**Jet and Rocket Propulsion**
**Prof. Dr. A. Kushari**
**Department of Aerospace Engineering**
**Indian Institute of Technology, Kanpur**

**Lecture - 11**

Welcome back. So in the last lecture, we started discussing Multi-stage rockets, we have discussed the nomenclature that we used to represent the performance of multi stage rockets. We have also discussed that why do we need multi stage rockets, what are the performance improvements that we can gain by going from a single stage to a multi stage design.

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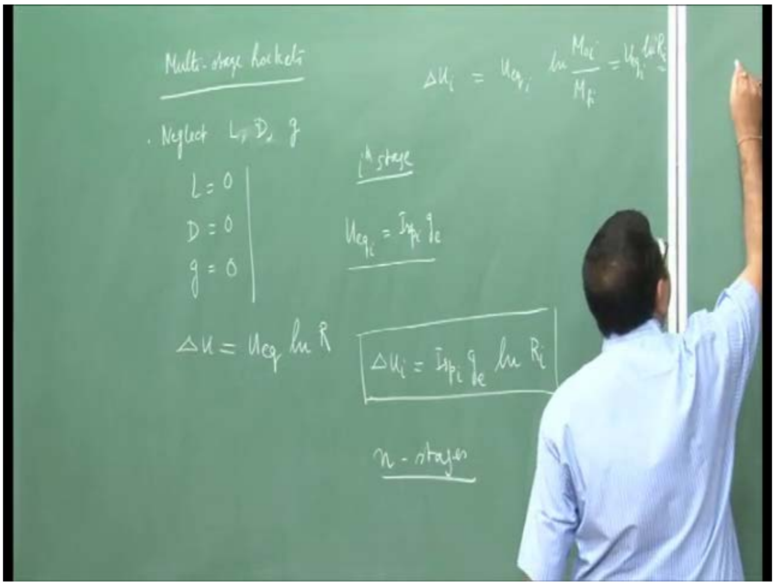


Next, what we will do now is to talk about the performance, we will derive the expressions which represent the performance of a multi stage rocket. Before we do that first let us make the assumptions that we will be talking about, we will be neglecting the lift, neglecting the drag and neglecting the gravity that is L is equal to 0, D is equal to 0, g is equal to 0.

Now, as for the last few lectures we have been talking about these assumptions. First of all, lift as we have seen typically works perpendicular to the flight path. So, if lift is non zero, the vehicle will start to deviate from the intended path sight path, therefore we always like to have lifted to be 0, because these are non lifting vehicles. So, therefore, this is not an assumption is actually a practical reality. Coming to the drag, on the ground or close to the ground, because the atmosphere is very, very dense we will have some drag.

But the point is that the rocket vehicles, travel extreme distances and as we go up high in the altitude the densities falls rapidly. So therefore, as we go high and high, the density will drop, therefore the drag is also going to go down. So, essentially the drag although towards the beginning in the booster stage is important, as we go further up out towards the edge of the atmosphere drag becomes a very small amount and in the outside the atmosphere, drag is practically 0.

Therefore neglecting drag also is something that is reasonable assumption and coming to the gravity, well gravity is going to be there, but the effect of the gravity, once we go to outer space is not very large. But, as when we come to space dynamics we can see that all our mission is actually dictated by gravity. Gravitational force is the most important parameter in rocket propulsion, so gravity is going to there, but the magnitude is not going to be very large.

So, for practical applications when initial design, let me put it this way the initial design when we are designing the rocket initially for a given mission, we can safely neglect the effect of gravity as well. And by neglecting particular drag and gravity our rocket dynamic equation, which is a differential equation becomes independent of time; therefore, we do not have to integrate over time, to get the velocity increment.

Therefore, neglecting gravity and drag actually simplifies our calculation and then as we have seen for the single stage rockets, the velocity increment, becomes a function of equivalent velocity and the mass inverse of mass ratio R. Now, this was the starting that we have proven for the single stage rocket.

