Foundation of Scientific Computing

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Module No. # 01

Lecture No. # 07

Today, in the lecture 7, we will be starting with discretizatrion of the Laplace equation and we will be talking about various ways of solving these elliptic partial differential equations.

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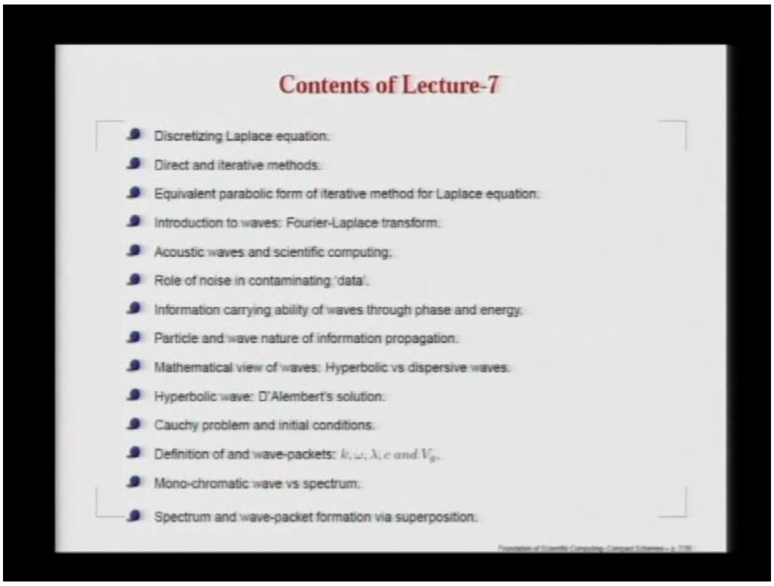

The two major categories are direct and iterative methods. We will note that if we are resorting to iterative methods, although the governing equation is elliptic, we convert to an equivalent parabolic form. That should tell us that numerically, all of us solve the problems which are somewhat different than their physical counterpart.

In this context, we will start our main journey in computing by introducing waves. To understand this wave, we need to understand it in terms of Fourier-Laplace transform.

We will give you an example of waves by the classical solutions provided by D'Alembert on acoustic waves and how this is related to scientific computing; we will emphasize on that aspect. We will find out that in capturing the waves, we need to worry about the noise or the error, because if they are present they are going to contaminate our signal. At times, in many problems, what we notice is that this signal or the noise is of same string; so, it is very essential that we understand the role of the noise.

Then, talking about propagation of signal by waves, we need to find out how really this information is propagated; we will identify that in the waves. The signal is transmitted through its phase and energy that brings us to this topic of dual nature of particle and wave aspect of information propagation. This will be followed by a mathematical description of waves and we would note that there are two different classes - major classification. One corresponds to hyperbolic partial differential equation. We have already talked about what constitutes hyperbolic problem, namely, the existence of real characteristics. In contrast, we can also have waves for governing equation. It could be either parabolic or elliptic or it could be even a problem which is governed by simple steady state equation. We will find out that those systems support waves and those waves are called the dispersive waves.

As an example of hyperbolic wave, we will talk about the D'Alembert solution of second order by directional wave equation. We talk about the Cauchy problem and initial conditions required to solve the Cauchy problem. We talk about the definition of these wave packets. Wave packets are nothing but interacting waves, which are characterized by the parameters listed here as the wave number k, the circular frequency omega, the wave length lambda, a phase speed c and the group velocity at which the energy propagates - that is given by Vg (Refer Slide Time: 03:48).

Whenever, we talk about wave propagation, we will understand that we could have the waves characterized by a single wave number of frequency; that is what we call here as the mono-chromatic wave. This would be contrasted with polychromatic waves which will have its distinct spectrum; that is what we will be talking here. Once we know what the spectrum is, we talk about how wave packet essentially forms as interaction of the constituents of the spectrum.

So, shall we start? See in the last class, we were classifying the partial differential equations with the idea that we should be able to get some generic rules for different types of equations, because they share some common properties. We classified those equations in terms of parabolic, elliptic and hyperbolic equations. Towards the end, I warned you that such classifications you obtained by mathematical tools may not mean very much when you come to compute. As an example, we discussed the heat equation and then we noted that it was a parabolic equation, but doing it or solving it explicitly, we noted that we oppose the problem not as a parabolic, but as some equivalent hyperbolic equation.

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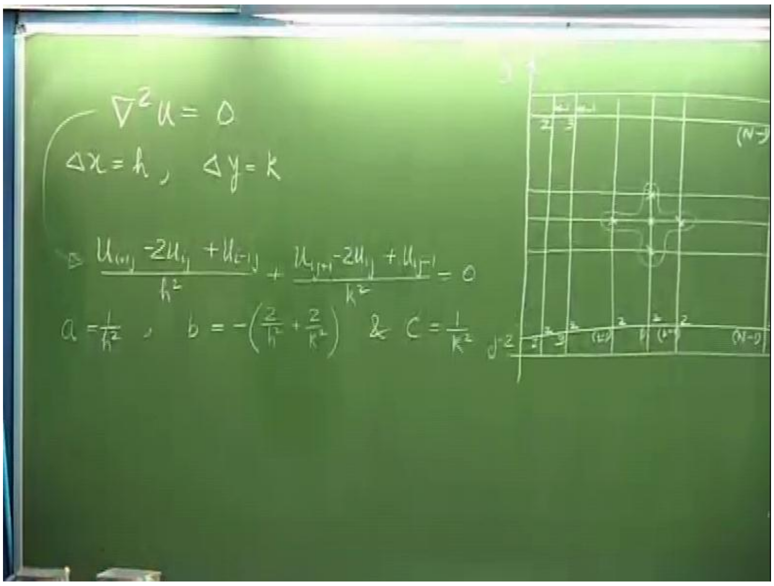

Then, we started talking about another example; that is why we stopped; that was this problem. We tried to solve this Laplace’s equation and I would not do it, but you can do it; comeback and tell me that this is analytic equation (Refer Slide Time: 05:47). So, analytic equation means, it has to be solved as a boundary value problem. I am going to come to elliptic equation much later, but let me remain within the theme of what we are trying to do. It is to discuss that irrespective of the classification of this equation that is, an elliptic equation, how we go about solving it?

So, what I have shown you here? Try to solve that problem in a rectangular domain with equi space point in the x direction, we have h and y direction; we have k as the spacing of the points. Then, write it in a Cartesian frame and using uniform grid we have seen that the secondary derivative could be written like this (Refer Slide Time: 06:40). We have talked about it earlier. Now, for the sake of simplification, let us define 1 over h square as a, b as minus of 2 over h square plus 2 over k square and c as 1 over k square.

Then, let us try to see how we start the unknowns. It is a boundary value problem, so we would be prescribing boundary condition. Let us say on all 4 segments of the boundary. Suppose, I start the points in what is called as a Lexicographic fashion; what it means is that you follow a structure in the way you define the points.

For example, I will start the unknowns like this. This is the first part, this is the second. So, what you do? You go from left to right, then bottom to top; that is the sequence, what we call as a Lexicographic sequence of starting points. Having done that what you could do is, you would write down this equation for each one of those unknown points. That would give you a set of coupled equation, which I have written it like this (Refer Slide Time: 08:17).

