Jet and Rocket Propulsion
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Lecture – 14

Welcome back to this lecture on rocket and spacecraft propulsion. For the last couple of lectures, we have been discussing the optimization of multistage rockets; first we talked about the performance of multistage rocket, then we have been discussing the optimization of those rockets. And we have stated that optimization essentially means that either if their delta v, that is the velocity increment is given and the final payload mass is given, how do we redistribute the masses among the rockets, so that the initial mass is minimum or we can put it differently that for the given overall payload ratio that is the final payload mass and the initial mass, how do we get the maximum velocity increment.

Then we had talked about three different cases; in the first case what we said was that the isp was constant for all the stages and the structural coefficient was also constant and for that we derived an expression for the velocity increment, which will give as the optimum value. And what we have seen is that in that case, all the stages must have same payload mass interaction in order to give the optimum performance. The second case which we looked at was when we still maintained same isp for all the stages, but we allowed for variation in structural coefficient, and for that case we optimized, and got expression for the in velocity increment.

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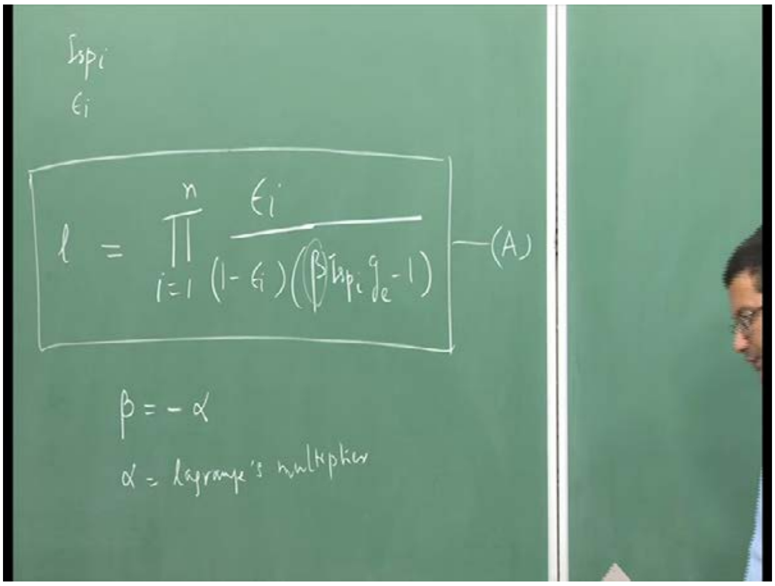

The third case we talked about was most general one where we say that I sp is different for all the stages and structural coefficient is different from all the stages. And what we did was we looked at the second approach for optimization in this, where we actually minimize the payload mass fraction or maximize the payload mass fraction, we minimize the inverse of payload mass fraction. So, for that we got an expression that l which is the payload ratio, overall payload ratio or overall payload mass fraction is equal to this, it derived this expression in the last lecture, let me call this equation (a), because this is a very important equation. So, in this equation epsilon i represents the structural coefficient of various stages and I spi it represents the specific impulse for different stages.

So, what is given to, these are the given values to us and then l is the overall payload mass ratio mass fraction which is also given. Now this parameter beta which is defined as minus alpha, where alpha is our Lagrange's multiplier; this is what we used to get the optimization. Then now let us look at this equation and see how do we actually solve problems. As I said this is the most generic case right any other case can be derived out of this; let us look at this expression.

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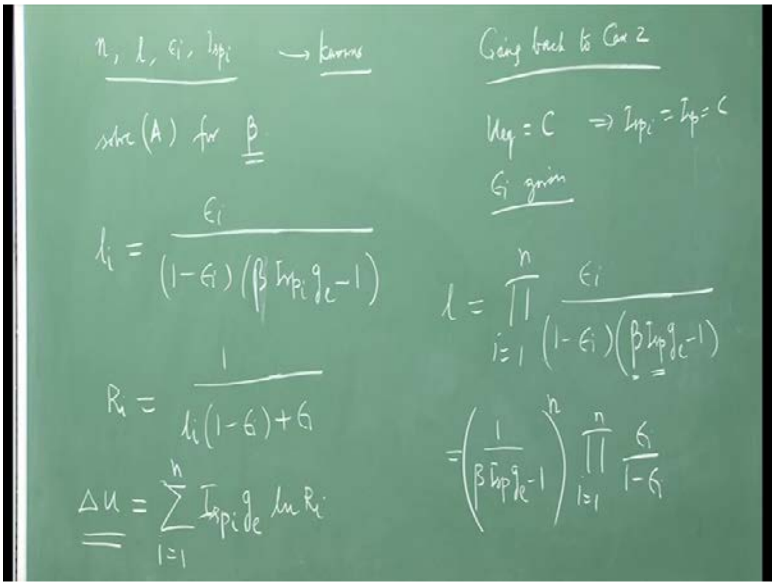