Now the same thing now can be applicable to a multi stage rocket, so if I write the same equation for a Multi-stage rocket, I will write it like this delta u i equal to u equivalent i L n M naught i upon M f i, which is essentially equal to u equivalent I, L n R i. Here, i represents the i th stage of a multi stage rocket, we have discussed the nomenclature in the previous lecture, that we have n number of stages.

Every stages is designated by subscript i can be 1, 2, 3 up to n depending on the number of stages. So, this is the expression for the velocity increment under these assumptions for the i th stage. So, delta u i then is the velocity increment for the i th stage, that is the difference of velocity at the end of burning for the i th stage minus the difference in velocity at the end of burning for i minus 1 th stage or the previous stage.

So, previous stage burns, it leaves the vehicle with certain velocity, and then the i th stage takes over from there, and gives an increment, so that increment is given here by delta u i, u equivalent i represents the equivalent velocity for the i th stage. So, that depends on the specific impulse, so I can write it here, as u equivalent i is i spi to g e. Where, i spi is the specific impulse for the i th stage and g e is the acceleration due to gravity at sea level, this is something that is important this is g e is all sea level.

Only in the gravity term, if we consider then that appears as g, so then with altitude it varies, but here it is only at the sea level. So, the equivalent velocity for i th stage is given like this, and then the mass ratio inverse of mass ratio R, R i is M dot i by M dot f, that is the initial mass of the i th stage divided by the final mass of the i th stage. We have defined all these parameters in the last lecture. Now, with this then if I put it back into this equation I get delta u i equal to I spi g e, L n R i, this is my expression for velocity increment for the i th stage of a multi stage rocket.

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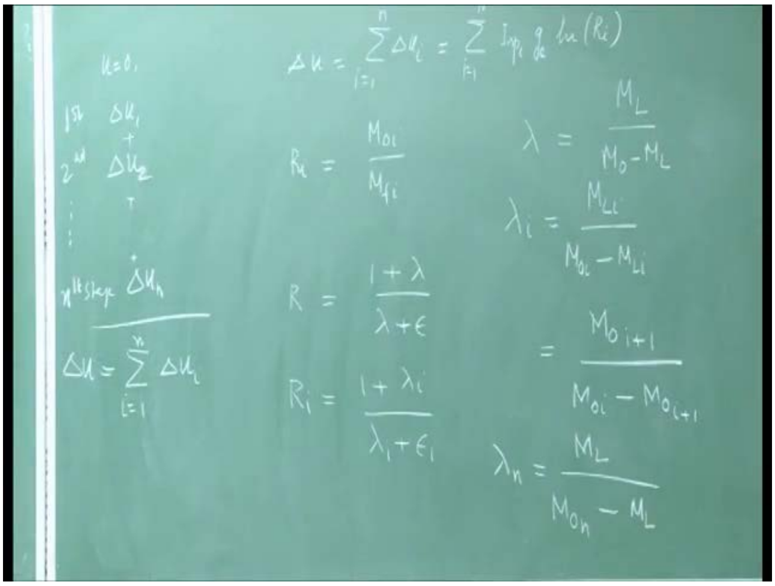


So, now if I have n stages, then what is happening is this the rocket let us say is starting from 0 velocity. At the end of first stage, it gains a velocity delta u i delta u 1, end of second stage it gains a velocity delta u 2, like that so to the end the n th stage, after the burning of n th stage it gains the velocity given by delta n sub n.

So, therefore the overall increase in velocity is nothing but, the summation of the velocity increment for all the stages, so the overall delta u is nothing but, sigma delta u i, i going from 1 to n. So, therefore the overall gain in velocity for n stage rocket is the sum of the gain in velocity in individual stages, so that is given as like this. So, therefore let me write the overall delta u now, in terms of this is equal to sigma i equal to 1 to n, delta u i, and my delta u i is this term. So, this is equal to sigma i equal to 1 to n, I s p i g e, L n R i, that is the expression for overall velocity increment for a multi stage rocket.

