Jet and Rocket Propulsion
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Lecture - 10

Welcome back, so far we have discussed the performance of a single stage rocket, next we will start discussing multi stage rockets, but we will borrow heavily from the discussions of single stage rocket. So, before progressing first, let us kind of recap what are the equations that we have derived so far for single stage rockets, which will be useful for multi stage rocket analysis.

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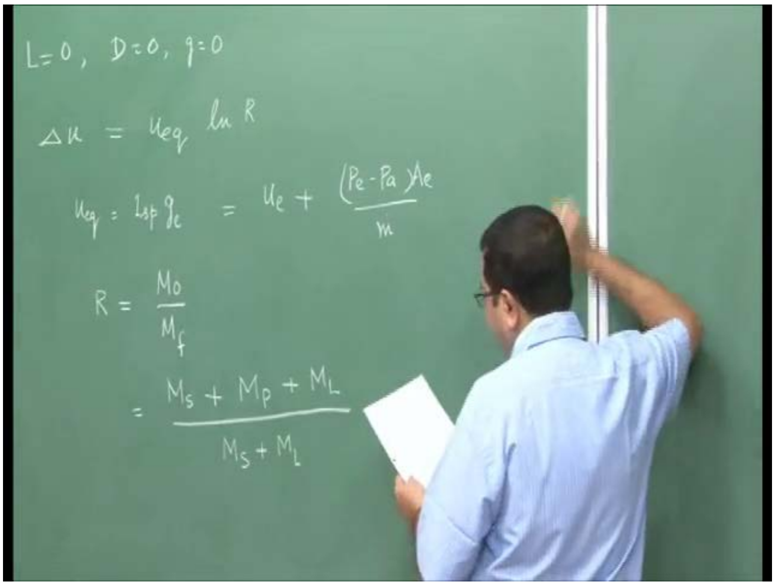

First and foremost the we will consider here also the case where lift is 0, drag is 0, and no gravity, we will consider this case, because this gives us an expression for the velocity change independent of time. Otherwise, time comes as a factor and then we have to integrate the expression to get the velocity field. So, this is the simplest one and it is typically applicable to outer space where there are no drag or gravity.

So, this gives us good starting point for analyzing the performance of a rocket, now for this case we had derived that the velocity increment delta u for the rocket vehicle is given as equivalent velocity of the exhaust times l n R; where the equivalent velocity is the specific impulse times acceleration due to gravity or this is equal to the actual exhaust velocity plus the pressure term given by this.

So, this we have proved and R is nothing but the initial mass divided by the final mass of the rocket vehicle and initial mass includes the structural mass M S, the propellant mass M P, and the payload mass M L. And the final mass will be the propellant will be used up as the rocket operates, so therefore only the structural mass and the payload mass will remain. So, therefore the final mass will be M S plus M L, this is something that we had already discussed so far. Now, what we will do is apart from this we had derived, first defined and then derived expressions for some more factors.

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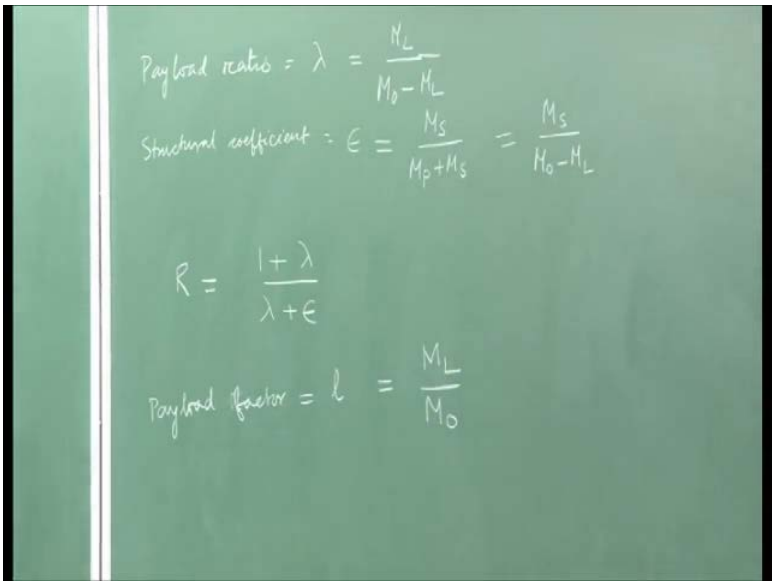

One of them was payload ratio, which is designated by lambda, this is defined as the payload mass divided by all the mass except the payload, that is how we define the payload ratio. And also we have discussed or defined structural coefficient designated by epsilon, this is equal to the structural mass divided by the sum of propellant mass and structural mass essentially, the ((Refer Time: 04:00)) here also means all the mass except the payload.

