Jet and Rocket Propulsion

Prof. Dr. A. Kushari
Department of Aerospace Engineering
Indian Institute of Technology, Kanpur

Lecture - 12

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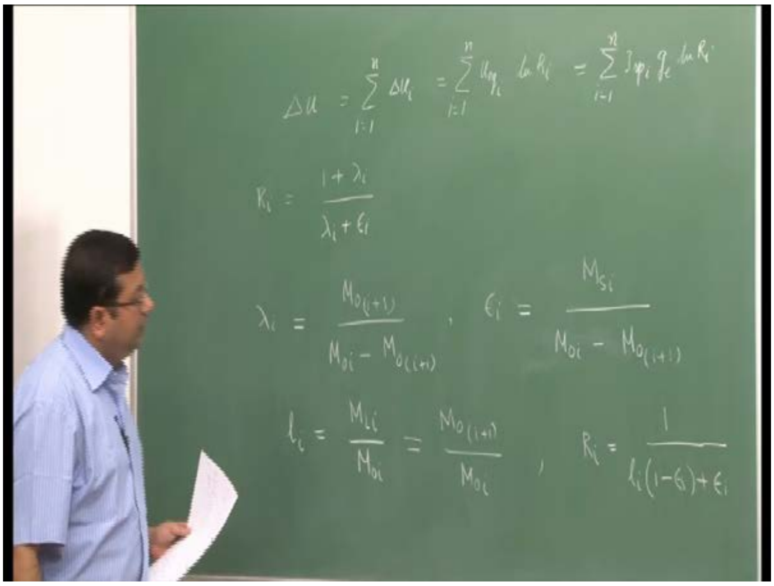

Welcome back. So, in the last lecture, we had discussed the operation of the multistage rockets, its performance and we derive an expression for the velocity increment for a multistage rocket. What we have shown is that the overall increment of the velocity of a multistage rocket is the algebra exam of the increment attain by individual stages, where subscript i represents the stage and a is the total number of stages.

And since we know that for a given stage, the velocity increment is function of equivalent velocity and the mass ratio given like this therefore, the total velocity increment can be written like this. And now the equivalent velocity we have shown again and again is the function of specific impulse of that stage. So, therefore, this is equal to i 1 to n I s p i g e l n R i, where I s p i represents the specific implies of a individual stages, we have shown this last time. Apart from that we had shown some relationships which we said will be useful in optimizing the performance.

So, before we go to optimization let us recap what we had discussed and list the relationships that we had obtained. So, one such relationship was relating R i to the payload factor lambda i and the structural coefficient epsilon i for different stages. We had shown this before where lambda i for a multistage rocket is the payload for i th stage which is equal to the initial mass for the i th i plus 1 it stage divided by all the mass of the i th stage except the payload, so given like this. So, this is the payload factor for i th stage. Similarly we had looked at the structural coefficient for the i th stage which is define as M s i divided by M naught i minus M naught i plus 1.

So, this is the structural mass of the i th stage, the initial mass of the i th stage minus the payload for the i th stage. So, this is the structural coefficient. we had also define the payload mass fraction for i th stage as M L i divided by M naught i where M L i is the payload mass for the i th stage and M naught i is the total mass for the initial mass for the i th stage. So, once again since this is the initial mass for the i l i plus 1 at stage, we can write this as M naught i plus 1 by M naught i. So, we had define this parameters and we had proof that R i is equal to 1 upon l i 1 minus epsilon i plus epsilon i. So, these are the things that we had discussed in the last class. Now, let us continue from here what we want to do today is optimization of multistage rocket.