Now, as we have been seeing so far, the parameters that are known, sometime the structural weight will be known, such time the propellant mass will be known, sometime the payload mass will be known, so there are different things that are known parameters. So therefore, although this is the governing equation, we may need to write it in different forms.

Secondly, when we come to optimization we will see, that we will optimize let us say with respect to the payload fraction of different stages, or let us say with respect to the structural coefficient of different stages. So, therefore for those cases then this equation have to be written, in terms of those variables, let us say payload factors structural coefficient etcetera.

So, what we will do next is this is our basic equation for velocity increment, we will just rewrite it this equation, in terms of the other defined parameter that we have defined. What are the parameters we have defined, we have defined lambda, we have defined epsilon, we have defined L etcetera, so we will try to express this, in terms of this parameters.

So now, let us look at this term R I, as we know that this is the inverse of our mass fraction or mass ratio, so this now R i, we will represent in terms of this parameters, lambda epsilon and L, because these are our optimization parameters. So first of all, let us look at back at R, we have seen that R is equal to 1 plus lambda upon lambda plus epsilon, now this is for a single stage rocket.

So therefore, for every stage of a Multi-stage rocket also this expression is valid, so we can write R i equal to 1 plus lambda i plus lambda i plus epsilon i, where R i is this term inverse of the mass fraction for mass ratio for the i th stage. Lambda i is the payload factor for the i th stage, and epsilon I is the structural coefficient for the i th stage.

So, then going back to our definition of payload factor, how did we define the payload factor? Payload factor is the payload mass divided by initial mass minus the payload mass that is, all the mass except the payload. So, we had defined this as, M L upon M L minus M L, M naught minus M L that is our definition of payload factor.

Now, if we use this definition for the i th stage of a Multi-stage rocket, then we will write it as lambda i M L i upon M naught i minus M Li, now comes the interesting part. So, this is the payload factor for the i th stage which is the payload for the i th stage divided by initial mass for the i th stage, minus the payload for the i th stage.

Now, from our discussion in the previous lecture, what is the payload mass for the i th stage as we have discussed in the last lecture, this is the total mass of the i plus 1 th stage, because this is something the i th stage delivers. So, therefore, this M L i is the total mass or initial mass, for the i plus 1 th stage that is the next stage, next upper stage.

Then, lambda i we can write as M dot i plus 1 divided by M dot i minus M naught i plus 1. So, the payload factor for the i th stage is the initial mass for the i plus 1 th stage divided by the initial mass of the i th stage minus the initial mass for the i plus 1 th stage. So, therefore, this will be defined only up to n minus 1, because for the n minus 1 th stage this is M l, but if i put it as n th stage sorry, it can be defined up to n th stage for the n th stage, this is the overall payload M L.

So, for the nth stage lambda n is equal to overall payload M L divided by the initial mass for the n th stage minus the overall payload, so the payload factor for the n th stage is given like this. Next let us look at the structural coefficient, now so far again the structural coefficient we have defined for a single stage. Now, when we come to the multi stage, then we define the structural coefficient for every individual stage.

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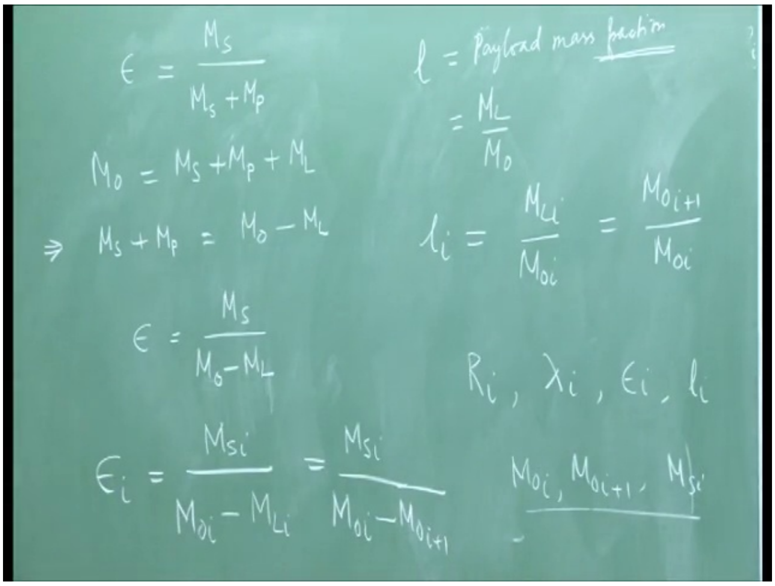