So, this is also equal to M S upon M naught minus M L, this we had defined where the structural mass will include the mass of the engine, the tank, the other control hardware everything. Now, after defining this two we had proved that R is equal to 1 plus lambda upon lambda plus epsilon, this is what we had proved; before progressing further proceeding further, let me define another non dimensional parameter, which is payload ratio, sorry this is payload factor not ratio.

This is payload ratio payload factor is given by L, which is defined as the mass of the payload divided by the total mass of the vehicle, so L is equal to M L upon M naught. Now, first what we will try to do is first let us get a relationship for R L and epsilon, because this is something we will be using for multi stage rockets. So, first what we will do is try to get an expression for this three, so first of all let us look back at the definition of our structural coefficient epsilon.

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$$\epsilon = \frac{M_S}{M_S+M_P} = \frac{M_S}{(M_0-M_L)}$$

$$M_S = \epsilon\left(M_0-M_L\right)$$

$$M_S+M_P = M_0-M_L$$

$$M_f = M_0-M_P = M_S+\cancel{M_P}+M_L-\cancel{M_P}$$

$$M_f = M_S+M_L = \epsilon(M_0-M_L)+M_L$$

$$= \epsilon M_0-\epsilon M_L+M_L$$

$$= \epsilon M_0+(1-\epsilon)M_L$$

Structural coefficient epsilon is defined as I have written there M S upon M S plus M P and M S plus M P, M S is the structural mass, M P is the propellant mass, this is equal to nothing but, the total mass of the vehicle minus the payload mass. So, therefore epsilon is equal to M S upon M naught minus M L. So, from this expression now we can get an expression for the structural mass in terms of epsilon, then the initial mass and the payload mass, we can write M S equal to epsilon M naught minus M L.

Now, from here what we are seeing is that M S plus M P is equal to M naught minus M L this is straight forward from here. Now, let us look at the final mass M F, what is the final mass of the vehicle? It is the initial mass minus the propellant mass M P, initial mass is the sum of all three that M S plus M P plus M L minus M P, then gives us the final mass or burn out mass.

So, you can see that M P can be cancelled of therefore, the final mass M F equal to M S plus M L, now in this equation if I replace this M S which we have derived here expression for M S equal to epsilon M naught minus M L, then we can write this as epsilon M naught minus M L plus M L. Let us know expand this, so this will be equal to epsilon M naught minus epsilon M L plus M L, we can very easily write it like this and now this we can write as epsilon M naught plus 1 minus epsilon M L.

So, M F now can be written in terms of the initial mass and the payload mass and the structural coefficient like this M F equal to epsilon M naught plus 1 minus epsilon M L. Now, let us now go back to our definition of R the way we had defined the R if you remember is the inverse of the mass fraction.

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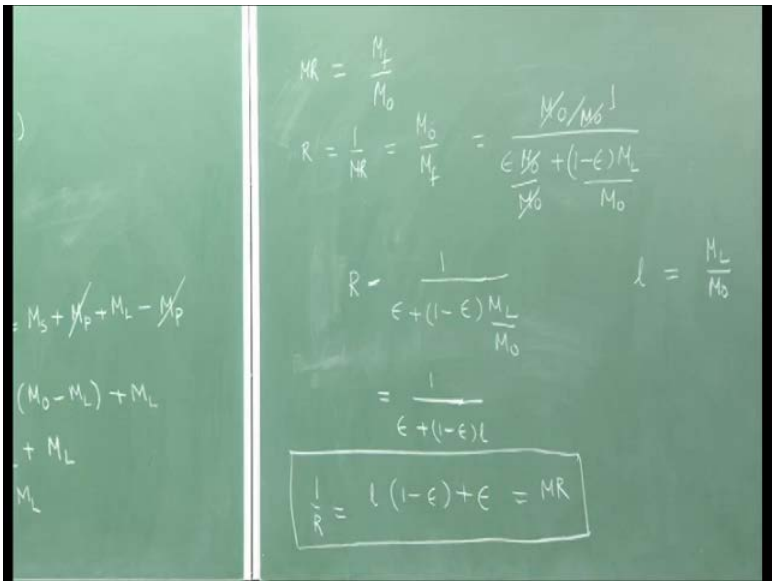