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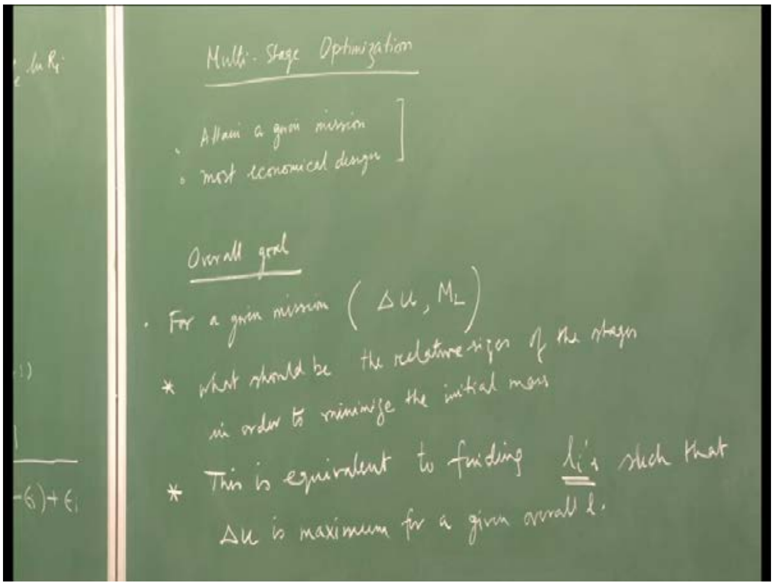

So, today’s topic is multistage optimization. Now what is our goal? When we are designing a rocket the goal is to get to the most economical design which attends a given mission. So, first of all to attain a given mission is the most important thing and we want to attain it in the most economical way. So, this is most economical design we gone to get the most economical design to attain a given mission. So, this is our mission objective or design objective. So, when we are doing optimization these are the things that we have to keep in mind. So, the given mission becomes our constrain and the economical design is our goal.

So, therefore, what is the overall goal for the design effort? First of all again for a given mission now, what is a given mission? Important to know, what is the given mission? What is the program director has given a designer? The given mission is to put a given payload into a given orbit. Now, that payload when it is in a particular orbit in order to remain in that orbit it must be delivered with the given velocity only then it will take that orbit and remain there. So, therefore, the given mission as for as that is designer point of view is consider, can be consider to be a given velocity increment delta u and the given payload M L ok.

So, this is my given mission is overall delta u and overall M L, the velocity increment and the payload. So, this is the given mission the overall goal is that, what should be the relative sizes of these stages in order to minimize the initial mass. Let me explain, what I mean by this is the design objective, we have two attain this mission, we have to get a certain delta u and sudden payload we have to deliver, but we want to do it in such a manner that our initial mass is minimum. If you have to carry the initial mass the lowest initial mass therefore, if I look at the mass ratio overall mass ratio are this factor R, this factor R is going to be higher right.

So, therefore, we get higher delta v or if you are reduce this it is more economical operation. So, therefore, if I have to, if I can reduce the initial mass, we can now, reduction in initial mass keeping the payload constant means either we reduce the structural mass or the propellant mass. If you are reducing the structural mass we have to carry less if you are reducing the propellant mass we have to carry less propellant at the same time we are saving on the propellant cost.

So, both of them taken together essentially improves the rather economy of the operation. Now, how do we attain that without compromising this that is the tracks of this optimization of design. We do not want to compromise on these, but we want to have the minimum initial mass, we can attain that by distributing this masses relative sizes of different stages we can distribute the structural mass and the propellant mass in such a way among different stages that the final overall mass of the rocket is minimum.

So, that is the design objective. Equivalently we can restate this that this is equivalent to essentially finding l i’s, l i is our structural sorry payload mass fraction.

So, is equivalent to finding the l i’s such that delta u is maximum for a given overall l. So, this statement here that, we want to minimize the initial mass is actually equivalent to finding the payload mass fraction for different stages, the optimum payload mass fraction of different stages. Such that, the velocity increment is maximum for the given overall payload ratio. So, essentially here, what we are saying is that, we keep this constant and M L is constant, we can reduce M naught. So, that we get the best possible performance here, what we are saying that? This is equivalent to saying that, we have the given M L we have the given initial mass M naught, what will be the distribution that will give us the maximum delta u.

Now, I am claiming that this two are equivalent states. Let us now, see whether it is truly correct or not are this two equivalent statement. For that what I am going to do is, I take an example. So, to show this equivalent of statement.