So, next let us define the structural coefficient, structural coefficient for a single stage rocket was defined as the structural mass of the rocket divided by the structural mass and the propellant mass, that was our definition of structural coefficient. Now, we know that the overall mass is the sum of structural mass plus propellant mass plus the payload mass, therefore the sum of structural mass and propellant mass is the initial mass minus the payload mass.

So, if I put it back into this equation, my structural coefficient is equal to M s upon M naught minus M L, this is for a single stage rocket. Now, we use the same definition for a multi stage rocket, so we define the structural coefficient for the i th stage of a multi stage rocket. So, that will be given by the subscript i, which is equal to M s i divided by now M naught becomes M naught i minus M L i, where M s i is the structural mass of the i th stage, M naught i is the overall mass or initial mass of the I th stage, M Li is the propellant mass, the payload mass for the i th stage.

Once again going back to the discussion here, the payload mass for the i th stage is the initial mass for the i plus 1 th stage, so therefore, this is equal to M s y, M naught i minus M naught i plus 1, so this is the definition of then structural coefficient for the i th stage. Next let us define l, l is our propellant payload mass fraction, so payload mass fraction, we had define this as M L upon M naught for the single stage rocket. This is payload mass fraction, this is the fraction of payload mass with respect to the overall mass, this is payload factor, lambda is payload factor these two are different.

So, as you can see here, lambda is defined as the payload mass divided by all the mass except the payload, payload mass fraction is the fraction of payload in total mass. So, lambda and L are two different parameters, so lambda is called payload mass factor, this is called payload factor, this is payload mass fraction. Now, so coming back to this l is M L upon M naught, so the payload mass divided by the total mass, this is for a single stage rocket.

So, once again if I use this definition for a multi stage rocket, then i represent this as L i, so L i equal to M Li upon M naught i, and once again L i M Li is equal to M naught i plus 1, that is the payload mass for the i th stage is the initial mass for the i plus 1 th stage. So, this can be written as M naught.

So now, if i re recap what we have done, so far is we have defined R i for a multi stage rocket, we have defined lambda i for a multi stage rocket, we have defined epsilon i for a multi stage rocket and L i these are the four parameters we have defined so far in this lecture. And, we have defined most of them as functions of M naught i, M naught i plus 1 and M si see these are the three masses that has been used.

So, we have defined the this four parameters for multi stage i th stage for single stage for a multi stage rocket, in terms of these three masses. The structural mass of the i th stage, the initial mass of the i th stage and the initial mass of the i plus 1 th stage based on this we have defined now, all these parameters.

Let us, now take a further step and write one more relationship here we have one relationship relating R lambda i and epsilon i, in the last class we also derived one relationship relating R L and epsilon. So, let me first write that expression for a single stage rocket, then I will write that for a multi stage rocket.