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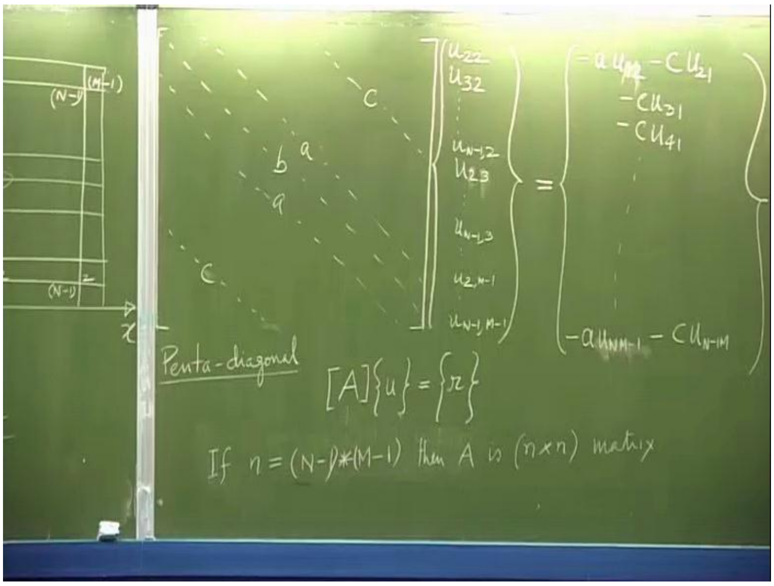

Please note the sequence U22, U33, U32, all the way up to this point U n minus 12. Then, you go over to the next line which I have not shown, that could be like this. So, you start with 23, 33 all the way up to n minus 1. Finally, go to the last line that will start with U2 m minus 1 all the way up to n minus 1 m minus 1. So, this is your unknown column vector which I am writing as U and these are those coefficients in these difference equations.

For example, the way we write it what will happen? This i j would go along the diagonal. The coefficient of the diagonal is given here by b - along the diagonal you will get b. What you notice that a point which is to the right of the point in question that would be given by i plus 1j, so that as a coefficients 1 over a square, that is what you get. The point that is to the left of that point that is also a, so that is what you get.

So, you get a kind of a tri-diagonal band here with the diagonal b. This super and sub diagonal are a (Refer Slide Time: 09:43). In addition, you notice that if I am looking at i j, then i j plus 1, this point has a coefficient of 1 over k square that is our c. So, we just move a pitch of n minus 1point, because of n minus 1 I will go back to the next thing. That is your pitch right, from 2 to n minus 1. So, what happens is, this distance is n minus 1 and that is where you get another diagonal element that is your c. You also notice that there is a point which is 1 pitch below i j minus 1; that would also be n minus 1 on this side.

This is a very rudimentary of this matrix, which I may call as a. This is called as penta-diagonal, but be aware that what we mean by penta-diagonal matrix; we may have a much better structure than this. All the elements would be side by side nuts stacked a part like this two sort of diagonal slices along lines, which are n minus 1 line apart; that is not strictly a penta-diagonal structure (Refer Slide Time: 10:40). Penta-diagonal structure would probably mean that all of these are together.

Now, as you can see that when I am writing this equation for this point, I would have this point, this point, this point and this point. These two points belong to the boundary, they contribute to these two terms (Refer Slide Time: 11:20). So, you can see that you do have a homogeneous equation, but its linear algebraic form comes out as homogeneous term and this right hand side is contributed by the boundary condition; that is what you do.

If the number of points is, let us say n minus 1 in i direction and m minus 1 in the j direction. Now, I can define n as the total number of unknown points that is n minus 1 into m minus 1 and then this size of the e matrix is n by n. So, I am trying to bring this thing to your attention that when you try to solve such a problem even by the simplest possible means of Gauss Elimination, which you may have come across before. How much operation would we require? All of you know that it is order of their n cube. So, if this is n by n that is roughly of the order of n cube (Refer Slide Time: 12:30). You think of the following, say n is 101 M, N. So, N becomes 10 to the power 4 and we are talking about 10 to the power 12 operations.

You can realize that even such a simple problem is going to be quite a bit of computations. So, what is usually done is you do not try to solve this equation like this. We do not try to solve this A inverse r; so, that is what your Gaussian Elimination tries to do. It basically gets you the inverse of the matrix, but as I told you that inversion operation or the elimination operation is proportional to n cube and that is a very arduous task, we do not like to do that (Refer Slide Time: 12:56).

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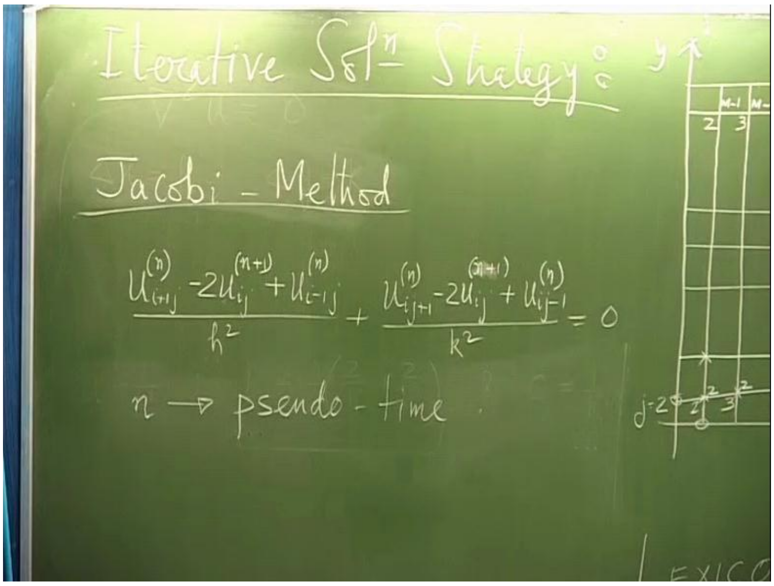

Instead, what is done historically, even today people prefer to do it in a sort of an iterative manner. What we try to do is not direct solution, but we take some iterative solution strategy. So, what we do? We make some kind of initial guess, which could be even the trivial solution, which you want to do. Then you try to figure out the next level by indentifying these two at n plus 1th level, while these ones are kept as the previous level. This is a simplest possible strategy you can adopt; this is what is called as a Jacobi method.

Now, why I am bringing this to an attention is the following. The moment we decide to go over from a direct solution to iterative solution route this n superscript that we are indentifying that I could associated with something pseudo time, it is like your time marching. So, you have marching in n direction, when will you stop? Stop when successive iterates will not change. That would be a wrong thing to do; I will talk about it later.

What we try to do is we will try to show that we will go to a level where we again compute this difference equations, sure that goes to 0. See that many times people make this mistake thinking that if I go from n to n plus 1 and if it does not change, we are converged. This is what convergence means in a crudes possible sense, but that is not correct, because you can adopt a very bad method which shows very slow progression and you come out with a wrong conclusion that I have got my converge solution.

What you should actually look at is the solution error. Solution error is take out the differential equation, write out its difference form and that should be satisfied. See, basically that is your solution strategy. So, you want to ensure that at level your solution has come to some tolerance level. It may not be exactly 0, you may decide to go to say precision 0, single degree precision or you can go to double precision depends on whatever you want to do, but you will have to look at there. Since, one of you asks me to give your practice problems I thought this could be a very nice practice problem for you to do.

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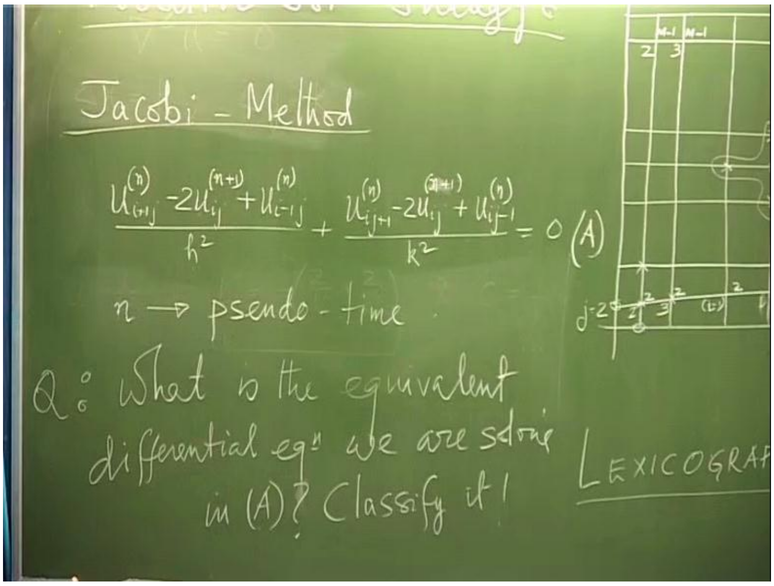

Let me call this as A; second, classify it. Well, I have this habit of announcing questions; I can tell you I will probably ask variation of this in your first method. So, it is your interest, you should try to solve it. I have given you some hint that it will not be elliptic equation; you will have to find out what it is? That will be with the theme of this course, what appears on the surface is not what we are looking for.