So, essentially the problem statement is we have given some parameters, some are known, what are known is the number of stages n is known to us. The overall mass fractions l is known to us, the structural coefficient for every stage epsilon I is known to us, and the specific impulse for every stage I spi is known. So, these are all known parameters; with this known parameters what we have to do is, we have to find out the best possible combination of essentially the payload mass fraction, so that we get that total overall payload is in the distribution is the best, so that we get the highest velocity essentially velocity increment.

So, what we do is the procedure is if I look at now this equation everything else in this expression except beta is known; only beta is the unknown. And as I have said towards the end of the last lecture, when why expand this depending on the number of stages we get polynomial in beta; if it is two stage we get second order polynomial, if it is three stage we get third order polynomial etcetera; so we get a polynomial in beta. So, therefore, this equation can be solved, this equation a can be solved for beta. So, the procedure is first solve a for beta. Once we have solved a and got this value of beta which is the inverse other than negative of the langrage’s multiplier.

Now what we do is we take this value of beta and put in to the expression for l i, now what is l i, l i is the payload mass fraction for individual stages which is equal to, this is the expression for l i. So, essentially it comes from the same expression here, this is how we get l i; once we get l i, then we go back to the expression for r i, r i we have proved earlier is equal to 1 up on l i turn 1 minus epsilon i plus epsilon i. So, the values of epsilon i are given; l i we have calculated from here after we have calculated beta by solving (A), now we have all the values of r i.

So, the next step then is to get the final velocity increment which is equal to sigma i equal to 1 to n I spi g e l n r i. So, these is the expression we used to get the velocity increment, where I sp is the specific impulse for the different stages, r i we are estimating from here. Now we can put all of them together to get the final velocity increment, so this is the solution procedure. As I have said that this is the most general case that we have talked about. Now let us see that we have started with this, we got this expression is the most general case, can be get back the other two cases we had discussed that is case one and case two starting from here. So, then if we just do this, everything else should fall in to place.

So, let us now look at the other cases, going back to case 2. What was our case 2? Case 2 said that equivalent velocity is constant for all the stages which implies I spi equals to i s p which is constant, but epsilon i is given it is varying; this is what case 2 was. Then for this case, we get l is equal to… Coming back to this equation, because this equation is also valid for this case also; so, l is equals to pi i equal to 1 to n epsilon is a variable for stage to stage, so write it like this. Only thing now here is that this I sp is constant. So, once again let see is the value of l is given in this case I sp is given, different values of epsilon i are given again we got a polynomial in beta which we can solve and get the value of beta. Once we get the value of beta, then what happens is that since here beta is a constant I sp is also a constant, what we can do is we can take it out of the product side right, because we know that beta is a multiplied it is an constant and is I sp is constant.

So, we can take it out of this product sign, if you do that, then this will be equal to 1 up on beta I sp g e minus 1 times n because this was repeating n times, so, will have at times basically to the rest to the power n, and then just this term. Now with this now let us simplify it little more, what we can do now is if the value of l is given and the number of stages are given we can actually get a course from expression for beta, by solving this.

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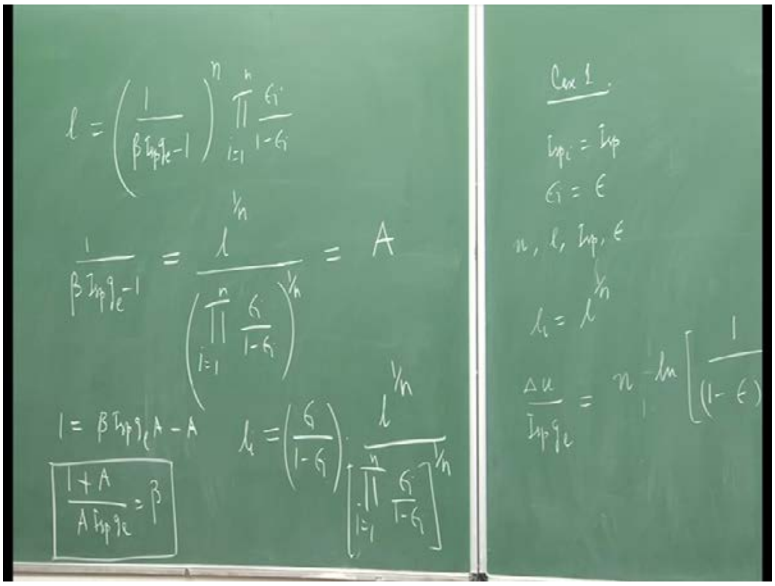