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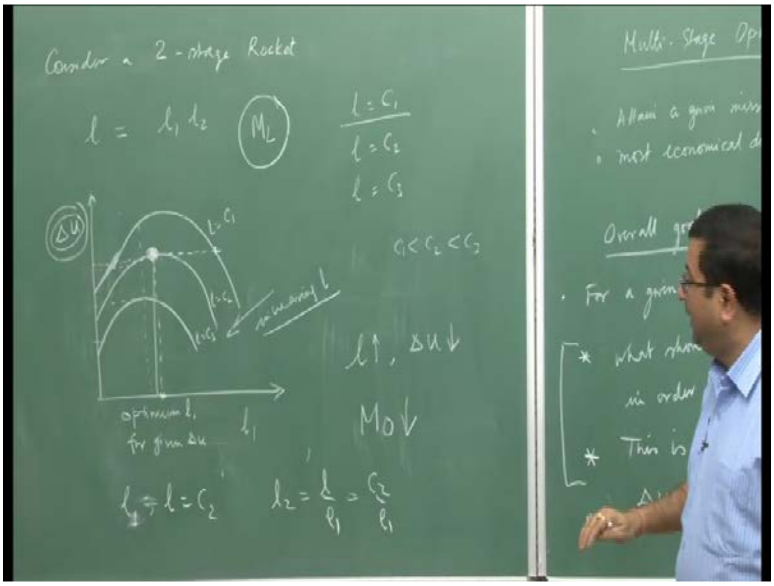

Let me consider a two stage rocket, I am considering a two stage rocket. Now, for this rockets since it is a two stage rocket, the overall payload mass fraction is given as the product of the payload mass fraction for the first and second stage this we have proved in the last lecture. Therefore, this rocket let us say what we do is? Let us say for a given value of l 1, what we do is? We consider a value of l different values of l. So, let they be consider l equal to C 1 or l equal C 2, l equal to C 3 etcetera. So, let say for a given value of l for example, let us consider l equal to C 1, this is a constant value of l. So, overall mass fraction is constant, what we do is, we vary l 1 and then estimate the corresponding delta u and plotted.

So, we see that the variation will look something like this. This is for l equal to C 1, let us now, increase l say C 1 is less then C 2 is less then C 3. So, now, we increase l and again repeat the same thing that is we vary l 1 and estimate l 2 for this given l and plot the variation in delta u. So, this will be like this and like this, this is l equal to C 2, l equal to C 3 now, this is our increasing l. let us see now, on this curve, what is happing? First of all let us look at the variation of for a given l 1, let us say let look at the variation of l, what we are seeing here is that? As we are increasing l, has we are increasing l, delta u is decreasing. So, has l increases, delta u decreases this is same for all the values of l. Now, how we are increasing l? How do we increase l? If you have to keep our the final payload mass M L to be given, we do not want to change this, then the only way we can increase l is by reducing M naught right.

So, here what we are seeing is that, if we have to increase l, if we increase l delta u decreases or in other words if M naught is increased keeping M L constant delta u decreases. Coming to this, this statement here, we want to keep this constant, we want to minimize the initial mass delta u M naught therefore, delta u will decrease. Now, the question is, but we want to keep this same, how can we do that? By we are decreasing it at the same time we want to keep this same now, that is possible if we work around a horizontal line like this right. If I work here let us along this horizontal line at this point this is l equal to C 1 right, from here and to if I come to these line, I have reduced l, I am sorry increased l which means I have reduce to M naught initial mass, I have reduced, but I have kept the same delta u right.

So, if I come from here to here to let us say, this point I have achieve this see attaining the delta u was by goal M L was given to me which I have to kept constant. So, I have achieve this statement by moving like this. Now, this is one of the conditions at the first statement that we have looked at. Now, this value here then which we are getting here is now, if I look at this point, this point is optimum l 1 for given delta u right. Now, if I look at this delta u value and compare it with l 1, what we see is that, if let us see, if you are on this curve, this value of delta u l 1 is here, but as I go up there is one point here at which l 1 there is a value of l 1 for which delta u is maximum right. If I increase l 1 beyond that delta u starts to drop.