Going back to that Harvelle's quotation that I gave you there are issues within issues which we would like to talk about. So, this is one such thing, computation means something which is somewhat different and then what you are taught in it; a focus maths course. So, maths tells you about what is the ideal situation. Here, we are talking about what exactly we do, how it bring that to what we want in a puritanical sense. This is about classification and to keep you aware that in computation things are different than what you actual expect.

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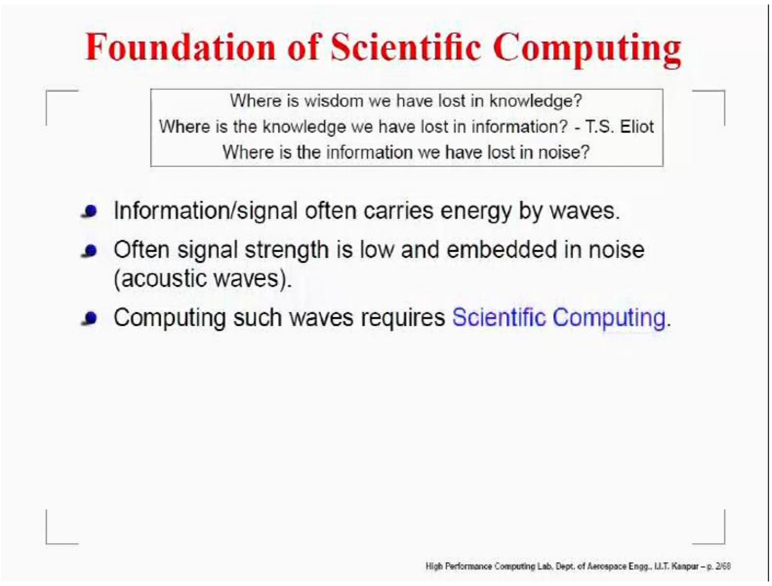

We go over to the next module of our discussion and this is on waves. I find that there are many people not only in this campus all over the world they tend to think that waves are something very special. This is not necessarily true because, information or signal a physical system carries in many cases they can be written explicitly by waves. We probably do not have a very clear definition of what wave is. We can represent those so called waves by in terms of Fourier Transform.

You would note that one of the beauties of Fourier's contribution or Laplace's contribution to mathematic has been able to express any periodic or a periodic functions in terms of Fourier component – Fourier Laplace component. So, what happens? Essentially, those are the photo typical building block of signals, which we call as waves. So, that is why I want to talk about it. Many times, this signal strength is very low they are embedded in noise, like what you have in sound signal. Then you have to capture them, it would be lot of hard work.

Well, we are not talking about the acoustics in auditorium, which is to loud. We are talking about some Spain agency in US trying to find out when a submarine comes out of a Russian port across the Atlantic; so you want to pick up such signals. Sometimes, you would like to be that fancy and it does happen all the time; so we are talking about that. See, one of the issues of some of this wave propagation problems are that you will have to find them out and it is noise.

I just quoted this from my favorite poet, this is very true. What I am trying to talk about here? Let us digress a little bit, going from computing mathematics to little bit of philosophizing. What we try to see in our perceived world we have different hierarchy of observations.

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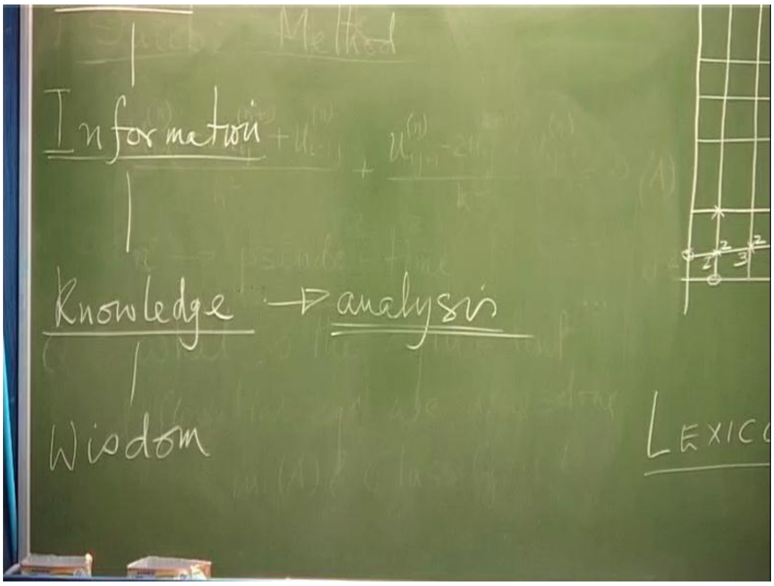

At the lowest level what we have is what we called as the data. If you look around you see that that is your data. Now, what you do? The next step is you try to put them in some kind of a sequence that would be your information. This is a very nice interesting subject that is being talked about in recent times.

Suppose you are in a railway station or airport, there lots of people are talking about; you pick up the acoustic feed. Then you go to a lab and you could hear some 10 people talking simultaneously on different things. Your task is to figure out or convert into 10 coherent stories, 10 groups of people are talking to each other. You basically have collected the data from your feed, now you are trying to put them in order. You try to convert them into a set of 10 coherent stories that is your information. So, it is a ordering of data that takes you to information level.

The next step is the knowledge, you analyze those data. You try to find out that what usually people talk about in this station, in this time of the day. Is there a pattern, you try to generate some kind of; if you are in market research, you try to figure out that. If we try to figure out what people eat by scavenging their garbage these days. So, there are lots of market strategies are there, I hope none of you end up doing that kind of thing.

But any sort of distillation of ideas, why analysis tool takes you to that level? That is essentially analysis is. Once you have analyzed you know at least that topic, this income group people living in this part of the city, eats this kind of food, so we should produce this kind of food more; that is kind of a knowledge you have gone out for your employer. That comes through some kind of an analysis.

So, highest form of all kinds of knowledge is wisdom. So, what is wisdom? I looked at around, when I found out the psychologist have one definition, philosophers have one, but there are no unique definition, but I suppose it is relates to something like synthesis. If you look around your physical world, you have generated lots of knowledge, you have not synthesized all those knowledge and something comes out then you say, if this is the unknown territory, this is the unknown region, I can project that this could happen; those are the things that you go to the wise people. So, wisdom comes there.

Here, actually Eliot asking where is wisdom? That has been lost in knowledge and where is knowledge? That we have lost in information, we are going one level below. There is data and data is contaminated by noise (Refer Slide Time: 24:40).

So, one of the theme in this course is basically to find out how those noise pollutes data and information? From there if you can get something coherent, which I may call as knowledge, it should come out from this course. We basically would like to do that. Now, as I told you that there are no definitions of what really constitutes a wave; this is taken from Whitham books on linear non-linear waves.

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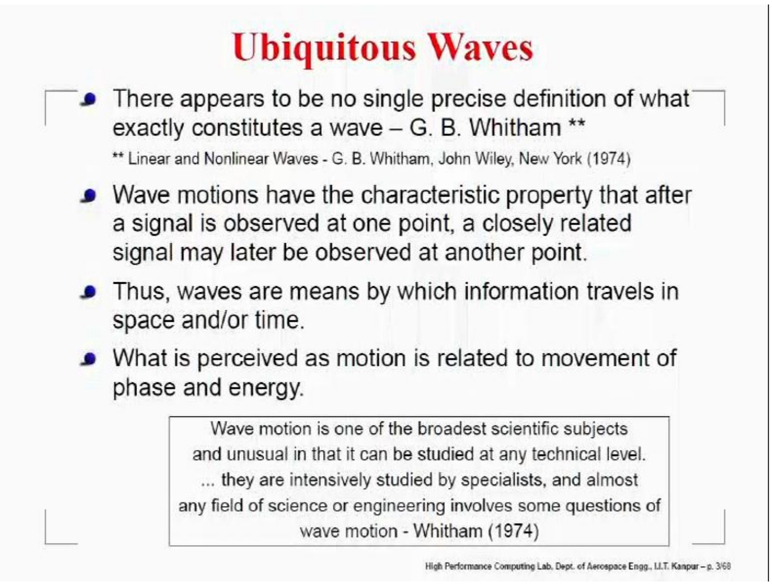

However, to understand what it is we should be able to say that wave motions have the characteristic property. After a signal is observe one point you would see a similar closely related event happening somewhere in the vicinity. So, there is some kind of coherence - correlation between the events happening at one point and its neighborhood. Then, we talk about waves as a mean by which information actually travels in space and in time.