Let us look at this expression little more clearly, carefully. So, what we have prove there is that l is equal to one up on beta I sp g e minus 1 to the power n pi i equal to 1 to n epsilon i 1 minus epsilon i. Let us not take this to the left hand side and bring l to the right hand side; while doing that what I will get is equal to l to the power one up on n because this was rest to power n. So, we take n th root of this then this becomes n to the power 1 by n divided by once again n th root of this term, so this is not n th root, that is 1 by n.

Now let us look at these expression l is given to us l is given to us epsilon i is given therefore, right hand side is a known quantity. And now we no longer have to solve the polynomial right, because this is say some constant A therefore what it becomes is that 1 is equal to beta I sp g e times A minus A, I can take A to this side so this is 1 plus A up on a I sp g e is equal to beta. So, I can directly solve for beta by using this expression, I do not need to solve for a polynomial now; and once I have solve the value for beta, I can now come back here and get l i, because l i is equal to nothing but we take this sign of this portion is l i. So, once we have got this, then we can get l i is equal to epsilon of up on 1 epsilon 1 i l to the power 1 by n and then and then pi i equal to 1 to n epsilon i 1 minus epsilon i to the power one up on n. So, we can directly get all those values. So, this is the advantage that we started off with complex thing, then now we are simplifying. So, of course, we are getting simpler expressions.

So, we get now the expressions for the mass fraction for every stage very payload mass fraction every stage directly by just putting this values because of l n l n known now this is case two. So, what we have done is we have got case two back here, next we go to the either simpler case which was case one.

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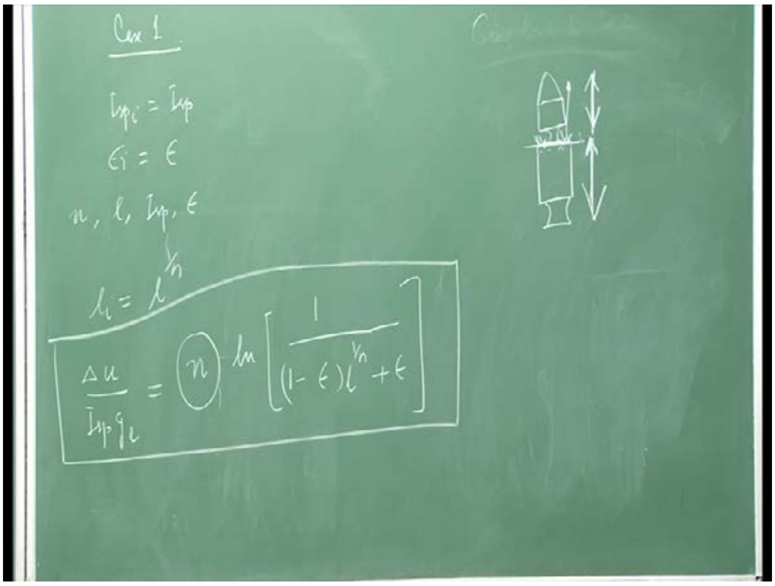

So, now, when we go to case 1, then we get what was our case 1? In case one what we considered is that I sp for all the stages were same and epsilon was also same. So, once again what were given to us was n l I sp and epsilon, these are the things are given to us. What we have shown for this case is that l i is equal to l to the power 1 upon n we have shown before. So, now, for this case we can directly get an expression for delta u by I sp g e is equal to n l n 1 upon 1 minus epsilon l to the power 1 by n plus epsilon; what we have done actually is we can taken that expression, and then put epsilon equal to constant. So, we get this from there.