Similarly, for this l up to this point if I increase l 1, delta u increases, it reaches this maximum then delta u starts to drop same here. So, therefore, if I look at this point them for this value of delta u this is the optimum delta u l 1 alright. So, for any value of now, instead of plotting like this, if I take any value of delta u, for that value of delta u there is a value of l 1 which will is a optimum for a given value of delta u. And therefore, this is the value we are trying to find what we are getting same here?

First of all that when we are increasing l sorry, when we are decreasing l, delta u is increasing right. When we are increasing l, delta u is increasing sorry, when we are decreasing l, delta u is increasing and that increase is that because how do we increase l? By reducing, just a second, it is from here to here l is decreasing, yeah when we are decreasing l for this constant, we are actually increasing M naught and then that in to essentially because of increasing propellant mass. So, we are having more propellant here. So, if you burning more propellant we get higher velocity right. So, therefore, this is obvious that increasing the propellant mass will give us higher velocity. On the other hand decreasing l, well reduce to reduction in will lead to reduction in propellant mass therefore, delta u will decrease.

So, now, from this curve we see that there is a point here, this point, this point here corresponds to both this statements right. First since, we are moving horizontally here it satisfies the first statement, if I look at this statement, we are finding at value of l i this is this optimum l i, l 1 right. Now, from this plot, if I can get l 1 and my l is fixed here, l is equal to C 2 therefore, l 2 is equal. So, l 1 is obtained from here, l is equal to C 2 according to this curve. So, therefore, l 2 is equal to l by l 1 which is equal to C 2 by l 1. So, l 2 is already is fixed once we have chosen this point right.

So, therefore, what we see is that, this point here also corresponds to at distribution of l i, that is l 1 and l 2 such that delta u is maximum at this point for this given value of l. So, therefore, this discussion shows that, both this statement are equivalent and that is what we want to use because minimizing initial mass essentially is something that is difficult to work with because then our expression for velocity increment it includes a initial mass. Because that becomes more complex, where as especial for delta u is pity state forward right. So, if you say that the objective function is maximizing delta u it is a straight forward approach.

So, therefore, we will look at this statement that will find out different distribution for which delta u is maximum. So, this discursion is for a two stage rocket, if you generalize it for n stage then we will have and n minus one dimensional plot. This is a one dimensional plot, if we increase to n stages there will be n minus one dimensional plot similar plot can be obtained. Now, with this discursion where, we have shown that the multistage optimization essentially leads to fulfilling this requirement, our objective function then is maximizing delta u and the constraints are either this or the overall because M L is known so over having the overall payload fraction this are the constraint. So, we have now, define the problem let us look at the mathematical way to optimize this problem. For that we will consider some specific cases, so first and for most let me consider a case where, we have say n stages with certain conditions.

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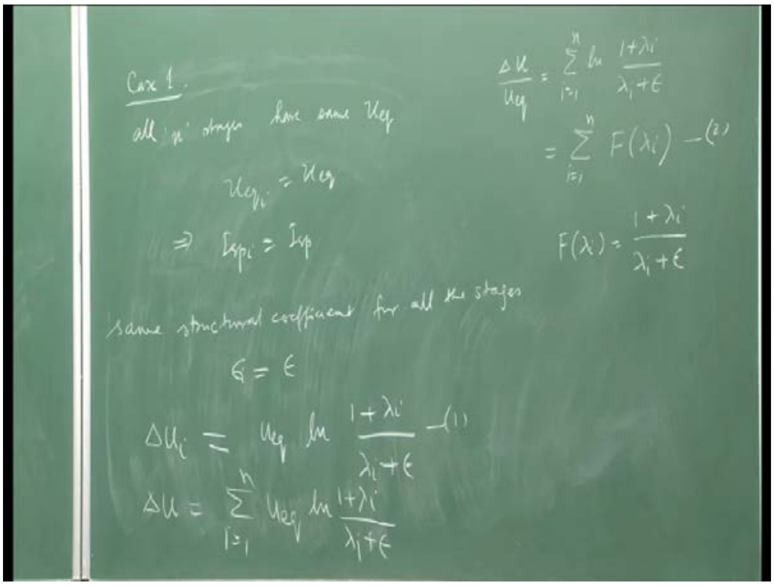