Now, what is perceived as motion is basically related to the movement of phase and energy. Because, all of you know, you have been told many times in your high school event that in many wave motions the particles which do not move at all, they simply carry phase information. In the process, it can also carry energy, which we will be doing shortly. So, look at it, wave motion is really a broad scientific subject, you can tackle it at any technical level. You can study it as a specialist in any field of science and engineering, which involves wave motion.

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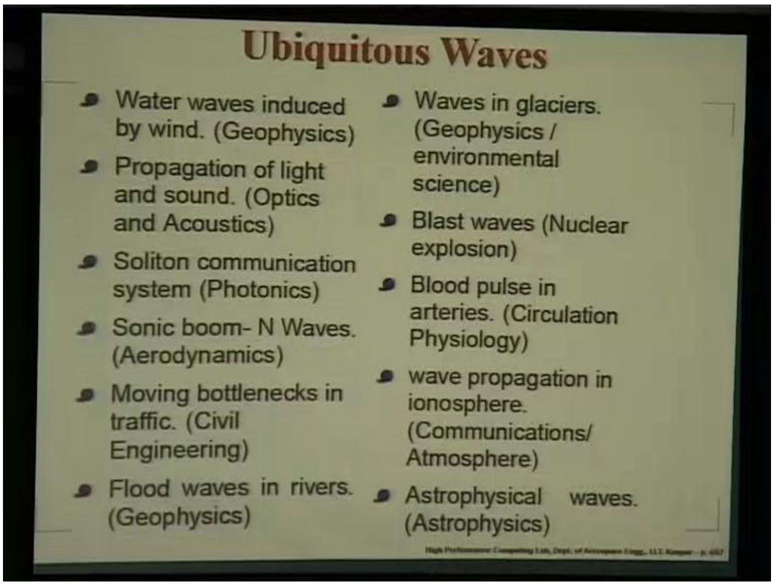

So, where do we see waves? We see waves everywhere, it is ubiquitous. Some of the common occurrences which you have familiar or you have knowledge about it is shown here. On the top, all of us are familiar how water waves are generated by wind, blowing over a confined source of waters and lake or even in ocean. Then, this is something which we have been told many times, propagation of light and sound. They are really nothing but manifestation of motion, although particle and wave duality that debate still continues.

You can also see in modern day devices where you have optical communication in photonics, solitary waves are used to pass on signal. We will be talking about it slightly in this course. People with the aerospace background will tell you about aircraft flying over head at a supersonic speed gives rise to some kind of rattling of the windows, which are associated with the waves. The wave has this signature of this letter N that is why these are called N waves and the phenomenon is called the sonic boom.

Those of you are from civil engineering you should be able to find out that some people have actually really looked at how this bottleneck in traffic congestion moves; it can be posted as a wave problem. Then, there are other things like flood waves, glacier waves, these are something which we have heard about. This is something which we have engineered in this last century, blast waves created by nuclear explosion. But, even if you are looking at the signal, the blood, the way it is passed in your arterial vein, they go as a pulse. In circulation physiology, this is sub studied as a wave phenomenon.

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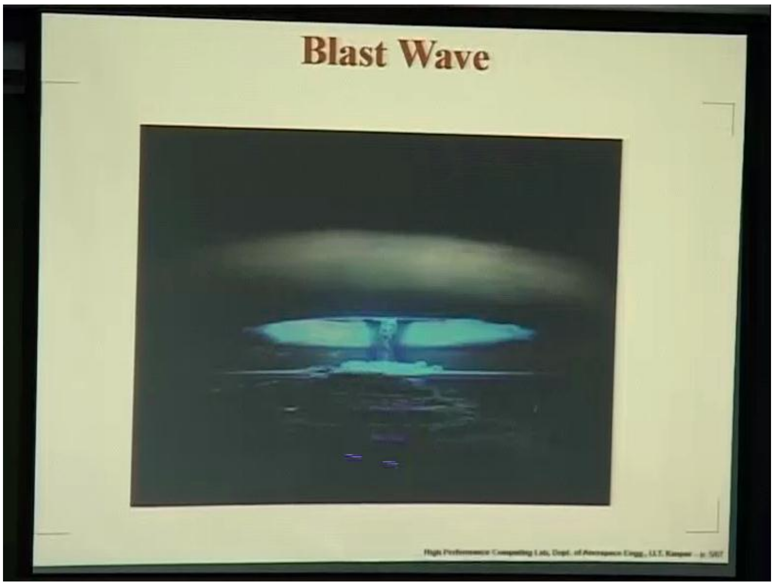

There are other wave propagation phenomena in ionosphere that affects communication system, so we are aware of it; even in astrophysics we do our study. Well, this is an engineered wave of last century blast wave; you can see a wave front, in the shape of a toroidal cloud it is rising above. The steam and the shape is what are called as a mushroom cloud and this is the signature of blast wave.

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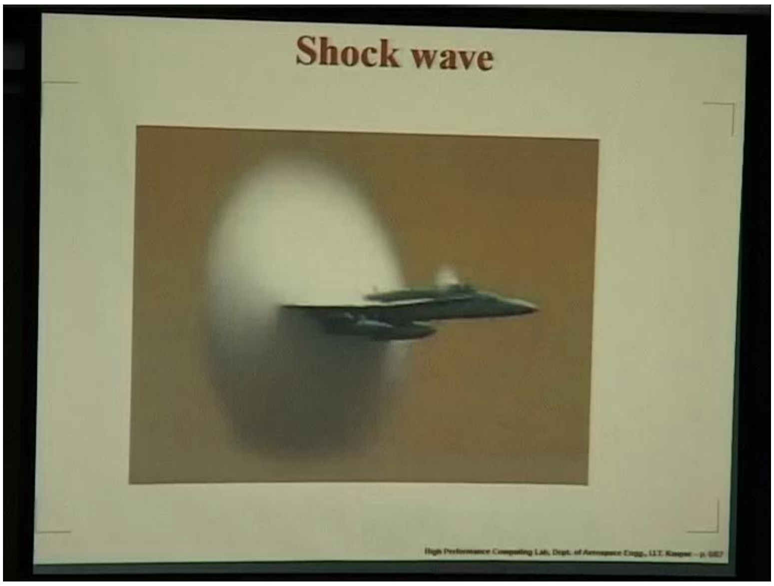

Now, this is something those of you; I am not familiar with aerospace engineering would know that for a long time people had thought that they will never be able to fly at a speed which is greater than the speed of sound. Because of this phenomenon, you see this photograph is taken exactly at the time when the aircraft was transiting through the speed of sound; this is what is called as sound barrier. As if there is a barrier the aircraft has to really go through this phenomenon. This is not condensation or any material thing that you have seen.

What you are seeing here is a basically the change in the optical property. It is not a sort of something that you may have seen some aircraft leaves a trail behind, those are due to condensation of humidity around, but this is not so. This is just due to the change in optical property and this is the major one that has circumvented, there is a smaller one just behind the cockpit of the pilot that also you can see a small wave coming up there. So, these are shockwaves, which we have to really understand in many computational frameworks.