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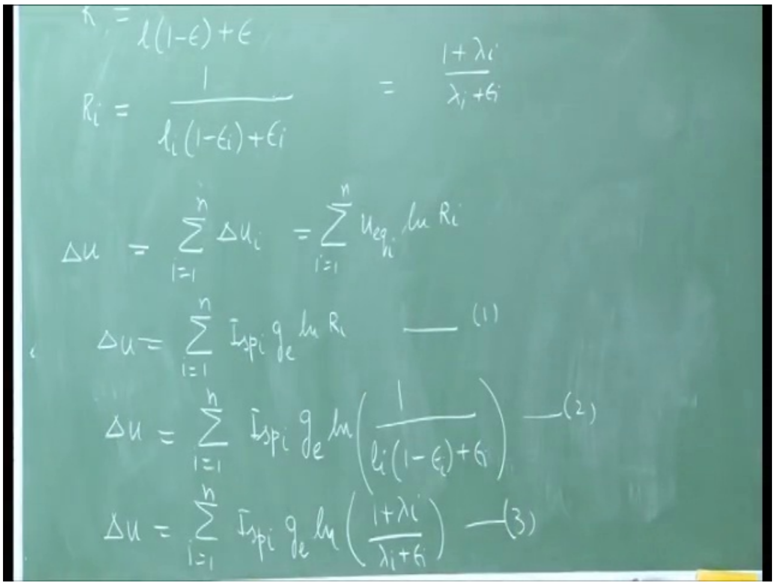


For a single stage rocket, in the last class we have derived this expression that R equal to L, 1 minus epsilon plus epsilon, this is one expression we had derived in the last class for a single stage rocket. Now, if I use it for a multi stage rocket, then for the R th, i th stage I can write R i equal to 1 upon l i 1 minus epsilon i plus epsilon i. Now, if I take this expression, and then go back now to our expression for delta u, and this is one another expression was 1 plus lambda i upon lambda i plus epsilon i, these are the two relationship that we had.

Now, go back to our expression for delta u for the multi stage rocket, we have shown that the overall velocity increment is the algebraic sum of the velocity increment of the n stages. So, this was equal to sigma i equal to 1 to n delta u i, and delta u i is equal to u equivalent i l n R i, and then u equivalent is equal to i spi. So, I can write it as I s pi g e l n R i, so this is one expression let me write it as equation 1.

Now, we can replace R i by this, so in the same equation now, we can replace R i by either this expression in terms of l i n epsilon i or we can replace it in terms of lambda i n epsilon i. So, therefore, I can get another expression for delta u, which will be equal to sigma i equal to 1 to n, I s pi g e, if I replace R i by this then is equal to l n 1 upon l i 1 minus epsilon i plus epsilon i, let me call this equation 2.

So, here in this what we have done is, we have replaced R i by this expression for l i n epsilon i or we can also replace R i, by this expression in terms of lambda I and epsilon i.

So, this can be written as I s pi g e l n, instead of R i can write 1 plus lambda i upon lambda i plus epsilon i, let me call this equation 3.

So, these three equations that I have written here are actually same, only thing is that different representation of the various parameters, depending on what are given to us, we can either estimate R or l i epsilon i or lambda i, then these are essentially the equivalent expressions. And, this becomes handy when we go for optimization, so these three equations we will be using for optimization purpose any one of them. Before we proceed further now, we will try to look a relationship for l i, l i is a very important parameter that is the payload mass fraction is a very important parameter for multi stage rockets. So, let us take a closer look at the payload mass fraction.

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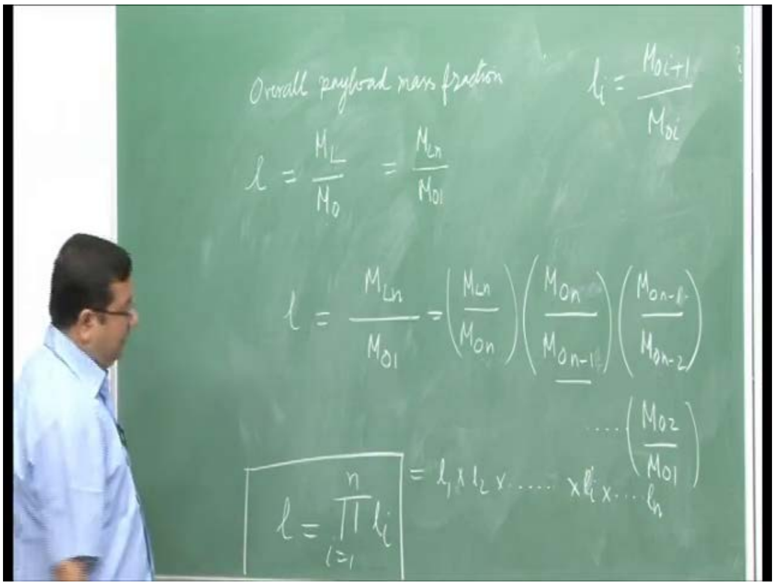