So this, what we see is that this expressions is an interesting an important expressions. Let us take a little closer look in this expression. What we are seeing here is the delta u is expressed as a function of the structural coefficient I sp e g e plus the number of stage, Now that brings us to a question so far we have been saying that in order to improve the performance, what we do is we go to more number of stages, but then the question arises that is it monotonic improvement? That we keep on using the number of stages on the performance keeps on improve improving or there is a limit beyond which, we probably would not get much of improvement? Then as the, because as we are increasing the number of stages, the complexity is increasing, because every time when we have to discard something that separation process itself is a very critical thing.

Let me take a little detail and talk little bit about the separation process. When the separation happens what is happening, this is one rocket on top of this, another thing is sitting around; this things case burned and then it has to separate. Now when this is burning, all the major controls at with this little stage, because this is the primary river now this essentially since there as a dump payload does not participating anything so all the controls are here.

Now at the point of separation we have to be very careful, few things happened. First of all when they are separated they are still moving with the same speed, so the entire thing is moving with the same speed, and this guy was caring this, suddenly this is separated, immediately the load acting on this is gone down and some force was acting on it; so suddenly the load goes down this needs to accelerate. So, this vehicle will try to accelerate in this direction; whereas for this it has separated, but the engine has was still not taken over. So, now what happens something was pushing it something was pushing forces gone. So now, it will have tendency slow down. So, this will try to move in this direction, this will try move in this direction, and the result is if you do not anything their going to collide and then break up part.

So, therefore, during separation it becomes very important that you have to prevent this from going this side and prevent this from coming down this side. So, what you have to do is, we have small rockets here; these rockets are fired in the opposing direction. So, this rocket when it fires it will push it downward. So, you create a opposing force. So, you are retro grading the rocket or the stage these are called retrorockets. So, they slowdown, they slowdown this version, does not allow it to go up; at the same time this guys this things has to be pushed out, but still you ((Refer Time: 18:48)) started in your smaller structures which essentially provide little force to compensate for sudden loss of the force that was occurring. So, you push with these small rockets, again attach to this in this direction.

So, this two has to fire simultaneously at the time of separation, so that they are actually truly separated, till they are truly separated and far off only then they can safely said that separation is complete. This is one problem; another problem is so far you are control was here now during the separation process we are retrorocket firing all rocket firing, but you are stage has not yet burnt started to burnt. Once this start to burnt by that time you should have the control transfer from here to here. There is a gap during the separation for the control remains here and then we have to go there. So, during this time this guys in free fall, it does not have any control or anything.

Now if you are control transfer is not proper then it may go out of the control loop at completely, out of the control completely if it goes beyond the sudden limit, the control later cannot bring it back to the path. So, therefore this control transfer is also very critical, because of this reasons even though we see that multi staging is good very good, but he cannot depend blindly on multi staging, because some these are very dangerous is a very tricky manure, let me put it this very tricky manure to say a separate stage and transfer the control, and then this also to relight, it has to relight after that and then move forward.

So, because of that, we would not like to have more number of stages then that is required, would like to gate an optimum number of stages which gives us required benefits. So, for that, lets look at this expression that is why now this expression becomes important, because this gives an idea of this n, the numbers of stages. So, if I look back at this, what I will do is, I will plot the payload fraction that we can achieve for with the velocity increment for different number of stages.

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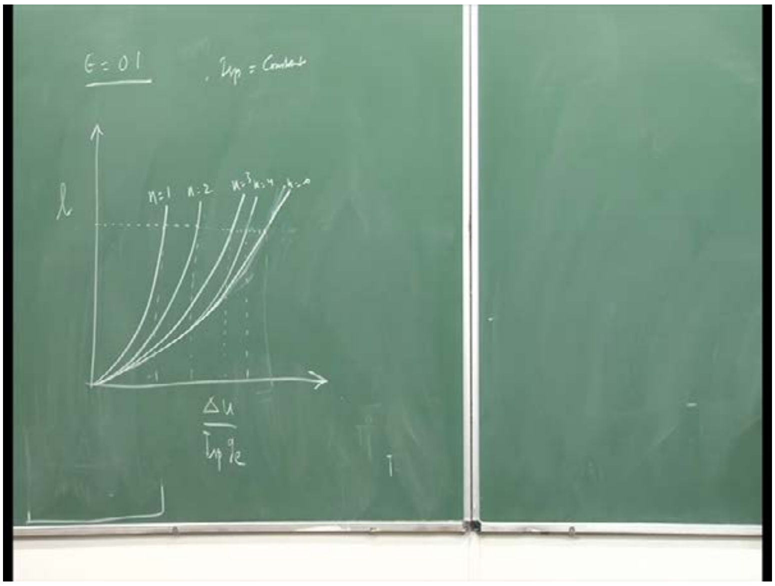