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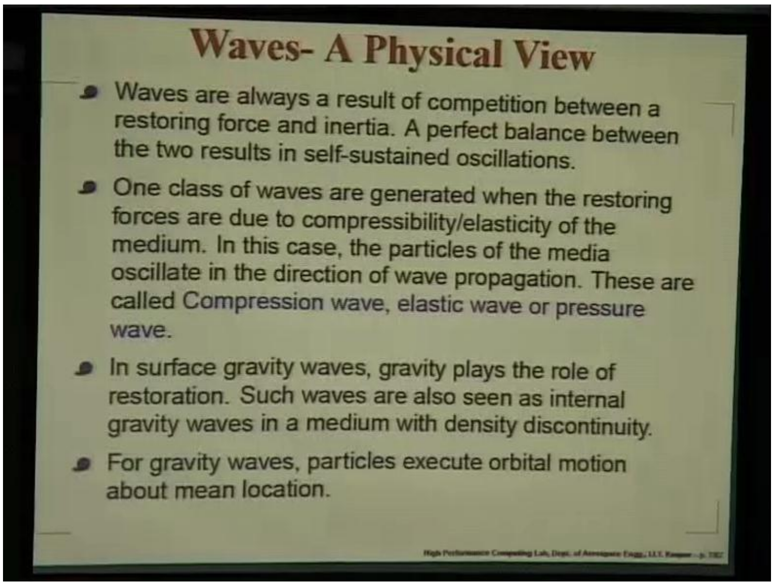

Now, if we look at waves from a physical standpoint, then as we have seen like any problem in dynamics and vibration it is essentially a competition between a restoring force and inertia. When you get a perfect balance then you get self sustained oscillation like swinging of a pendulum; that is why restoring forces is the gravity; inertia, if they coupled together in a perfect balance you will get a steady oscillation of the pendulum.

Now, one class of waves is generated when the restoring forces are due to compressibility or elasticity of the medium. In this case, the particle of the media actually oscillates in the direction of wave propagation. These are called the compression waves, elastic waves or pressure waves. You may have also talked about it as what? Longitudinal waves. You also classify according to whether it is a transverse or a longitudinal waves, this correspondence to a longitudinal wave.

Whereas, the waves those are created in water - body of water those are called the surface gravity wave, their gravity plays the role of restorations. Even you can see such waves in the interior of the adequate. If you have some kind of a density discontinuity those propagate as internal waves. What is interesting is that here the particles actually are neither describing longitudinal motion nor transfers motion, but it is a combination of the two. So, the particles will describe let say, ellipse or circles depending on the depth of the liquid, we will see shortly.

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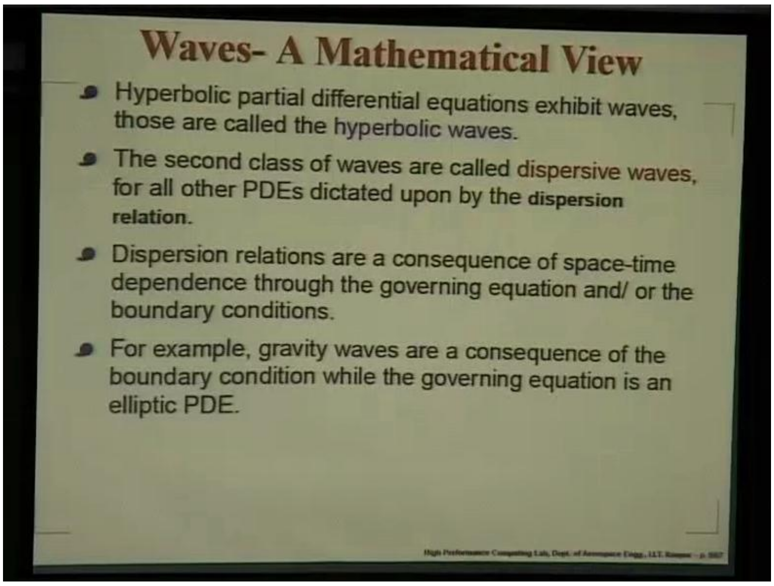

So that is what we look at a mathematician view. A mathematician will actually tell you that wherever you see waves they must be a consequence of hyperbolic PDEs, when you have such waves you call them as hyperbolic waves. However, there are also second class of waves, which will be called as dispersive wave, they are there all possible kinds of PDEs. They are dictated upon the space-time dependence and that space-time dependence is called the dispersion relation when we look it in the spiritual plane, but in the physical plane, you have the governing equation itself.

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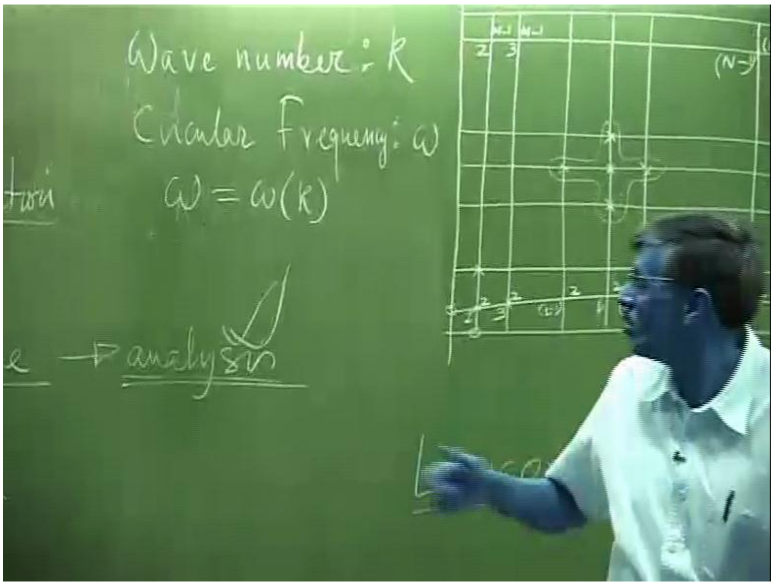

It is an interesting thing for you to note that if you look at hyperbolic waves this dispersion relation will provide you with some kind of real relationship between the wave number and the circular frequency. So, dispersion relation is something like I will tell you how this omega and k are related. That is what happens in a hyperbolic PDE, You have the governing equation in terms of space time derivatives.

So, there you can convert the special dependence in terms of wave number k or the time dependence in terms of a circular frequency omega and put them in the differential equation; that gives you this dispersion relation; that is what we are talking about. However, if that is there through the governing equation and that happens to be a real relationship that is what we call as the hyperbolic waves. In some situation, you will find that the governing equation does not even have time dependence but you still get waves.

One of the examples is the gravity waves. We just now saw the governing equation for the gravity waves are nothing but the Laplace equation, let us say in a simplest form that is an elliptic equation, but it still sustain waves. That comes out because of the boundary condition. So, it is rather interesting for one to understand that those classifications that we have done in the last class, they are related to classifying the equation based on what we see in the differential equation.

The boundary condition are not been even considered, whereas you can see dispersive waves out of any PDE that may be a consequence of a boundary condition. So, a gravity wave is a very good example for you to understand that the governing equation is an elliptic PDE, but it still supports waves.

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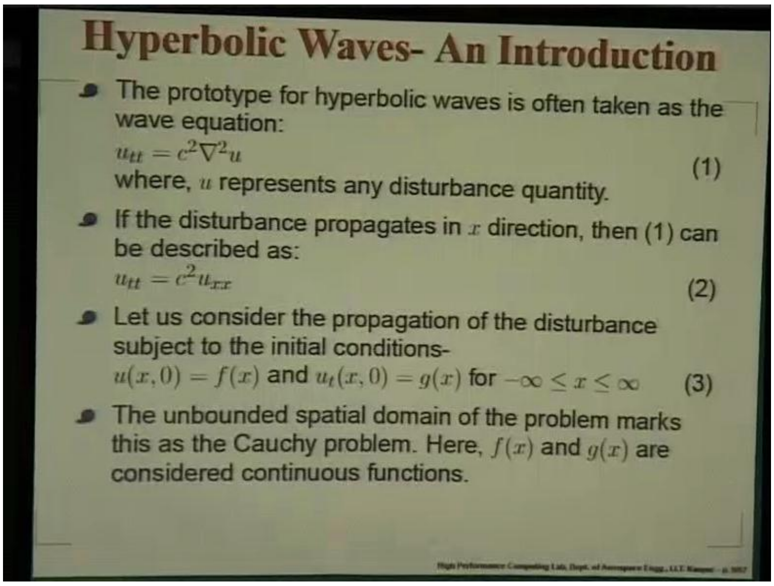

Let us do the easier part - hyperbolic waves. This goes way back into history, where D'Alembert really first look at this problem and try to obtain the first solution. So, we talk about this simple wave equation that you must have done in your math course. This tells you that U any disturbance, its second derivative in time is related to the Laplacian of the variable in space multiplied by C square, C is the some kind of a speed.