So, first let us look at case 1, in case 1 what we will says that? Let us say, all the stages all n stages have same equivalent velocity which, essentially means that u equivalent i is equal to u equivalent and we know that equivalent velocity is essentially i s p which, implies that i s p for all the stages as same. Specific impulse for all the stages as same so for this then and will have another condition that we have same structural coefficient for all the stages which, essentially means that epsilon i is equal to epsilon all the stages have same structural coefficient. Then we want to get the distribution of l i or lambda i which, will give us the maximum delta u.

So, for this delta u i which is the velocity increment for i th stage is equal to u equivalent l n 1 plus lambda i up on lambda i plus epsilon right, this is something that we have shown before. Now, if that is the case then for the n stages for the all the stages the velocity increment total velocity increment is delta u which is equal to sigma i equal to 1 to n u equivalent l n 1 plus lambda i up on lambda i plus epsilon at epsilon is epsilon i is constant. Therefore, we write it as epsilon on the now in this equation u equivalent is constant. So, we can take it out of the summation sign and then take it to the left hand side then, we can write delta u by u equivalent is equal to sigma i equal to 1 to n l n 1 plus lambda i lambda i plus epsilon.

So, in this expression the only dependences on the stage comes through lambda i right every other parameters independent of the stage. this term is essentially a function of lambda i nothing else. So, we can write this then as sigma i equal to 1 to n a function of lambda I, let me call this, first of all let me call this as equation 1, let me call this equation 2. Here what is f lambda i? f lambda i is equal to only this term 1 plus lambda i up on lambda i plus epsilon. So, here f lambda i is a function of only lambda i given as 1 plus lambda i up on lambda i plus epsilon. Now, before we progress proceed further i would like to derive another expression relating l and lambda I, why we want to do that because we have a constraint on l like we have discussed here the product of all l i is equal to l. So, we have a constraint at l. So, therefore, let us first derive an expression relating l and lambda i because here our objective function is in terms of lambda i.

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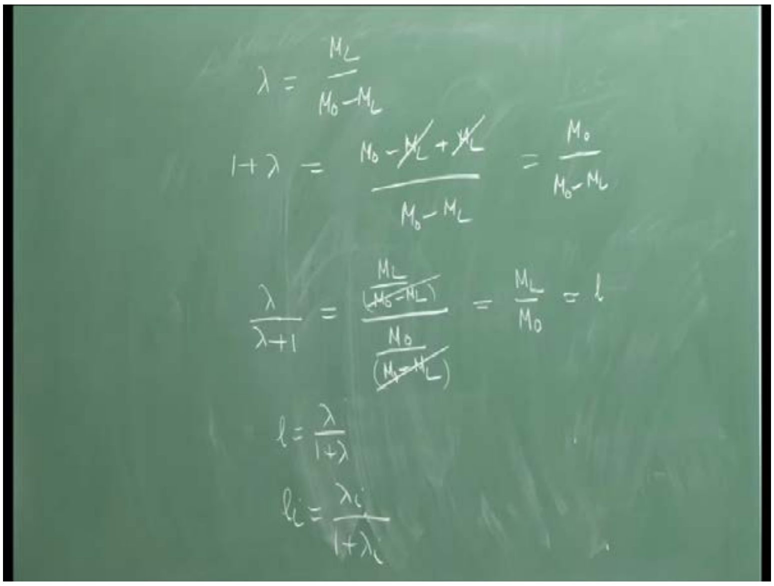

So, let us first define a get an expression relating l and lambda i for that let me first see what is lambda i? lambda I, lambda by definition is M L upon M naught minus M L right. That is how we have define lambda if I add 1 to this then this is equal to M L minus plus M L upon M naught minus M l. So, M L we cancel of all. So, therefore, 1 plus lambda is equal to M naught upon M naught minus M L then if I do now, lambda upon 1 plus lambda then, that will be equal to M L upon M naught minus M L divided by M naught upon M naught minus M L.