Let us consider that we take a case, where the structural coefficient is same for all the stages 0.01, let also consider the I s p s constant. So, our governing equation is this, which is dictating the performance. Now for this case let me plot the variation of delta u by I s p g e which is the left hand side of this expression versus l, l is the overall payload mass fraction. So, for a given value of this we get a sudden, we can move sudden mass fraction in the optimum manner. Now let us do that for different values of n if I do that for different value of n, this is what I am going to get this is n equal to 1 n equal to 2 n equal to 3 n equal to 4 n equal to infinity.

Now, let us see what we have here; let first of all we know that as been increased delta u v l is going to increase first point; we had also in discussed let for a single stage rocket, there is a limit, to which you can increase it, beyond that the acceleration becomes too high etcetera so you do not want to upper it. So, n equal to 1 we can get let us say a particular payload fraction moved optimal wave. So, in order to move go to higher more higher velocity for the same payload at fraction, then we go to multi staging.

So, let us say we go to two stage, so let us say this is the payload that we are trying to move, so from one stage we go two stage, immediately we can see that the same payload, we can now move (( )) velocity, because of the fact that we are discarding some of the dead weight. So, we are able to move too much high velocity if we increase the number of stages, we are still improving we are still going to higher velocity, because we are discarding this weight also. But the increase that we achieve between 2 and 1 and between 3 and 2, the increase that the incremental between 3 and 2 is less that between 2 and 1 we are improving it. But now the incremental gain is less, we go to four stages again we are getting some improvement, but once again the incremental improvement is less. As we keep on the increasing the number of stages, we keep on getting some increment or other. But this advantage is reducing is like law of (( )) written as we keep on increasing number of stages we are getting advantage, but the quantum of advantage is reducing.

So, beyond the particular number of stages we see that actual negligible change in velocity, where we keep on number of stages. So, at the infinity there is a maximum velocity we can get we cannot get more than that, there is a optimum value, we cannot get more than that. Then the point is that we have to make a call going from four to infinity where actually not earning much, but we are increasing the complexity a lot. So, then is it what going for this complex system, instead we can just stop at 4.

So now, that question is how do we decide, what is over optimum number? So, once again to summarize this, where we are higher number of stages we get greater delta u, but the pay of diminishes with n. So, we consider that parameter called terminal velocity. So, terminal velocity is the maximum velocity that we can get with a number of stages let us say that this is the delta u n is the terminal velocity, when everything is burn, this terminal velocity that we get.

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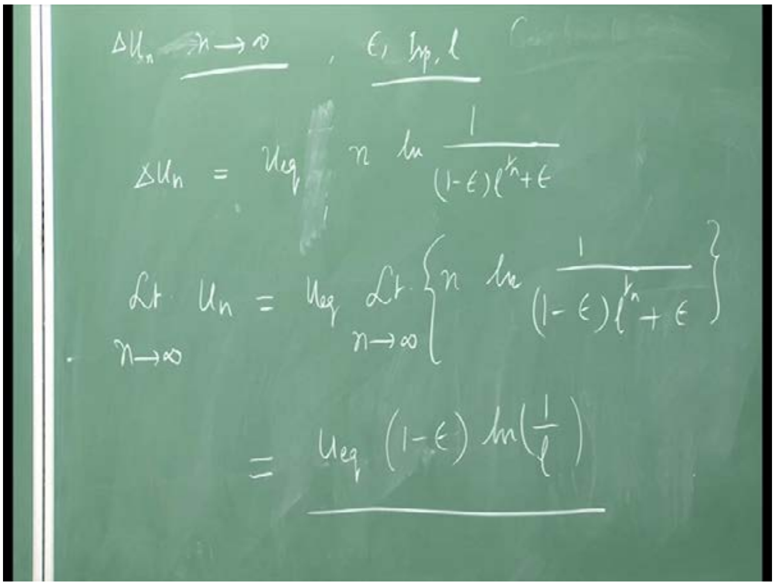

So, let us look at u n or delta u n when n tends to infinity and all the stages are same. So, we are looking at case 1, case 1 all the stages were same structural coefficient was same I s p was same, the payload mass fraction was same. So, same thing we are looking at, same epsilon same I s p, same l for all the stages only variation is the number of stage. So, we are considering a rocket where all the stages are same, but you have infinite number of stages. Then the velocity increment will be given as limit u n equal to u equivalent n l n 1 upon I will just write some more thing, once let me write the expression this is the general expression for the velocity increment with n number of stages.