So, initially what we had done is we had defined the mass fractional, mass ratio as M F upon M naught, this is initially what we have defined the mass ratio as, so mass ratio is the final mass divided by the initial mass. Then, what we have said is R is inverse of mass ratio, why we have done this, because this is the term that was appearing for our expression for delta u and with a minus sign, so when we defined R the minus sign went away.

So, therefore we have defined R like this, so this is equal to M naught upon M F, now what we do is we take this expression for M F and put into this equation so this is equal to the M naught upon epsilon M naught plus 1 minus epsilon M L. So, essentially we have just replaced M F by the expression that we have derived there. Next, what we will do is let us divide both the numerator and denominator by total mass M naught, then as we can see here this cancels of we get 1, this cancels of…

So, our R is equal to 1 upon epsilon 1 plus epsilon plus 1 minus epsilon M L upon M naught and we had defined the payload fraction M L as equal to M L upon M naught is our payload fraction. So, we can put it here so our R is equal to 1 upon epsilon plus 1 minus epsilon times L, so the expression for mass ratio is 1 by R which is equal to L 1 minus epsilon, times epsilon which is our mass ratio.

So, this is an expression that will be helpful in optimization of a multi stage rocket as well it is applicable to a single stage rocket, if you recall in the last class we have solved a problem where the payload mass was given delta V was given we wanted to estimate the initial mass M naught and we used this relationship actually to get that value. So, these are the equations, then that we will be needing to solve for a multi stage rocket with this background. Let us now, come back to a multi stage rocket, so we are done with the discussion of single stage rocket.

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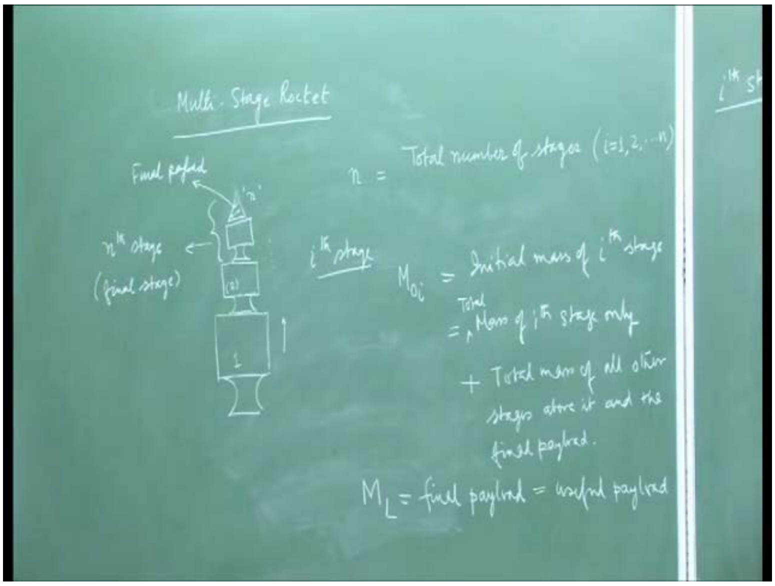

n = Total number of stages $(i=1,2,\dots n)$

M_{0i} = Initial mass of i^{th} stage

= Total Mass of i^{th} stage only

\+ Total mass of all other stages above it and the final payload.

M_L = final payload = useful payload

Let us, now talk about a multi stage rocket; first let us look at the nomenclature what are the terms that we will be using for a multi stage rocket so for, that first let us look at the nomenclature of the rocket system, this is the schematic of a multi stage rocket where we have many stages stacked on top of each other like this. So, let us say that n, the small n is the total number of stages. So, which means that, we have many stages starting from i equal to 1, then 2 like that to n.