Then, if the disturbance is simply propagating in one direction try to make it simpler, this could be a 3D problem. Let us look at its solution in the 1D that is what D'Alembert did. So, try to solve U tt equal to C Square U xx and then you need to solve it subjected to some initial conditions.

Let me try to get you out of this comfort zone. Always try to think that any PDE can be separated the way you want. What happens is most of the time you end up by getting ODEs for each of the independent variable and then you have the tendency to think of in terms of number of boundary conditions.

See, if we have noticed in the beginning of this class, I was continuously saying, this problem according to your separation of variable I need two boundary conditions in x and two boundary conditions in y, but when it comes to PDE you would be saying that I need only one boundary condition.

What happens is this whole thing which you are imagining as four segments essentially constitute one boundary. So, I will not go into this, leave it to our mathematician friends to talk about any elliptic PDE of order 2N requires N boundary conditions. So, if I am looking at a Laplace equation I need only one boundary condition that would be this everywhere (Refer Slide Time: 38:10). So, you know it is very difficult for one to say by looking at this equation how many initial conditions, how many boundary conditions you would know, after we have separated?

So, it is something that I am telling you right now, which we will verify later. This equation that you are seeing here requires two initial conditions. Initial condition is adequate for you to define this solution, we will derive it shortly.

Let us say, we are trying to solve this problem in an infinite domain. So, x goes from minus infinity to plus infinity, such a problem we will call it as a Cauchy problem. Cauchy problem means, we are solving the problem in an infinite domain. This initial condition for the disturbance and its time derivative - f of x g of x are considered as continues functions.

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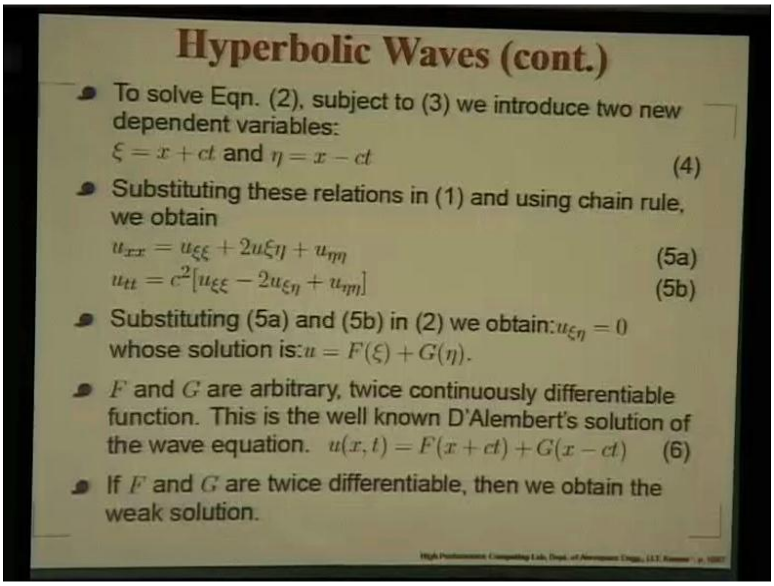

Now, if we try to solve that 1D wave equation subject to those two initial conditions, let us define, two new independent as dependent; well, this is independent variable, there is a mistake, it should be independent variable, which we will call as psi and eta. Phi is x plus ct and eta is x minus ct, if we substitute this relations in that governing equation, use the chain rule, then you would note that U xx would be use U phi phi plus 2, U phi eta. This is subscript phi eta and this is U eta eta. Similarly, U tt would be C square times the C, similar quantity expects that the middle term appears with a minus sign.

Now, you plug this 5a and 5b into, then you get the governing equation by simplify to U phi eta equal to 0, this is what D'Alembert did. So, what happens is you can integrate it twice. If you integrate it twice, you will get two functions f of phi and g of eta. These are kind of arbitrary, twice continuously differentiable functions, this is what D'Alembert obtained and the solution works out to this.

Because phi is x plus ct and eta is x minus ct, this is your solution. All you have to do is figure out what this F, G are. If F and G are not even differentiable we can obtain weak solutions. So, we will have to worry about weak solutions for the time being.

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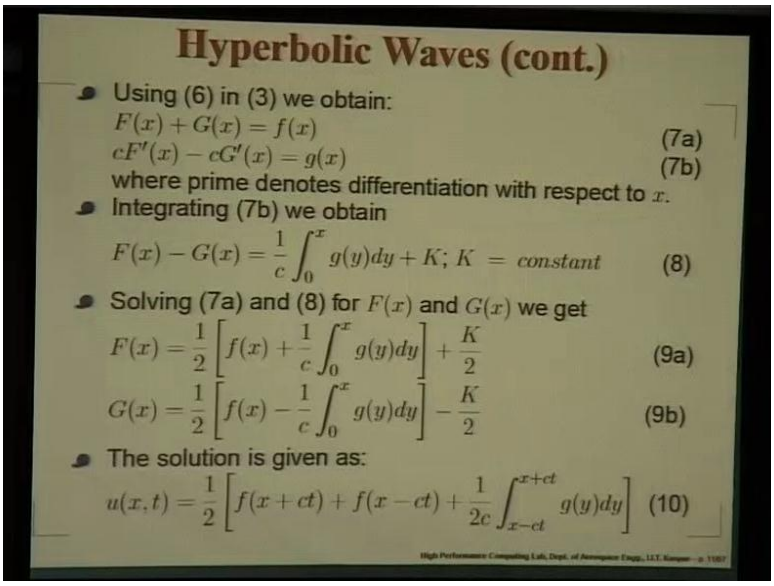

Now, what we need to do is substitute that generic solution that we have just now seen here, 6 into our initial condition, so u will give us this and ut will give us this, where prime denotes differentiation with the argument. If I integrate this equation, then I will get F of x minus G of x is this and k is some kind of constant. Then, what we could do is we can solve 7a and this 8 to get F and F of x and G of x in terms of f x and g x. So, effectively you get the solution like this.

Now, what you are noticing is that the initial solution that was given at t equals to 0 was this f of x. So, the moment you look at its time propagation you see that it splits into two. This part x plus ct moves in the negative x direction. Whereas, x minus ct part goes in the positive x direction and this is the contribution coming from the del u del theta.

So, you know it is a very interesting equation, because it tells you one thing. Whatever the initial condition you have given at t equals to 0, whatever f of x that you have; let us considered its simpler case, let us say this part is 0, g of y is 0 then everything is determined by this only.

So, the initial condition tells you that it splits into two parts; one goes to the left, other goes to the right. This is what we call as a left running wave and a right running wave. So, x plus ct is the left running wave, x minus ct part is the right running wave that is going in the positive x direction. So, it has some very good property, it tells you that the solution at a subsequent time does not attenuate, because what you have done at t equal to 0 the same functional form is propagating in both the direction by taking half of the initial part of the solution.