Now as n goes to infinity then what happens, we get limit n tending to infinity u n equal to u equivalent limit n tending to infinity, sorry this sign was not there, delta u was only this, so summation sign was not their limit n tend to infinity n l n 1 upon 1 minus epsilon l to the power 1 by n plus epsilon this is the 1. Now if I expand these and take the limit was second I am not going into the detailed of the mathematics, I will give you the final number. The final expression the final expression comes out to be like this.

So now, notice one thing that the maximum velocity increment that we get is actually independent of number of stages, it does not have n anywhere. So, when it makes n tend to infinity it becomes independent of number of stages. Now here l is a constant given value epsilon is a given value u equivalent is a given value. So, therefore this is the maximum velocity increment we can get, no matter how many stages we put. And then once we decide on this, then the choice comes in, that for a practical application where do we pitch our self, how many number of stages we should kept; that can be essentially decided on by looking at how much increment we were getting by increasing the number of stages. So, this keeps of then the optimum number of stages as well. So, with this we come to the end of our discussion on multi stage optimization.

Next what we will do is before we go to space dynamics, let me solve couple of problems. So, let me solve one or two problems on this optimization. So, we have already solved a problem on case 1, next let us solve a problem on case 2, what was case 2 I s p was varying, now I s p was constant epsilon was varying.

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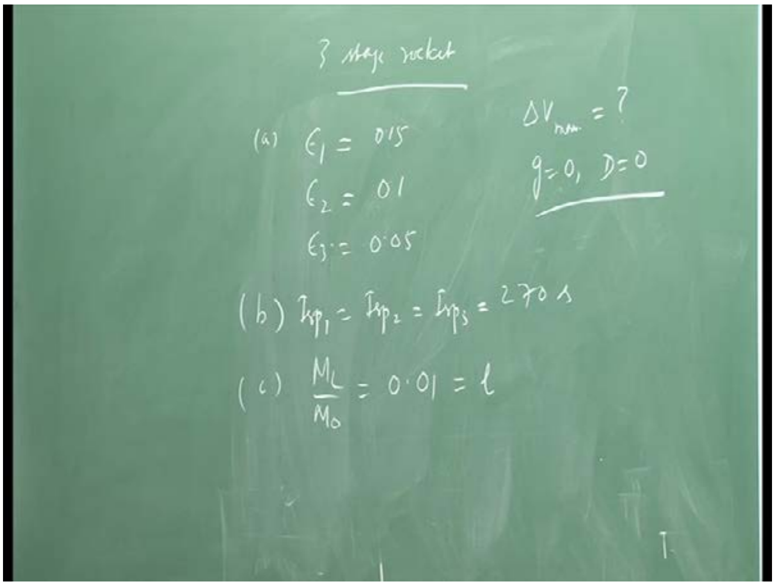

So, we solve a question on case 2. Again we consider a three stage rocket, these are the given conditions epsilon 1 is equal to 0.15 epsilon 2 is equal to 0.1 epsilon 3 is equal to 0. 05; and I s p is same equal to 270 seconds, and the overall mass fraction is also given 0 .01; for that we have to get the maximum delta v space delta v, g space means we do not have gravity, we do not have drag. So, this is the problem that we want to solve. So here this is a quite simple a problem, first of all what we do is we had already formulated this.