So, if I look at this picture this is my first stage, so i is equal to 1 and then we go up towards the payload which is my second stage like that finally, we come to n th stage. So, if total number of stages is n then we have i equal to 1 to n representing different stages. So, here this is these two together that is this part and this what I have shown here a little triangle is our final payload, that needs to be delivered and then this stage which is the payload plus the rocket motor is our n th stage or the final stage.

And then what we have is different stages stacked on top of each other like this, now let us consider any stage I, let us consider any i th stage between i equal to 1 to n, we are considering any stage then the initial mass of that stage is represented by M naught i where M naught is our initial mass as far as our nomenclature. So, far and I represents the stage we are talking about so M naught i is the initial mass of the i th stage.

So, this is initial mass of i th stage which essentially means the if I consider just that stage which essentially means, if I consider just that stage and forge the rest, then M naught i is the initial mass of that i th stage, plus all the mass of every other stage and the payload. So, this is initial mass of the i th stage which is equal to mass of i th stage only or let me put it total mass total mass of i th stage only plus total mass of all other stages above it and the final payload.

So, why we write it like this that this is the initial mass of the i th stage, so the i th stage actually has to lift the entire mass which is the mass of that stage alone plus the mass of all other stages and the payload i th stage has to move all of these. So, that is how we define M naught I which is the initial mass of i th stage, then let us look at M L, M L is the final payload or the useful payload to be delivered. So, this is the final payload or useful payload that needs to be delivered to the required for the required mission, so, this is represented by M L.

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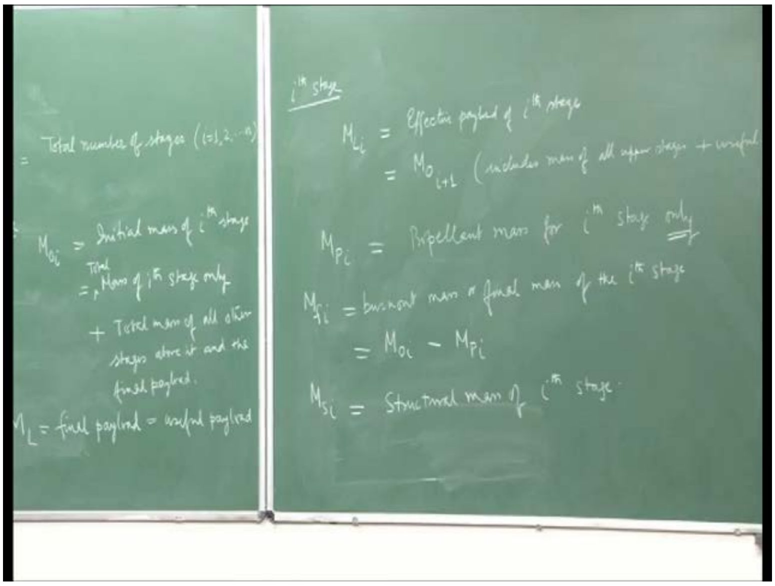

= Total number of stages $(i=1,2,\cdots n)$

M_{0_i} = Initial mass of i^{th} stage
= Total Mass of i^{th} stage only
+ Total mass of all other stages above it and the final payload.

M_L = final payload = useful payload

i^{th} stage

M_{L_i} = Effective payload of i^{th} stage
= $M_{0_{i+1}}$ (includes mass of all upper stages + useful

M_{P_i} = Propellant mass for i^{th} stage only

M_{f_i} = burnout mass or final mass of the i^{th} stage
= $M_{0_i} - M_{P_i}$

M_{S_i} = Structural mass of i^{th} stage

Now, if I look at again the i th stage, if I look back at the i th stage then once again go back to i th stage. Let me, define the effective payload of the i th stage given by M L i our nomenclature says M L is the payload mass, i represents the i th stage. So, therefore, this is the effective payload of i th stage, now what is the effective payload of i th stage, if I look back it is a first stage.

So, i equal to 1 this portion here is the structural mass propellant mass which essentially burns out and then discards, this portion has to be delivered by this so everything above is has to be delivered by first stage to certain up to a certain altitude with certain speed. So, therefore, this portion here which is excluding the structural mass and propellant mass of first stage happens to be the payload for this stage.

