epsilon equal to 1 minus r p upon a. If I solve this, this is equal to 0.4. So, immediately we see that epsilon lies between 0 and 1. So, it is an elliptic orbit. So instead of giving the apogee height, if my epsilon was given and perigee was given, I could have solved for 'a'. Then we could have found out how much should be the apogee. right. So, the problem could have been stated as given epsilon.

So, now I have got the eccentricity epsilon. Let us proceed further. We had to prove that 'a' is equal to minus G M dash upon 2 upon E by M. This was our equation J in the list of equations. So, from here we can get E is equal to minus G M dash upon a divided by 2 a. Now, the value of G is the universal gravitational constant. So, that is the known value. And M dash, now we are saying is our planet is earth. So, you have to know the mass of the earth M is the vehicle payload, which is given as 1000 kg. 'a' we have estimated.

So, now we need to know the value of G and M dash. So, at the beginning of this discussion, two lectures back, I have given the value of G; which is the universal gravitational constant is equal to 6.67 into 10 to the power minus 11 Newton meter square per k g square. And the mass of the earth is equal to 5.975 10 to the power 24 k g. This is the mass of the earth. So, now in this equation we have G, we have M dash, we have M, we have a. everything is known. So, we just plug in all these values and estimate the value of E, which is the energy in the orbit; orbital energy. So, that comes out to be equal to minus 1.155 into 10 to the power 6 joules. No sorry, 10 to the power 10, not 6. It is 10 to the power 10 joules. Once again, I would like to point out here that we had discussed that in an elliptic orbit, the orbital energy is less than 0; because kinetic energy is less than the potential energy. So, that is proved here. So, now we have the value of E. We take this and put it back into the expression for our orbital energy. Then only unknown is v.

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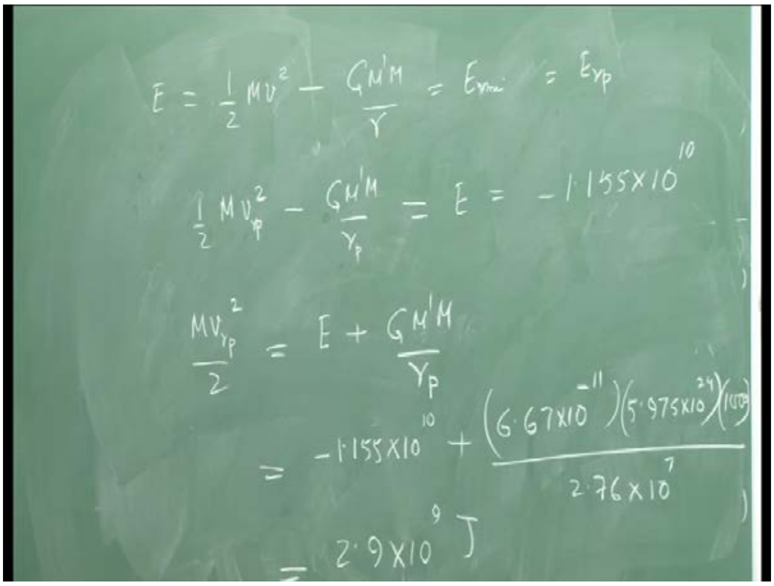

So, let me do that. Go back to our expression for the orbital energy. E is equal to half M v square minus G M dash M upon r. This is what we had already obtained. Now, this is equal to E r min; which is equal to E r p because the energy is constant everywhere. Therefore, the energy at the perigee is equal to energy everywhere.

So, I can write it now as half M v r p square minus G M dash M by r p, where r p represents the distance of the perigee is equal to this E. E which is equal to minus 1.155

into 10 to the power 10. If I look at this equation now, this mass is given. This velocity is what we have to find out. G is known, M dash is known, r p is given, E we have estimated. So, now we just put all of these back here and first of all I get M v r square by 2 equal to E plus G M dash M by r p. So, this is equal to minus 1.155 into 10 to the power 10 plus G is 6.67 into 10 to the power minus 11, M dash is 5.975 into 10 to the power 24, M is 1000 divided by r p is 2.76 into 10 to the power 7. right. So, all these things are given to us. I can now calculate first the kinetic energy in the orbit, which comes out to be equal to 2.9 into 10 to the power 9 joules.