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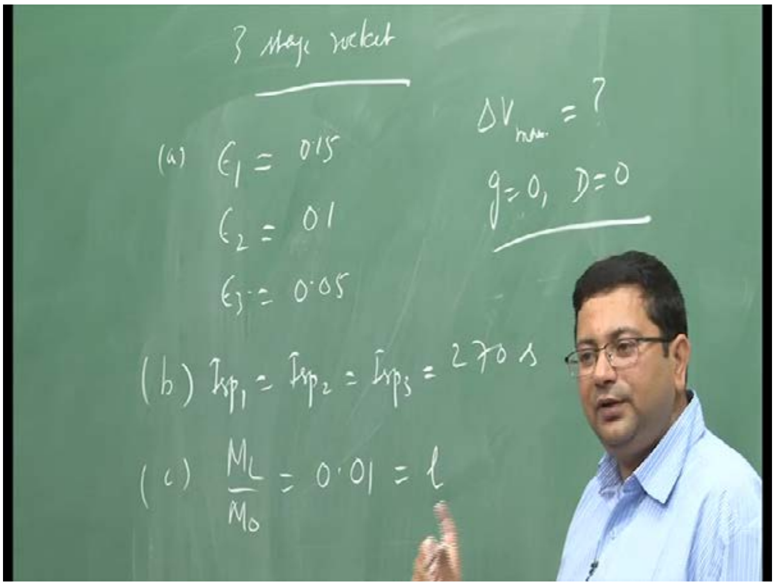

So now, I will just I am use the equations. We had just few minutes back derived an expression for these case for variations in l 1, l i, l i according to our estimations where epsilon i 1 minus epsilon i l to the power 1 by n by pi i equal to 1 to n epsilon i 1 minus epsilon i to the power 1 by n. This was our expression for l i for this case. Now we have l 1 l 2 l 3, n is given to be 3, l is given to be 0.01 and first we put epsilon is equal to epsilon 1 we get the value of l 1, this is equal to once we solve it 0.3762.

Next we put epsilon is equal to epsilon 2 taking epsilon 2 from there, again we solve this equation we get 0.2369, and the third again we put epsilon is equal to epsilon 3 and solve these, this is equal to 0.1122. Now what we do is we estimate delta v, which is equal to equivalent another I s p g e then sigma i equal to 1 to n l n 1 upon l i 1 minus epsilon i plus epsilon i, we have proved this. Now what we do is we have got to the value of l 1 l 2 l 3 I s p is given epsilon 1 epsilon 2 epsilon 3 are given we just put 1 at a time and add them we get the final value, this is 32780 meter per second. So, you can see that it is not very difficult to get it.

Let us now look at another case, which is a combination of let say case 1 and 2, what I will do now is that I will consider the same constant value of s or epsilon and I will allow I s p to vary, and do the same thing, because if we recall the case 3 was most generic, so then now we can use the case three equations to take any type of variation.

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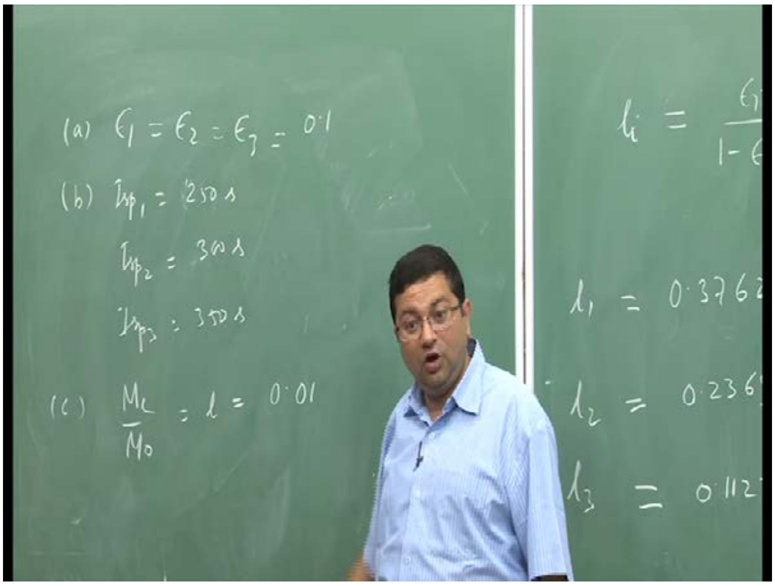

So, next what I will do is similar problem, but with little difference. So, once again we are talking about estimating the maximum three stage velocity for a three stage rocket. These are the condition epsilon 1 is equal to epsilon 2 is equal to epsilon 3 is equal to 0.1, this is what I am going to consider. But, now what I am saying is that my I sp values are different. So, what I have here is that the value of structural coefficient all the stages are same at they are equal to 0.1; and I sp for the first stage 250 second, for the second stage is 300, for the third stage is 350 second. And I maintained the same overall mass fraction l, which I have been doing for the other problems 0. 01.

Now for these, I have to get the delta v. So, see we had not discussed this case, what we have discussed was rather everything was constant or this was constant this was varying or everything was varying. So, since this was not discussed, let us considered everything varying case and there we will put this, so that is going to work for this case also. So, since we are talking about that case, where everything is varying.