So, next we will look at, first of all the overall payload mass fraction, this is the overall payload mass fraction for the entire rocket, entire multi stage rocket, so it is given by l. So, then this l is equal to nothing but M L upon M naught, where M L is the actual payload, and M naught is the full initial mass, which includes all the stages, and this is the actual payload M L is the actual payload.

Now, in the nomenclature what we have seen is the actual payload actually is the payload for the n th stage also, so therefore, this M L can also be written as M L n, so it can be written as M L n, and the total mass is the initial mass for the first stage. So, I can write it is M naught 1, so then from this discussion, M L is the payload mass for the n th stage divided by the initial mass for the first stage.

Now, I can write this as the payload mass for the n th stage divided by the initial mass for the n th stage, then the initial mass for the n th stage is my payload mass for the n minus 1 th stage. So, I can write it as M naught n minus 1 divided by the initial mass for the n minus 2 th stage, just a second initial mass, this is the initial mass for the n th stage which is the payload mass for the n minus 1 th stage.

So, let me just redo it, this is the payload mass for the n th stage divided by the initial mass for the n th stage, and we have said that the initial mass for the n th stage is the payload mass for the previous stage, so this is the payload mass for the previous stage divided by initial mass for the previous stage. And then we continue, so now this becomes the payload for the n minus 2 th stage, and then initial mass for the n minus 2 th stage.

And, we continue like this till we come to the first stage, M 02 is the initial mass for the first initial mass for the second stage, which is also equal to the payload mass for the first stage, so we continue like this till we come to the first stage. So, now, if I look at this equation, and look at the definition of l i, l i is the payload mass fraction for the i th stage, which we had defined as M naught i plus 1 upon M naught i, we have defined it like this.

So now let us look at this term, what is this M 02 initial mass of the second stage, which is the payload mass for the first stage divided by M 0 mass 1 which is the initial mass for the first stage, so therefore, this is nothing but l 1. So, this is equal to l 1 then multiplied by l 2 continues till if I come to any stage in between it will be l i into what is this the payload mass for the n th stage divided by the initial mass for the n th stage. So, that is L n, so that is payload mass fraction for the n th stage.

So now what we see here, is the overall payload mass fraction is essentially the product of individual payload mass fraction, this is a very important observation this gives us as a constraint, and it is true no matter what, this is the constraint with which we will be working. So, I can write as l is equal to product of i equal to 1 to n, l i this is a very important result.

So, let me call this equation 4, when we are using these equations, when we are using these equations l i is varying here, typically the overall mass fraction will be a known quantity from there we can estimate l i, I will come to that later or when we go to optimization there will be a specific relationship, which can be shown. So, therefore, no matter what by the way for a multi stage rocket, this expression is always valid. Because of the fact, that the payload for the one stage is the initial mass for the next stage, therefore this equation is always valid.

Now, with this then let us now solve a problem for estimation of essentially velocity increment for a particular rocket with particular given conditions. We will take a very simple example, later on when we do optimization, and then we will go for little more complex problem. But, very simple problem, essentially what we are trying to do is use these equations use these equations to get some information.