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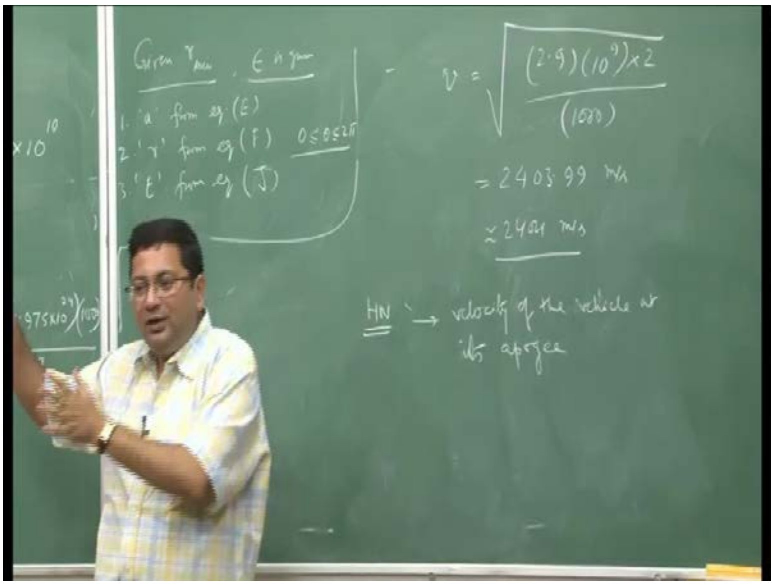

Now, I can get the velocity which is equal to; the velocity will be equal to square root of this kinetic energy 10 to the power 9 into 2 divided by the mass, which is 1000 k g. Now when I solve this, I get the velocity at the apogee, sorry, at the perigee; as say roughly 2404 meter per second. So, this is the velocity that the vehicle must have in its perigee in order for it to remain in that orbit. It should have a different velocity in the apogee, which can also be calculated. You can do that as a home work. What should be the velocity of the vehicle at its apogee? We have got at the perigee. You can find out what should be the vehicle velocity at its apogee. And this is how we estimate the required velocity. So, now you can appreciate that first we carry out this exercise, where the orbit is given. Once we know the orbit, then we estimate this velocity.

Now, we go to the flight mechanics part. Find out what type of configuration, what type of multi staging will give us this velocity increment. Notice one thing. So, now what is given is M l payload. Now, we have; payload is given, the velocity is given. Right. These are the two parameters we needed for estimation of the distribution of the mass. So, from where now we can get the mass distribution. Now, the flight mechanic design is complete. Now, we have the design parameters which need to be given to the rocket motor designer to design the rocket motor.

So, the next thing what we are going to do is start now with the rocket motor. What we are going to do first is that; remember that I have pointed out one thing. That, one of the thing is that made the rocket flight possible was the invention of Goddard of putting a de Laval nozzle after the rocket chamber or combustion chamber.

So, we will start from there. We will first look at the nozzle performance; because the most important parameter in the rocket is the nozzle. So, we will start with the nozzle. We will look at the nozzle performance in detail. Derive the equations that how the nozzle will give us the exit velocity. Once we have done that, then we go to the chemical rocket performance in more detail.

So, we will start first from the nozzle discussion. We will discuss the nozzle. First what we will do is, basic Aero Thermodynamics discussion of the nozzles. So, we will discuss the nozzle performance. One point is that I have mentioned that the over expanded nozzles, under expanded nozzles, ideally expanded nozzle. At that time, we had a very cursory look at those. Now, we are going into more details of those that what will be the prevailing conditions for those cases.

First we will do that and then we go to the nozzle performance. Discuss in detail about the nozzle performance. There we will see that the nozzle performance to certain extent depends on what is coming to the nozzle, the inlet conditions; which comes from the combustor. So, first you design the nozzle. So, we will look at the nozzle design which will... So, essentially now what we are looking for is the equivalent velocity. Now, we have seen delta v. We know what delta v is required. Then we go to the flight dynamics. We look at; for getting this delta v what is the equivalent velocity is required, which is the job of the rocket, engine designer and what is the ((Refer Time: 47:44) required, which is done as the flight mechanics part. So, that part is done.

Now, we go back to the equivalent velocity. So, the equivalent velocity is essentially coming from our nozzle. So, we will focus on the nozzle in detail. Get the expression for the equivalent velocity and do different case studies. After that, we will see that the equivalent velocity is a function of chamber pressure and temperature. So, once we have done with estimation, we go back and then we talk about how do we estimate the chamber pressure and temperature that brings us the combustion part. So, then that will complete.

So, as what I am saying is that the design is going backward. We will first start with a Space Dynamics, Orbital Mechanics, go to that specifies a machine requirement. From this we estimate what should be the velocity, what is the payload we can put. Then we go to the flight dynamics part. Look at in order to put this payload, what should be the mass distribution and equivalent velocity. Then we try to estimate the equivalent velocity by going to the rocket nozzle. Then in order to rocket to function like that, we need to have the pressure and temperature which comes from the combustion. So, we will look at the combustion.

So, that will complete our discussion on the chemical rocket part. So, next thing is now what we are going to talk about essentially is chemical rocket. So far, we have not discussed. So, specific of any specific type of rocket; what we had was essentially payloads, velocities, etcetera. We have not talked about how it is produced. So, now we have to go to how it is produced. So, that will be our next topic. So, with that we complete our discussion on Space Dynamics. So, in the next class we will start from rocket nozzles.

Thank you.