So, the payload for i th stage is the all the initial mass of i th stage minus the structural mass of that stage and the propellant mass of that stage. Now, when I come to this stage then means after first when I go to the other portion, then for the second stage this is essentially the initial mass. So, therefore, the payload for first stage is initial mass for the next stage. So, therefore effective payload of i th stage is equal to M naught initial mass of i plus 1 th stage.

So, as usual this includes mass of all upper stages plus useful load, so therefore, the effective payload for i th stage is the initial mass for the stage just ahead of it that is I plus 1 th stage. Let me, also write the M P i, M P i is the propellant mass for i th stage only is the catch phrase here remember that the propellant is distributed among all the stages some propellant is in one some propellant in two like that.

So, M P i is the propellant mass in the i th stage only not all the propellant mass, now apart from that we need to have some more definitions one is M F i, as we have defined F as the final mass or the burn out mass. So, therefore, M F i is the burn out mass or the final mass of the i th stage. So, after the i th stage burn out where all the propellant in i th stage has been consumed the remaining mass is the final mass of that stage, but remember one thing that this includes the structural mass of i th stage also.

Because, we have still not discarded the structure I will come to the why multi stage is required and what is the advantage. So, this is essentially is after the burn out or propellant of propellant present in the i th stage has been used out. So, then going back to our definitions of the initial mass and the propellant mass, the final mass is then the initial mass of i th stage minus the propellant mass of i th stage.

This is our definition we have been following and one more mass we have to define is the structural mass, so M S i represents the structural mass of i th stage, so M S i then represents the structural mass of i th stage. So, with this we have defined all the relevant masses with which we are going to work and like the single stage rocket we have seen that the mass ratios are the most important not the most important, but one of the important parameters in dictating the velocity increment, the other important parameter is the equivalent velocity.

So, now since we are talking about different stages, every one of this stage can have different equivalent velocity of its own or all of them can have same equivalent velocity, it depends on the propellant choice as we have seen. So, it depends on I S P if you are using the same propellant in every stage, the I S P will be same, so the equivalent velocity will be same for all the stages or if you are using different propellant. Then, the I S P is going to be different, so equivalent velocity is going to be different, so we will not comment on this right now.

So, now we have defined all the masses that are relevant, next let us see that why do we want to go for multi staging what are the advantages that we gain by going for multi staging and let me talk about the multi stage performance.

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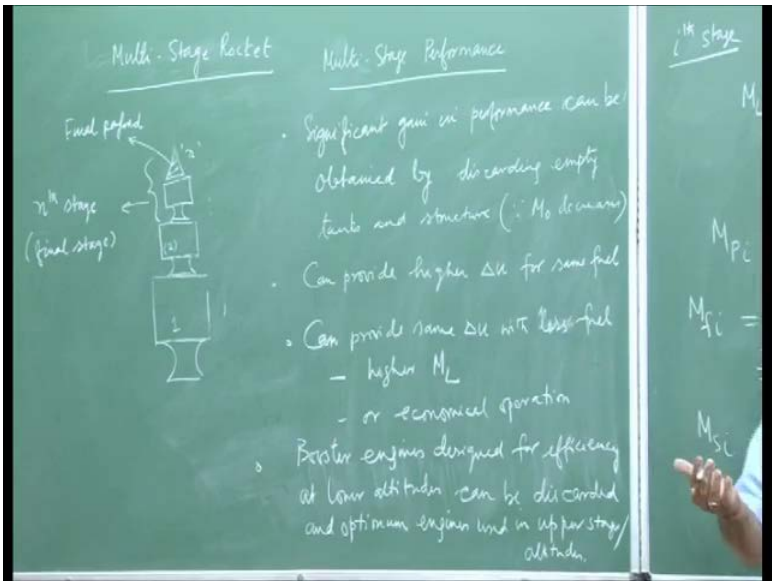

Multi-Stage Rocket

Multi-Stage Performance

- Significant gain in performance can be obtained by discarding empty tanks and structure ($\therefore M_0$ decreases)
- Can provide higher Δu for same fuel
- Can provide same Δu with less fuel
  - higher M_L
  - or economical operation
- Booster engines designed for efficiency at lower altitudes can be discarded and optimum engines used in upper stages/altitudes.

i^{th} stage

Now, what is exactly is multi staging? Multi staging is we as I have shown here we design the rocket consisting of various separate rocket motors and a payload. Now, what happens as we have seen that the overall velocity gain that we get depends on the mass that it needs to carry, now if it was the same single stage like we have been discussing so far. Then, all through the burn out it is carrying the entire structural load and then it finally, delivers the payload to the orbit or whatever the destination.