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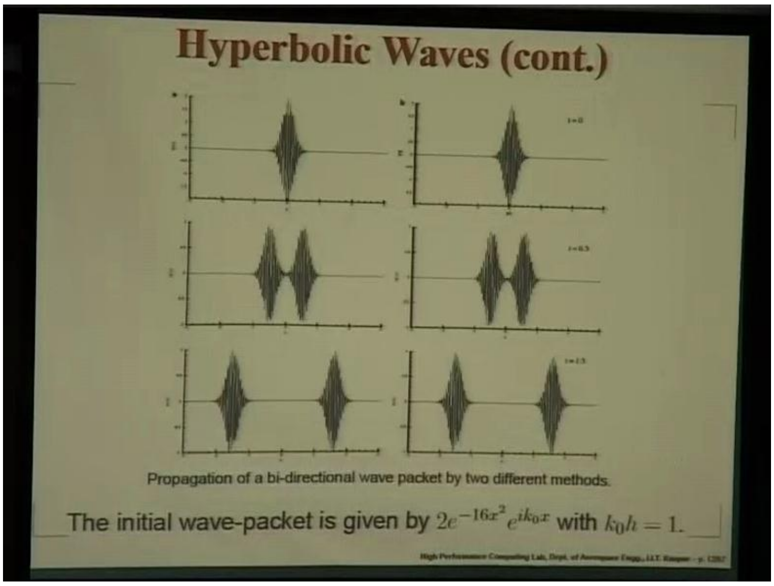

This is a very interesting property; it does not suffer any attenuation. So, this problem can be solved numerically and we did it. What we did? At t equal to 0, we assume this kind of a wave packet here. Wave packet is something like this. It is a basically a wave e to the power i k0 x. What we do additionally is we make its amplitude decay with around the central point, so this is going to be, the central point here x equal to 0; that is what we have done.

Either in plus or minus direction the amplitude decays by this exponential factor. It is a Gaussian distribution, minus 16 x square; that is what happens. You have the peak amplitude here, with x decreasing or x increasing the amplitude comes down. Whenever, you have this you call this as a wave packet. So, we will be taking about this wave packet quite often. Now, what we have done is we have taken delta x as the spacing that is equal to h. So, I have the wave defined by k0 and that we have taken it as equal 1; this is the way we have defined this.

So, we have solved this problem in a finite domain for a finite amount of time, but you could do it for a very large domain for long time and we try to solve the Cauchy problem. What you notice? As we have seen in the exact solution at t equal to 0 we have one wave packet. At a later time, at t equal to 0.5, this wave packet has split into 2; one part is going to the left, this is the left running wave packet and this is the right running wave packet.

It splits exactly into identical half of the original. Subsequently as you can see, it goes on separating, it is all. As I have told you, the moment it splits into two subsequently, it just simply does not change its shape, it just goes as coherently as it was originally defined.

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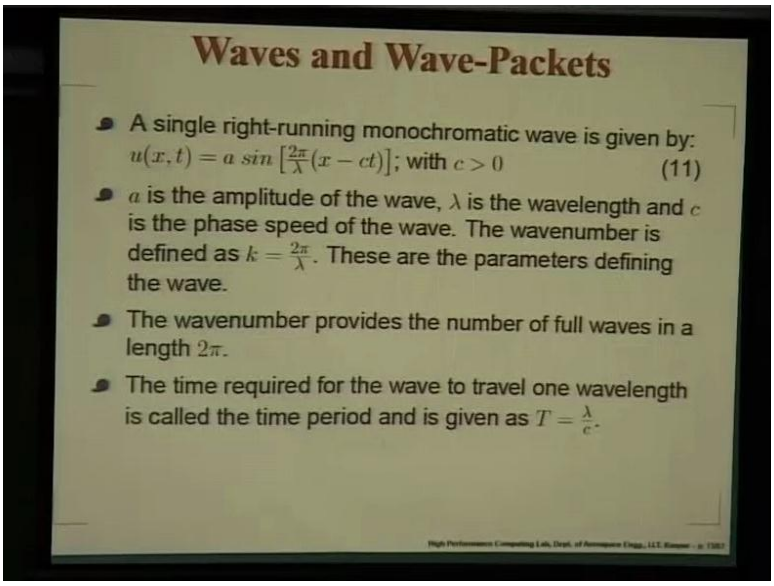

Well, basically, we would try to solve it by one of the method; we have proposed along with a little older method, we show that these are essentially the same. Anyway, what we noticed that even the simplest problem gives us a very interesting set of solutions. Suppose, I am looking at only the right running a monochromatic component, we will define it as U of x comma t in terms of the amplitude a, in terms of its wave length lambda. This is constant c, let us considered c as positive.

So, c is basically telling you what? We will write it as e to the power i k x or we will write it in terms sin or cosine, it means the same thing. So, that exponent or this argument defines here the phase of the function. So, you can see with time the phase keeps on changing, the rate at which the phase change is given by c, because its x minus ct.

So, with time the phase keeps changing at the rate of c that is why c will be called the phase speed. Now, I just want tell you about wave number, wave number is 2 pi by lambda. Wave number is nothing but count the number of waves in a phase of 2 pi. If you have a length of 2 pi you calculate how many waves you have there; that is what we will call as the wave number.

The time required for the wave to travel one wavelength is the time period; we will call it as t that is nothing but lambda by c. So, lambda is distance, c is the speed at which it is propagating, so lambda by c will be the time period. Having obtained the time period you can always talk about the frequency.

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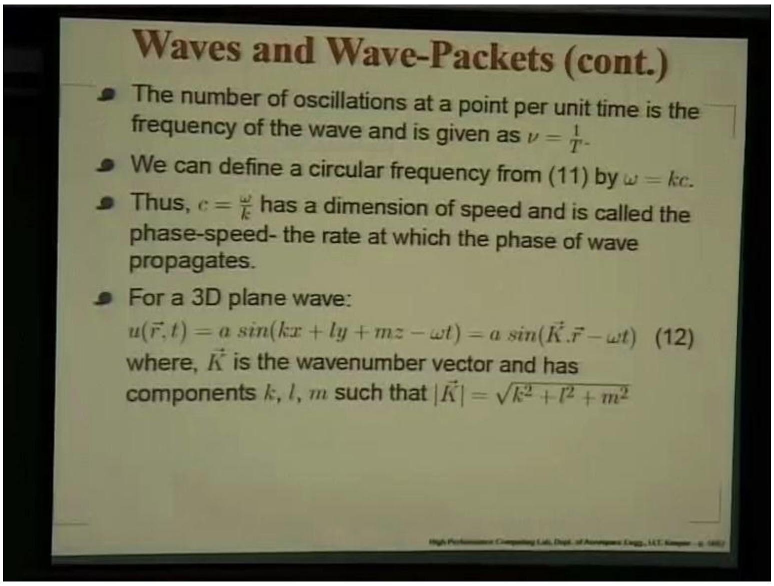

Frequency is of course, the inverse of the time period that is this. We can define a circular frequency that is why we have talked about omega, which is nothing but k times c. If you look back, 2 pi by lambda is k, so we have k x minus k ct, so k c is termed here as omega. Basically, what we are writing is sin of k x minus omega t - in this form; that is your case description (Refer Slide Time: 49:00). So, circular frequency comes in the way it is changing. This is nothing but omega is equals to 2 pi nu; the nu is there, so 2 pi nu is your omega.

What we have now so far defined is for one dimensional wave. We could also describe it for three dimensional plane waves, so we will be having variation in the all the three directions in a Cartesian frame; I will write it as k x plus l y plus m z omega t is that time dependent part and we can write it in a Vectorial notation of this kind; k vector has this component k l and m such that the modulus is given by this.

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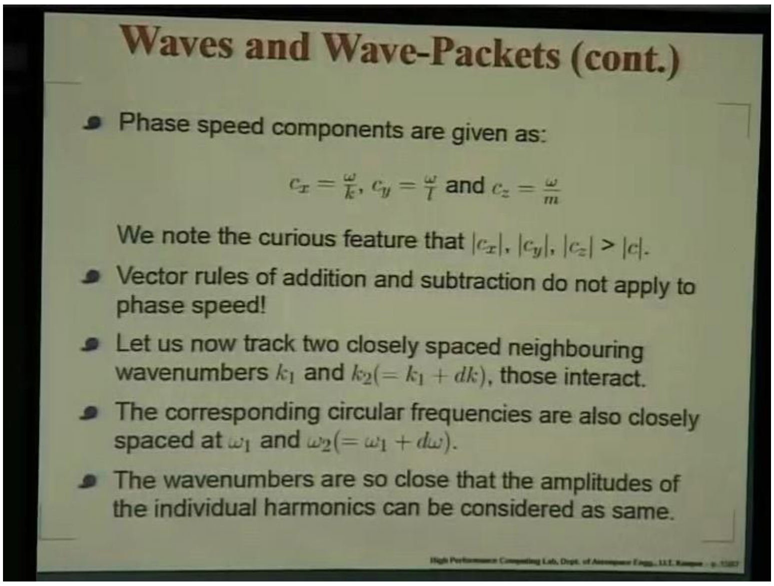

Now, there is some interesting property. If I look at the phase speed here, we have defined for this hyperbolic wave v, in general for wave, we have got omega equal to k c. So, what we have is basically for 1D wave, c was nothing but omega by k. So, in 3D something interesting happens. I will have three components of phase speed; I will call them as cx, cy, cz; cx is omega by k, cy is omega by l and cz is omega by m.