You can see that the numerator denominator for both of this will cancel off. Then what we will be left with is equal to M L upon M naught and that is equal to l the payload mass fraction. Therefore, this shows that l is equal to lambda upon 1 plus lambda, which is now, applicable to any stage therefore, I can write that l i equal to lambda i upon 1 plus lambda i. So, this relationship is going to be useful, when we try to maximize this delta u. So, now, let me look at the expression for l overall l we have proved a relationship between l and lambda i.

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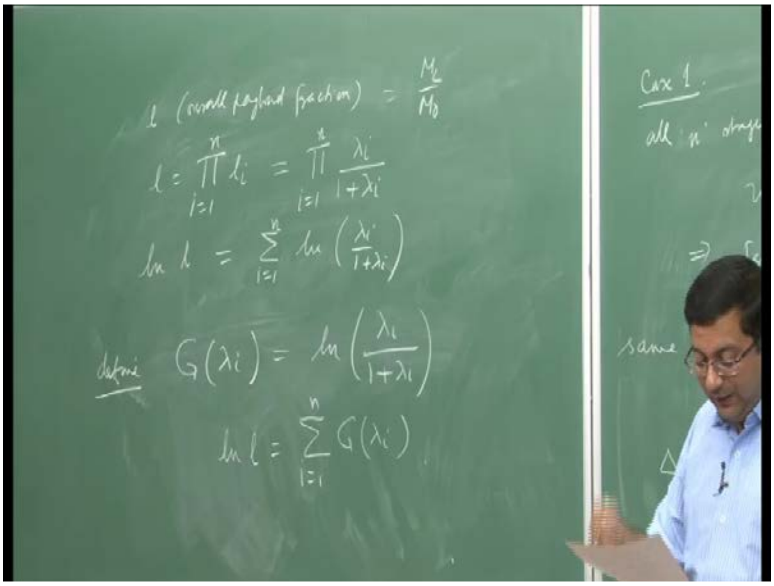

Let us now look for l l, l is our overall payload fraction. Now, this is something as for as optimization is consult is the given quantity. So, this is equal to M L upon M naught this is given to us and we know that we have proved that L is equal to the product of the payload mass fraction for all individual high stages. So, therefore, L is equal to pi i equal to 1 to n l i. Now, if I put this relationship into this equation then this is equal to pi i equal to 1 to n lambda i upon 1 plus lambda i. Now, this is now, our constraint because we have said that this relationship is always valid for any multistage rocket therefore, this must be satisfied at the same time we are saying that l is given to us. So, then this becomes a constant. So, this relationship now, what we are saying is that we want to maximize this delta u, but this maximization must be according to this rule we cannot valid this at the same time we have to maximize delta u. So, how do we achieve that, what we do is, we will form a different function before doing that, I will simplify this expression little more, what I will do is, I will take logarithm of l, l n l equal to now, this is a product right. So, logarithm of product is essentially some of logarithm of independent individual variables.

So, therefore, l n l will be equal to sigma i equal to 1 to n l n lambda i plus 1 upon lambda i. So, now what we get here is l n l is equal to the sum of all of this function lambda i upon 1 plus lambda i. Let me say that, we define another function g of lambda i which is nothing but lambda i 1 plus lambda i. We are defining this as a function. Then this constraint that we have now can be written as l n l equal to sigma i equal to 1 to n g lambda i.

So, now what we are having here is our what we want to maximize is delta u which is nothing but sigma i equal to 1 to n F lambda i, where F is a function of lambda i here this is our constraint that l n l equal to i equal to 1 to n g lambda i, where g is again a function of only lambda i. So, now, what we can do is we can define a new function which includes both of this. So, we will define this because essentially what we want to do is we are trying to maximize this. This is what we are trying to maximize with this constraint. So, we will define a new function, let us call that l which, is also a function of lambda i which includes both this maximization variable as well as the constraint.