So, the entire engine mass and tank mass has to be carried all the time, essentially what it means is that you required to have a higher structural coefficient and we have discussed that for the single stage performance higher structural coefficient will reduce your velocity increment. Because, we are carrying lot of weight which is essentially a dead weight, it is not taking part in the operation, so instead of doing that if we break it up into multiple smaller modules.

First, let us look at the first stage we essentially what we do is because the velocity anyway is not going to increase in a jump, it is going to increase continuously. So, we can design the system in such a way that after every let us say x velocity increment or delta u velocity increment we reduce the mass somehow. Then, for the rest of the flight path it has to carry less weight or less mass.

So, it will due to higher velocity because it is not carrying all the load like that if we split the mission into smaller modules then finally, since it is not carrying all the dead weight all the time it will move to a higher distance or a longer distance that is the primary idea of multi stage rocket.

So, now if I look at this the first stage of course, takes off with all the things after the first stage is burned out all the fuel is consumed what we do is we discard the first stage immediately, the weight of the rocket motor the tank required for the first stage the engine everything is reduced. So, the structural mass for the remaining stage comes down so that is the overall, now this portion is moving like a fresh new rocket, so it has its own initial mass which is less than the mass it had before.

Now, once again the second stage is burnt out we allow this to separate because this finally what is going on so we allow this to separate, so once again there is a reduction in overall mass, when we come to the third stage and so on. So, therefore in every stage separation, there is a reduction in mass that the for this the upper stages need to carry because of that, if I look at the total picture when we come to the last stage which actually contains the actual payload.

The last stage is not carrying all the weight that will come with the previous stages all the dead weight of engine and tank etcetera, this is not carrying that it is carrying only its own propellant and the payload. Now, when this propellant is burnt and by the by that time it reaches here, we have already reached relatively high velocity, so as we have seen that we are interested in delta v.

So, if you start with the higher base and now delta v is equal to u equivalent l n R, so u equivalent let us say same with relatively smaller value of R, we can get high value of delta u. So, starting from the ground to the final mission we are getting a very large velocity increment, but the velocity increment for this stage is not required to be very high.

So, therefore a small velocity increment here will give us the required performance parameter and then the payload can be delivered into the required location. So, that is the biggest advantage of multi stage that, we can go to higher distances by actually using up equal or may be less amount of fuel. This total propellant, if I see distributed among all of them may be less than the single stage to give the same performance.

And now if you are carrying less amount of fuel what we can do is we can put more payload, so the payload carrying capability can be higher. So, it is a more economical or efficient operation that is why multi staging is something that is particularly, when we are talking about long missions like say we have to put a satellite in a polar orbit or a orbit you have to have multi stage rockets.

Single stage rockets are essentially used as sounding rockets where you go to pretty low orbit, but if you go to high orbit like low earth orbit or high earth orbit, you need to have multi stage rockets to get the velocity increment that we need. Now, what are the significant performance indices, first of all by carrying out this process where at the end of every burn out, we discard the dead weight, structural weight, we can get significant gain in performance can be obtained by discarding empty tanks and structure, because the overall mass decreases because this is no longer being carried by the rocket.

So, the significant gain in performance can be achieved by doing this, second point is it can provide higher delta u for same amount of fuel. So, now if I looking at particular rocket and let us say make it a multi stage rocket, then it will give us carrying a same amount of fuel higher delta u, on the other hand if that is the case then can provide same delta u with lower fuel or less fuel rather not lower less fuel.

So, therefore the multi stage can provide same delta u with less amount of fuel which essentially means provides higher M L payload carrying capability or more economical operation or economical operation. The operating cost will be less if we have to put the same payload compare a single stage and a multi stage rocket, the operational cost will be less for a multi stage rocket compare to a single stage rocket.