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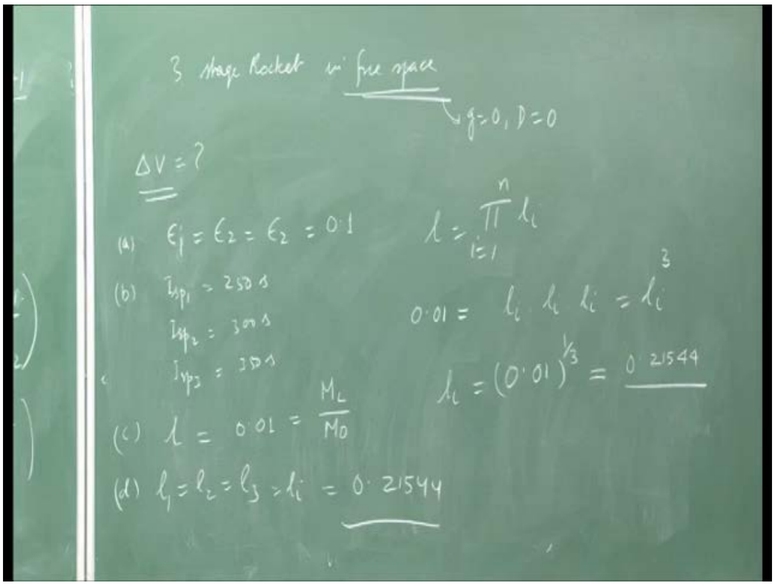


Let us consider, three stages rocket in free space, so essentially first of all what does this term free space means that g and D are 0, gravity and drag are 0. For this three stage rocket we are supposed to find out the velocity increment, delta v, the conditions given are a, that the structural coefficient for all the stages are same, that is epsilon 1 is equal to epsilon 2 is equal to epsilon 3, that is equal to 0.1, this is a given condition.

Secondly, what is given is that, the I sp values is given that the I sp value for the first stage is 250 second, I s p value for the second stage is 300 seconds, and that for the third stage is 350 seconds, these are the three values, that are given, it is also given that the overall payload mass fraction l is equal to 0.01.

And, another condition that is given is that the payload mass fraction for every stage is same, these are the conditions that are given we have to get the velocity increment for this rocket. First let us come to this l is equal to 0.01, which is the total final payload divided by the initial mass, and we are given l 1 equal to l 2 equal to l 3.

First, what we do is let us get these values what are the values of l 1 l 2 l 3 for that we use this equation, what we see here is that first lets write l 1 equal to l 2 equal to l 3 is equal to l i, then from that equation l equal to pi, i equal to 1 to n l i, and therefore, this value is given 0.01 equal to l i into l i into l i, so this is equal to l i cubed, therefore l i is equal to cube root of 0.01.

Now if I solve this is equal to 0.21544, so what we are said doing here is that, the problem gives us the overall payload mass fraction, and it also states that every stage all three stages have the same payload mass fraction. Now, if the payload mass fraction is same for all the stages then from this equation, we have just proved that this equation is valid for any multi stage rocket which says that the product of individual stage payload mass fraction is equal to the overall payload mass fraction.

So, we have used this equation here l is equal to i equal to 1 to n l i, this value is given we have from there we can calculate what is the payload mass fraction for individual stages. So, now, this is equal to 0.21544, so this is the first stage of the problem done. Now let us come to the second part of the problem, with this now we have to estimate delta v delta v is our velocity increment for that we go back to our expression for multi stage velocity increment.

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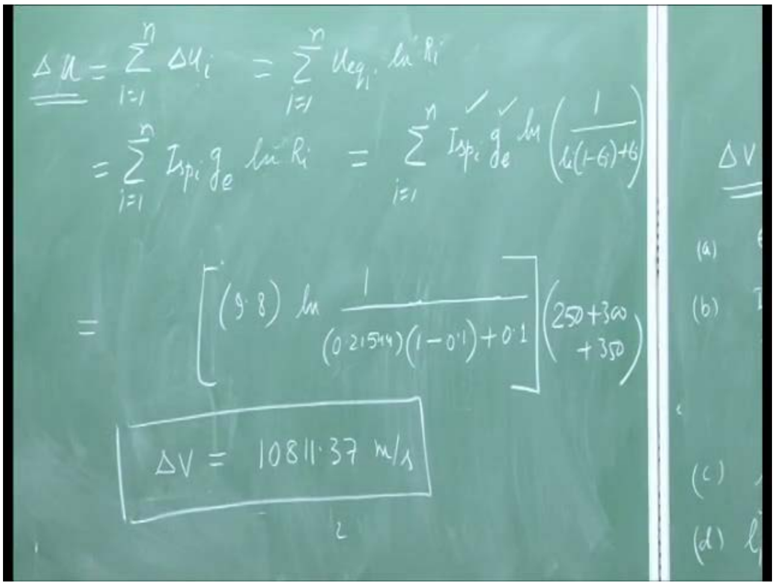