Look at the curious feature that all these components are greater than the resultant. So, this is something, please do not try to apply vector rules to phase speed - never ever. Now, that is simple part; what is important for us to know is that it is hardly likely we are going to come across this scenario, where you will have monochromatic waves, you will not have a single component of k or etc.

What you would have instead is a spectrum. This is something which we must keep back of a mind, in all realistic system if I do not do it specifically, design it in a lab; I am not going to see a monochromatic wave. What I am gone to see instead is what I will call as the spectrum.

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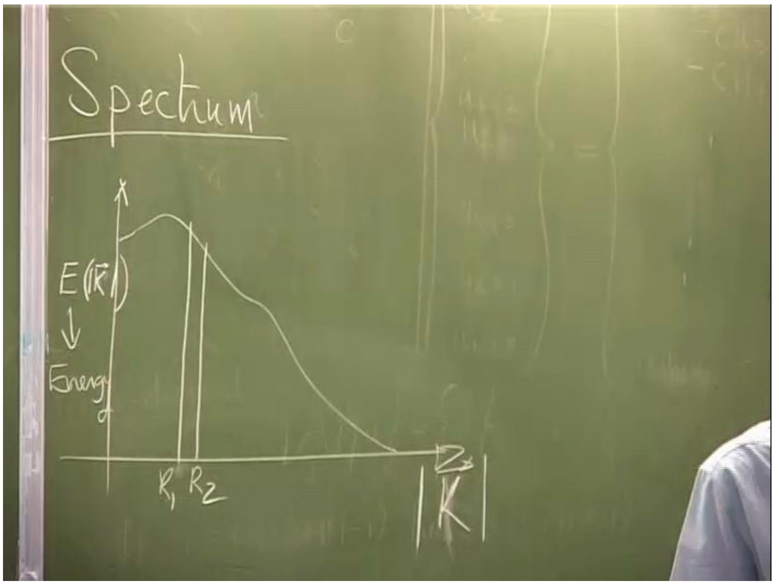

Spectrum is very integral to any of our discussion of a physical system. So what happens is, let us talk about; this is what I have just written there as K. Let me try to write E of modulus K vector, we are doing this; we are just showing with the amplitude of k. It is like in the K space (Refer Slide Time: 51:45). We are talking about the whole possibilities, K modules can go from 0 to infinity and we are looking at spherical shelves. It is like onion ring kind of a thing, you can peel it. This delta k will tell you in that band how much of energy is distributed.

In general, you would find that any system with a finite amount of energy would have a spectrum like this, it is a typical property, it just does not go to infinity; because, we are talking about physical system, it will have a finite energy. So, integrate this whole energy that tells you about the total resident energy in the system. So, that has to be finite that is what you get.

Earlier, we were talking about a discreet single component. Now, what we are talking about is two neighbors; I just call them as k1 and k2. These two neighbors are there and we are talking about how does these two neighbors being together affects the system. That is what we are trying to find out. We are trying to track two closely spaced neighboring wave numbers k1 and k2. So, k2 is slightly displaced from k1 by this amount dk - small amount and they are there simultaneously. So, what you are going to do? You are going to do simply add on simple position that is how they would be.

Now, we also talk about the dispersion relation. We have seen this is one such thing for hyperbolic wave the dispersion relation, omega in k related by that question omega equal to kc. So, the moment I have k1; I have a corresponding omega 1. In the same way, if I talk about k2, I should have a corresponding omega 2, so this comes from our dispersion relation.

So, what we are saying that this is a well behaved system, if k is related by a small displacement dk, I have the frequencies d omega. It is a very specific request, but it can be probably extended, for even if there is a finite jump between omega 1 and omega 2, we can do that. But, let us keep the arithmetic simple and say that circular frequency is also closely spaced.

This wave number is so close to each other that this amplitude; see the energy is proportional to what? Amplitude square for a wave that we know. If this E of k is closely spaced, the amplitudes also going to be closely spaced; so, we will make a mathematically simpler by taking an amplitude also same.

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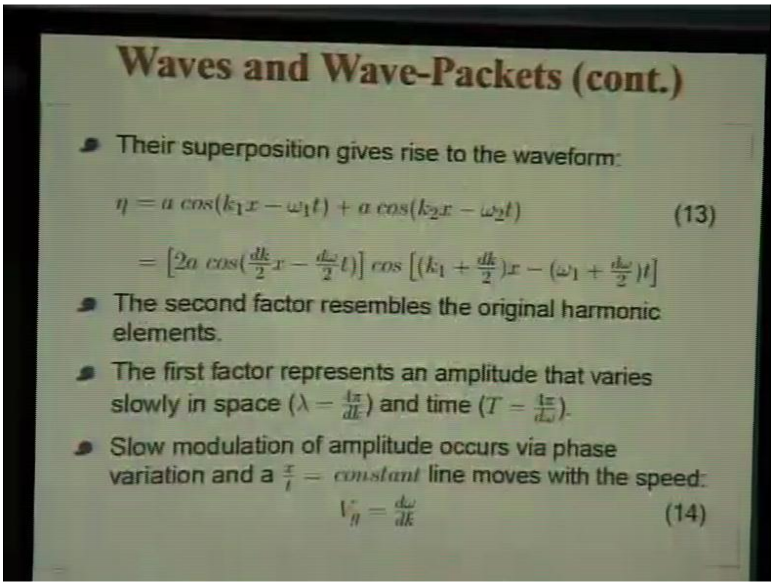

If I do that then, what do I get? I get the super position like this. So, a is the same amplitude that I am writing then, I get cos k1 x minus omega 1t being superposed with a cos of k 2 x minus omega 2 t. Are you all aware of trigonometric identity? Then, it will be 2a cos of this time and this part (Refer Slide Time: 56:10).

Now, that is what happens when you have two entities close to each other. They tell you the super position, gives you one part which is this second factor and that is almost like the original, because the original was k1 x minus 1 omega 1t. All you have done, you have picked it by dk by 2 and d omega by 2, it is a kind of average between k1 and k2, and omega 1and omega 2.

So, that is what it is? This part is not interesting, what is interesting is this first part. First part actually tells you the amplitude has not just simply added to 2a, but it is going to be 2a times this cosine of this function. How is this function changing? This function is changing slowly in space and time. In space, how is it changing? Remember, that k was the wave number; from there I could calculate the wave length that was 2 pi by k. In this case, 2 pi by dk by 2 will give me a wave length of 4 pi by dk and the time period is going to increase to 4 pi by d omega.

So, what has actually happened is that two simple waves interacted with each other that gave rise to the phase dependent part, almost the same, but the amplitude have started changing slowly in space and time, this phenomenon is called as modulation. You will notice that this modulation is happening along this. If I am trying to track this constant phase part, what should be the speed at which I should be going? That should come out from x by t here equal to x by t equal to constant. That will be something like your d omega dk.

So, in a very simple minded fashion what we found the amplitude is changing at a speed which is given by d omega by dk, d omega t. This quantity is what is called as the group velocity. So, what we have just now figured out is that, in a realistic system where there are more than one wave numbers and frequencies involved, we do get groups of events colluding together to show that central part. What is the central part? k1 plus dk by 2 corresponding omega was this that actually travels with this kind of amplitude variation.

So, amplitude is related to energy. In a real spectrum, you will see it is the group velocity at which the energy propagates and that is a very fundamental relation, we should keep in mind. With this I think I will stop.