Now, these are some of the points the next point is we can now since we are doing multi staging, we have discussed one thing that the performance depends on your equivalent velocity right and equivalent velocity is a function of atmospheric pressure and the exit pressure P e minus P a. If, you are considering a same fuel I S P is same the equivalent velocity will be dictated by P e minus P a, so far for a single stage rocket I have discussed that when we are at the ground we have relatively higher pressure as we go up the ambient pressure decreases.

Now, you cannot possibly vary the exit pressure depending on the ambient pressure variation. so what happens is that you design for certain condition and most of your operating operational domain you are actually away from the design pressure, but with multi staging you know that what will be the operating range of each of these engines. So, you can design each engines for that ambient pressure range.

So, that essential and we have seen that if you design the engine such a way that P e is equal to P a your thrust is optimum. So, therefore, designing these rockets for different stages actually will give us the optimum performance, because we design for the given altitude. For example, a booster rocket solid propellant booster rocket they are supposed to be most efficient on ground which we have discussed earlier.

Rather, whereas when we go up we need other types of rockets like liquid propellant ((refer time- 34.43)) etcetera, their efficiency is different at different locations. So, therefore, what we can do is we can have different design for example, the first stage we have step on boosters, then the first stage may be a solid propellant, then we go to the second stage which may be a liquid propellant like that we can choose different I S Ps.

This is another advantage that we can choose different I S P for different stages and we can get the optimum performance by distributing the propellant as well as having different equivalent velocity for different stages which are optimum. Because, we are designing it for the given altitude, so we can get the given pressure. So, therefore, I can say that booster engines designed for efficiency at lower altitudes can be discarded and optimum engines used in upper stages slash altitudes.

So, we have discussed the solid propellant rockets typically which are used as step on boosters, they are efficient at lower altitudes, so we can have booster engines at the lower altitudes, another point like to point out here we have talked about one more parameter which is the burning time. If, the burning time is longer we can get lower delta v that is burning time is shorter we can get higher delta v, but acceleration will be higher, so there is an optimum burning time for a particular rocket.

So, let us say a booster if you allow it to burn for a long duration a slow burning velocity increment is going to be low, but we cannot allow it to burn very fast as well accelerations will be so high that, there will be tremendous load on the equipment and everything. So, let us say 10 second, 15 seconds burning in booster we attain the required delta v, and then we discard the boosters, now we go to the other stages burn it for longer duration because now when as we are going up and up we do not need very high delta u.

Because, we are already starting with a higher base we already have achieved a certain velocity we have to increase from there we are not starting from zero, so now in the later stages we can give longer burning time and attain the given mission requirement. So, that is why we can also have different burning time for different stages and we can optimize this burning in such a way, that we get the best performance which means either the highest possible payload or the most economical operation.

By essentially, tailoring the burning time for all the stages and the last point or the last advantage that we can talk about is once again essentially the coming back to this since, the boosters are very, very heavy why because at the beginning you have lot of weight right it is the whole vehicle weight with all the stages we have to lift. So, by far the heaviest stages are the boosters and the lower stages.

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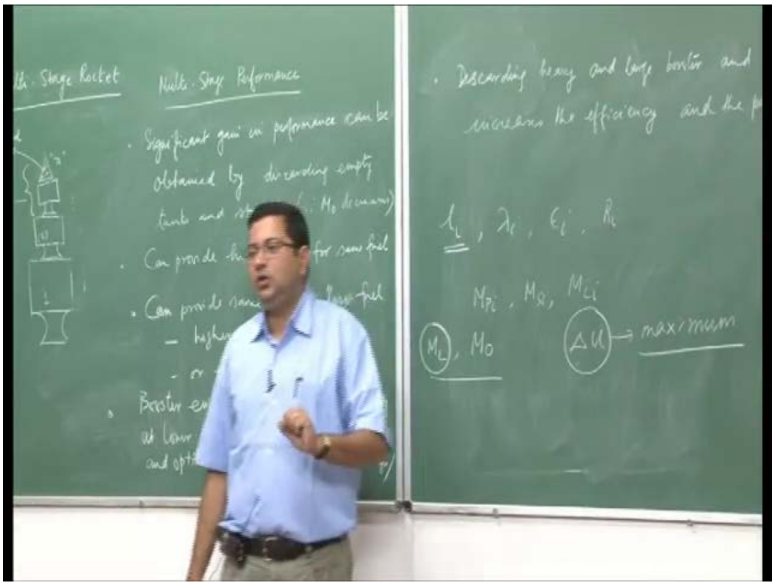

So, this discarding this heavy and large boosters and lower stages now, when we are inside the earth atmosphere the drag is also higher. So, you have to produce more thrust in order to produce more thrust, you have to burn at a very high speed high rate and produce high thrust. So, therefore you need to have pretty heavy boosters which will burn at a high speed and produce the required velocity, but as we go up high in the altitude we do not need so much of thrust.