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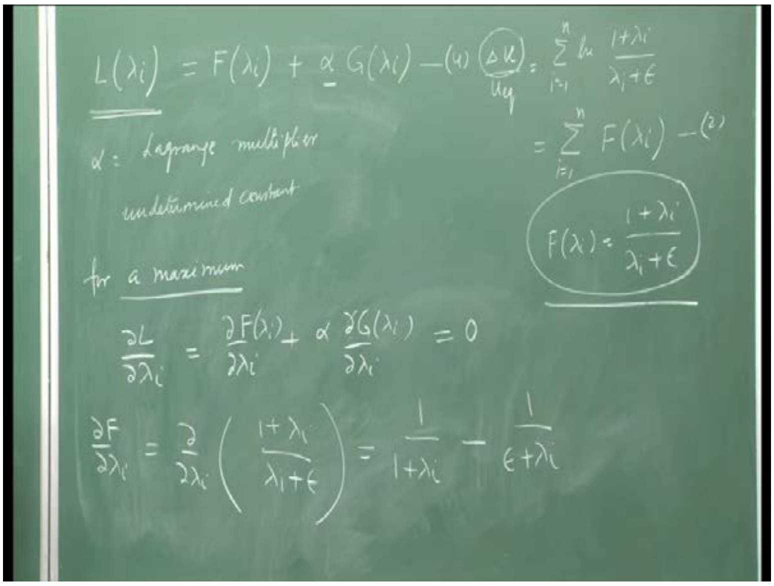

So, if I define a new variable here are the sorry, a new function here, we can write is as L lambda i is just an algebraic sum of F lambda i plus sum fraction alpha times G lambda i ok. What we can say is that, there is a linear relationship relating this to get a new function L. Now, this equation again let me call this as three and let me call this equation has equation four. Here first this parameter alpha is called Lagrange multiplier, this is an undetermined constant this is an undetermined constant Lagrange multiplier.

Now, when we combined all this, we get this equation as we see that is a function of only lambda i there is a function L lambda i, which is the sum of f lambda i plus alpha G lambda i. So, alpha is a variable or a parameter which will actually dictate what is going to be the maximum value for this rather the, what will be the maximum combination will be dictated by alpha. So, we can maximize this equation and once we maximize this will find a relationship between alpha and lambda i right. So, then for a given value of lambda i we get the alpha. So, that is how once we get the alpha, we had the corresponding best possible combination of lambda i that is what is the optimization all about. So, essentially now, this becomes our objective function L lambda i which, we want to maximize. So, for a maximum what we do is we will differential L with respect to lambda i right. Now, as we can see that is a linear function this and this are adding together the linear operator.

Therefore, d L d lambda i is nothing but d F d lambda i plus alpha d G d lambda i. So, in order to maximize then, what we do is? We differentiate this expression with respect to lambda i and then we equated it to 0 that gives us the maximum. So, now, what we have is essentially d F d lambda i plus alpha d G d lambda i equal to 0 where, our F is given by this and our G is given by this. So, now, let us look at the first term here d F d lambda i this is equal to nothing but d d lambda i of this term 1 plus lambda i upon lambda i plus epsilon right. So, now, we can differentiate this, this with respect to lambda i this will be equal to 1 upon 1 plus lambda i minus 1 upon epsilon plus lambda i. So, if I differentiate this, this is what I am going to get. So, now, the first term here is taken care of it. Next look at the second term, the second term here is this G lambda i multiplied with n alpha. So, first of all what we do is let us get d G d lambda i.