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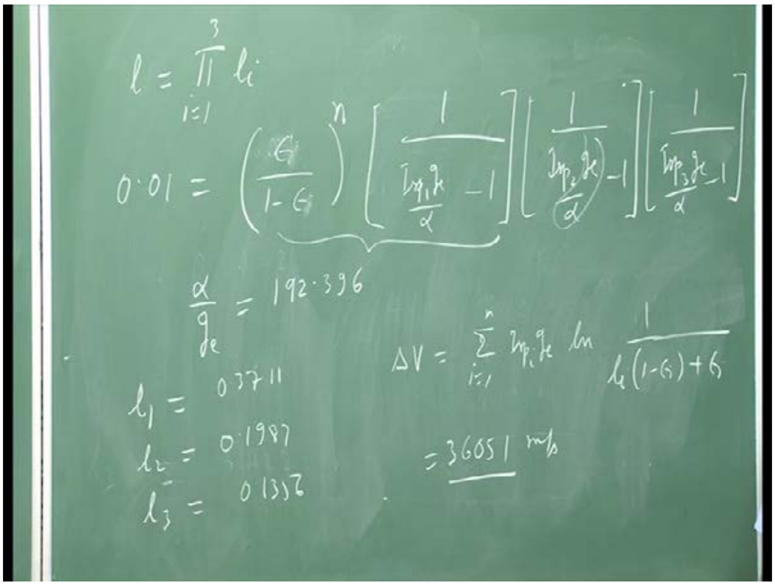

So now, we are to deal with the optimization what we have is l is equal to pi i equal to 1 to 3 l i, and l is equal to 0. 01 and l i is epsilon i 1 minus epsilon i to the power n at because this is constant epsilon i is constant, so this earlier it was total. Now since it is constant, it will be just the rest to the power n, and then 1 upon I sp g e upon alpha minus 1 1 upon I sp 1 I sp 2 g e upon alpha minus 1 and 1 upon I sp 3 g e upon alpha minus 1.

We had done in it terms of beta, but again that was a constant, here I am writing beta as 1 upon alpha. So, that is also a constant so I can use this definition. Now once I expand it, I get a polynomial in alpha after solving this polynomial, we will get the value of alpha. Actually what I will do is, I will get an expression for alpha by g e, because g is also constant. So, I can get an expression for alpha y e these. So, alpha by g e after solving these comes out to be equal to 192.396.

Once we have that now look at these term, this is if I put is equal to i this is essentially l i right. So now, I can get an expression for l i. So, l i is l 1 is only this portion without this n, l 2 is these times these, l 3 is these times these. So, I can now get the value of l 1 which is equal to 0.3711, l2 0.1987 and l 3 equal to 0.1356. Once I have this, now I can get the expression for delta v if delta v is equal to sigma I is equal to 1 to n I s p g e l n 1 upon l i 1 minus epsilon i plus epsilon i. Now everything here in known I just put them and get the value. So, 36051 meter per second. So this gives me the velocity increment for this specific case.

Now I would like point out something here. We solved three problems, for all of them the mass ratio was same, the average I s p was also same, what we had was different cases for different structure mass distribution and pilot mass distribution. And as we can see for all these cases we got different velocity increment. So, this shows that how we distribute the masses dictate for velocity increment we get, that is why the optimization we were talking about is very, very critical. So, with these let us now complete our discussion on the multi stage rockets. So, essentially the flight dynamic spot is more or less (( )), we know now what should be the mass distribution for a given I s p and structural coefficient excreta to get a certain velocity. So, from this discussion, what is emerged is that the velocity is important parameter; we have to attain certain velocity.

Next thing what we do is we focus on that, at why do we need certain velocity that depends on the space dynamics or vehicle dynamics. So, the next topic that we are going to start is space dynamics, where we will see that in order to complete a given mission which is let us say putting a suddenly like to an orbit or given orbit to do certain things and not only putting into an orbit, we have to also have to specify how much is going to be period of that. So, in order to achieve that how much velocity is required, that we will try to estimate from using loss of gravitation; so, that is what space dynamics is. So, the next topic that we are going to start is essentially space dynamics, the rocket vehicle dynamics we have covered in (( )) detail. So, that where I will stop here now, so next lecture will start space dynamics.