Now, carrying this entire booster and the heavy engine the lower stage engine adds to the weight. So, therefore discarding this heavy large booster and the lower stages which are essential on the ground to provide the initial lift of the initial velocity gain is essentially going to improve the efficiency, so lower stages increases the efficiency of the entire rocket and the payload ratio lambda.

So, discarding this heavy boosters essentially is going to improve the payload ratio how much payload we can carry improves the efficiency, another point also is that the structural load is also going to be different at different operational speed because now all your vehicle is moving at different operational speed operating at different operational speed working for different time, so acceleration is different for different vehicle.

So, structural load is also going to be different, we can distribute the structural coefficient also in the manner such that the optimum performance is achieved. So, now what we have is we have l i which is the payload ratio for the payload fraction for all the stages, we have lambda i which is the payload ratio for all the stages, we had structural coefficient for all the stages and we have R i for all the stages.

Now, based on this we can also find out how much will be the propellant mass for every stage, how much will be the structural mass for every stage and how much will be the payload mass for every stage. So, if we are given the final payload to be delivered and the initial mass that is there for the rocket, we can find out what should be the distribution of this mass, the total mass among different stages. So, that we can get the best performance for example, what is the best performance as far as this rocket performance concerned the best increment in velocity.

So, we can find out how we will we should distribute this masses among all the stages, so that our delta u is maximum we get maximum velocity or we can repose the problem in such a way that let us say at the end of the operation we want to get certain delta u certain value of delta u because that is the mission requirement. We know that if we have to put a satellite in orbit that satellite should be delivered into the orbit with certain velocity and altitude.

So, let us say this is specified if this is specified and some of this parameters we know we can find out what is the maximum payload that can be delivered with this velocity to the required orbit starting from the initial mass M naught. So, therefore now this becomes important because this information the rocket designer can provide to the satellite designer that your satellite weight should not be more than this, it should be within this or if the satellite is already predesigned, it has to be delivered to certain orbit.

Then, the rocket designer has to play with these parameters M Pi, M S i, M L i etcetera, to find out what whether the mission is possible or not and then if it is not possible with the given fuel, we can change the fuel to get different I S P and redo the calculations to find out with a different type of rocket can we achieve this or not. For example, P S L V is a four stage rocket, G S L V is also a four stage rocket, but P S L V cannot deliver a satellite to geosynchronous orbit.

Whereas, the G S L V can why because G S L V different stages have different configuration, it is have different specific impulse different propellant mass. So, it is for a different mission at the same time if I compare S L V and P S L V they are also different. So, even though S L V I think it is a three stage rocket, so P S L V and S L V are also different.

So, every stage has different burn out time has different propellant mass has different I S P and that is why for a particular mission we choose a particular rocket because these distributions are specified. So, that the rocket designer knows what is the optimum performance can be obtained from a given rocket. Now, I have said during the history part that earlier what Russians did Russians started to convert I C B Ms into rocket launchers.

So, that is exactly what they did they had the I C B M they can do this calculations and find out if they are launched vertically how far they will go and from there they identified the orbit at which they can be launched. So, essentially this multi stage performance is very, very important in determining the performance of the rocket because now a days we very rarely have single stage rockets single stage rockets are essentially used for short range missiles most of the rockets are multi stage rockets.

So, with this we come to the end of this lecture, in the next lecture now so we are saying that this parameters essentially dictates the performance, now we will see how this parameters are related, so that we can identify the performance of the parameters of the rocket. So what we will do next is the performance calculation for a multi stage rocket, once we do that, then the next will be the optimization like I have been talking about that we can have different distribution, so that we can optimize the performance of the rocket.

Thank you.