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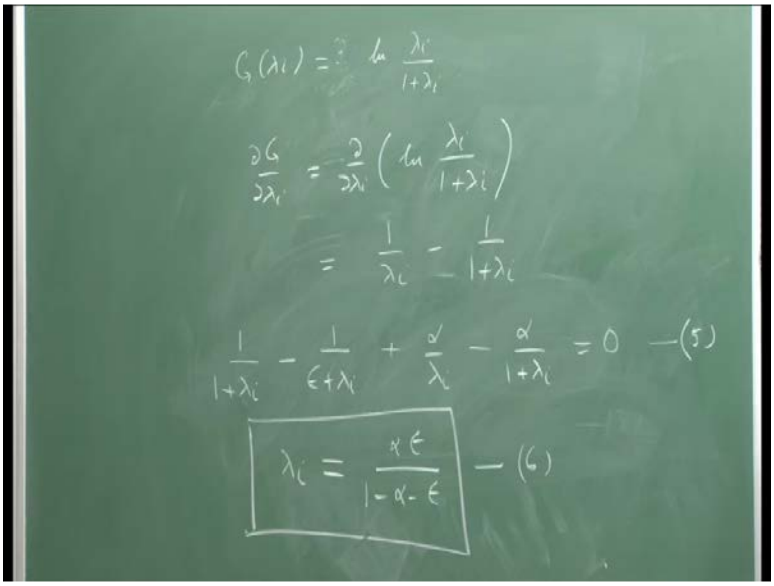

So, in this expression, let me first write G lambda i then we will do d G d lambda i. So, G lambda i is equal to l n right, l n lambda i 1 plus lambda i sorry, here one thing is missing l n time. I forgot to write the l n here, plus and l n that is why we get this expansion yeah, this is an l n. Sorry this thing is missing from these expression there will be a normal coming here in this expression. So, this will have an l n term.

So, now here also we have l n d lambda i l n lambda i 1 plus lambda i. Now, if I do the, if we are differentiate this expression with respect to lambda i and I will get this is equal to sorry, d G d lambda i equal to d d lambda i of l n lambda i 1 plus lambda i. So, this is equal to then 1 upon lambda i minus 1 upon 1 plus lambda i so after differentiating if we get this expression. Now, let us take this and this and put it back into this expression for maximization. Then what we get is 1 upon 1 plus lambda i minus 1 upon epsilon plus lambda i plus alpha upon lambda i minus alpha 1 plus lambda i is equal to 0. Let me called this equation 5.

We got it by putting this and that expression back end to this equation. Now, we have these expression here, what we see is that is a function essentially relationship between lambda i and alpha for constant value of epsilon. So, this can be reduce to then after some algebraic manipulation to alpha epsilon 1 minus alpha minus epsilon. Let me call these equation 6. So, the optimum value of alpha will be obtained for like this. So, now, if lambda is given we can get it like this.

So, this expression then gives us the value of alpha. Another if alpha is known we get lambda I, if lambda i is we get alpha and I will come to that, how do we get this two different parameters little later. So, just to conclude what we discussed today is that this is our expression for the final velocity that, we will get which in terms of lambda i and epsilon is given like this. What we say is that? This term here with in the summation sign is function of lambda I, we defined it as F lambda i equal to l n 1 plus lambda i upon epsilon plus lambda i therefore, if you maximize this, we maximize delta u as well. Now, this is our objective function and then we had the condition that L i s the product of L i s is equal to L overall payload fraction. From there we got relationship relating G and lambda i then, what we say is this our constraint. So, the objective function and the constrain together multiplied by Lagrange multiplier gives us a new objective function which, we want to maximize.

So, in order to maximize this, we differentiate this expression with respect to lambda i, well lambda is our variable, differentiate this and set it equal to 0. Once we do that, we get this expression which after simplifying gives us lambda i as a function of epsilon alpha, alpha and epsilon. So, therefore, this is the optimum combination that we are looking for because now, the objective function is optimized with the constraint. So, this gives us this is the method called optimization and here what we have considered is a specific case where, equivalent velocity was constant for all the stages; that means, we have the same specific impulse for all the stages and the structural coefficient was consider to be same of all the stages. So, under this assumption this is the optimization. So, let us stop this lecture at now. In the next lecture, we will continued from here and we will discuss little be more on the multistage of optimization, we look at some other cases us well.

Thank you.