So, we have seen that delta v is equal to sigma i equal to 1 to n, delta u delta u i, this we have seen and this is equal to i equal to 1 to n u equivalent l n R i, we have also seen this u equivalent we can write it as I sp, so we can get it as I spi g e l n R i. Now in this problem what are the parameters given to us, after we have solved this, what is given to us are epsilon 1 epsilon 2 epsilon 3 and this is epsilon 3 and l 1 l 2 l 3.

So, essentially this R is given in terms of l i and epsilon i we had derived a relationship relating R and epsilon R i l and epsilon. So, now, if I put it back here, this is equal to sigma i equal to 1 to n, I s pi g e l n 1 upon l i 1 minus epsilon i plus epsilon i, this is the relationship that we had derived. Now let us look at this equation, this is what we want to estimate, what is given to us is I s p 1 I s p 2 I s p 3, so these are given, g e is known acceleration due to gravity at sea level is a known parameter, epsilon i is given so that means, epsilon 1 epsilon 2 epsilon 3 is given.

L i, we need and we have estimated L 1 L 2 L 3, so therefore, all the parameters in our right hand side are now known. So, if I now just put it back here, this becomes equal to first epsilon I s p 1 is 250 times 9.8, which is our g e times l n 1 upon 0.21544 times, 1 minus 0.1 plus 0.1. Now, if I look at this the first term for i equal to 1 is this, for i equal to 2, this term remains same because our l i is same for all three stages, and this terms also remains same because epsilon i is same for all three stages.

So, only thing that is going to change is I spi, so what I will do is the I will rewrite it as this term is same, and here I will add the I s p values 300 plus 350. So, essentially what I am doing is I have this value as a constant, and then to this I am multiplying sum of 3 I s p values 2 of 250, 300 and 350 seconds.

Now, after this then if you solve this delta v value comes out to be equal to 10811.37 meter per second, so this is the value of delta v for this problem. So, now we know how to estimate delta v for given conditions, later on in the next lecture, what we are going to see is how do we optimize this, that is we are given some parameters how do we distribute it, so that we get this value maximum. So, we will be given some constraints under this constraints, we would like to maximize the velocity increment that we can attain for this multi stage rocket.

This example, I have given is a very simple example where all the structural coefficients were equal, and the stage payload mass fractions were equal, but in reality you may not have the same condition, here only thing that was differing was the I sp. In reality you may have different distribution, depending on the mission requirement you may need to have different distribution, so that we can optimize the performance of the rocket.

So, far we have seen how a multi stage rocket works, we have considered this for free space because as I have said that the drag and gravity can be neglected for most practical applications. So, we work with this expression, we have expressed R i in terms of different parameters, so depending on what are the values given we choose for that is why I have taken this example. In this example what we have seen is that the epsilon values were given, the l values were calculated, so we used this form of the equation.

But, if let us say instead of this something else were given, then we use a different form of the equation, so therefore, there is a depending on the given parameters we choose which is the equation that needs to be used. And then essentially the governing equation is this you put it here, and get the final expression for our velocity increment, that is not going to change. Only thing is change is going to be that what form of R we are using, so let us stop here today, and in the next class we will take up the optimization of multi stage rockets.

Thank you.