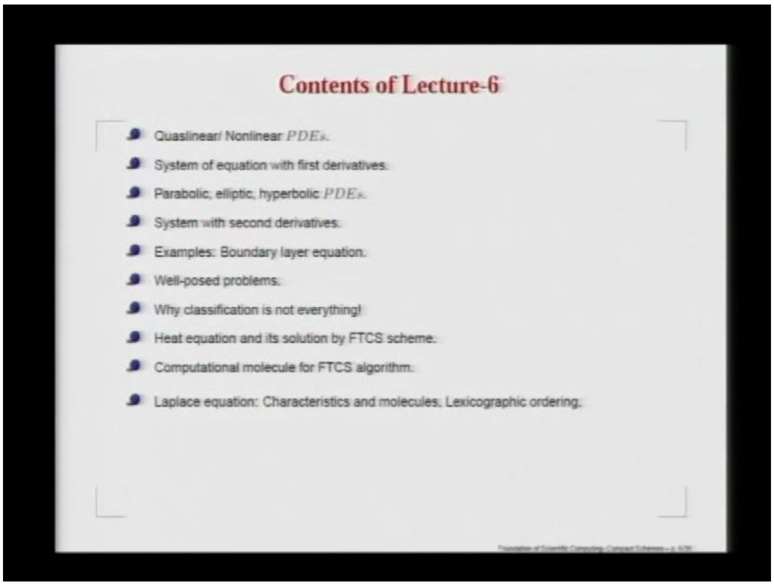
Foundation of Scientific Computing
Prof. T. K. Sengupta
Department of Aerospace Engineering
Indian Institute of Technology, Kanpur

Module No. # 01
Lecture No. # 06

(Refer Slide Time: 00:16)



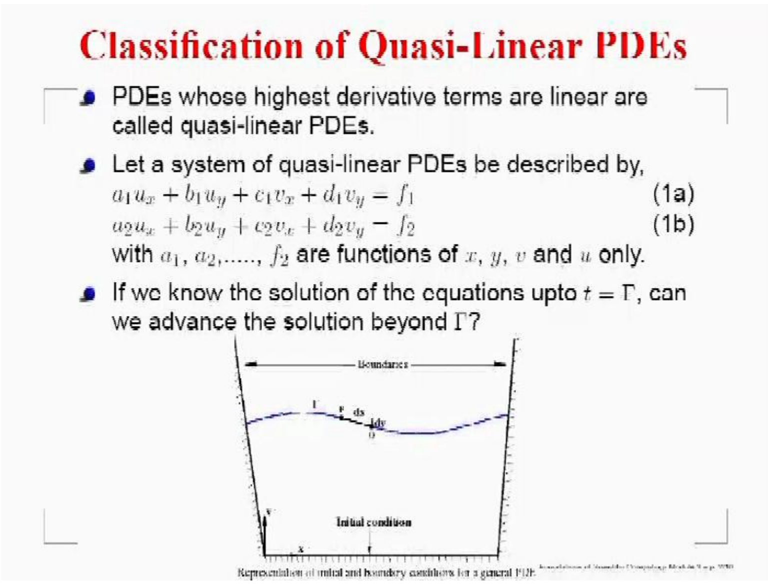
The contents of lecture 6 consist of the following: we will begin by classifying Partial Differential Equations or in terms of quasi-linear or non-linear Partial Differential Equations. We will see that most of our problems are quasi-linear in nature and we will talk about the classification of PDEs by taking various examples. We will start with a system of equations involved in first derivatives. We will have more than one dependent variable and of course, there are more than one independent variable for all PDEs that you know.

We will see that quasi-linear PDEs would be classified into three types: they are parabolic, elliptic or hyperbolic partial differential equations. Having defined what these are and what are their properties in terms of the characteristics, we will talk about systems with second derivatives; and as an example, we will explain this with the help of boundary layer equations. Having classified the PDEs, we will talk about the basic requirements of solution methods. One of the prime requirements is noted here; that is - this problem has to be well posed.

Despite this classification and well-posedness of the problems, we will also notice that classification is not everything, because of certain features by which numerical solutions are obtained. We will quote a few examples. For example - heat equation will be solved by one of the easiest methods which is called the FTCS scheme or Forward in Time and Centered in Space schemes. will identify, what is known as the computational molecule of this particular algorithms.

We will also take another example from elliptic PDEs and that is the Laplace's equation will once again show how its characteristics is obtained, what is its molecule and how the unknowns are ordered in solving the problem. Elliptic PDEs can be further classified if they are linear. That is very easy to see that dependent variables in all the terms appear as linear functional, whereas nonlinearity is also equally understood. In contrast, you have the third class which we call as the quasi-linear PDEs.

(Refer Slide Time: 03:17)



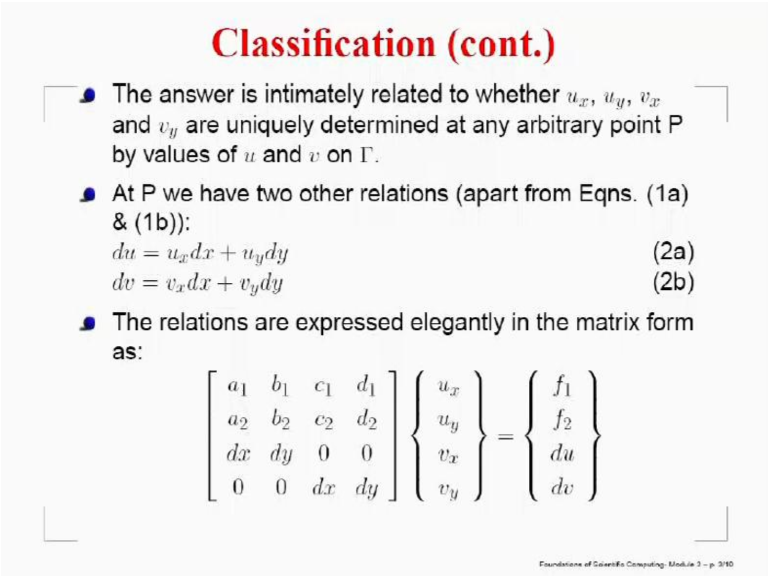
Quasi-linear PDEs are essentially non-linear PDEs, but in the highest derivative term they appear as linear. This may appear as a kind of a severe restriction, but you would find most of PDEs that we come across in various branches of Science and Engineering happens to be quasi-linear PDEs. And quasi-linear PDEs have been easily classified - a systematic theory exists - whereas non-linear PDEs are not so easy, but some specific classes of non-linear PDEs have also been analyzed.

Let us look at classification of quasi-linear partial differential equations - as I told you that these are the PDEs where highest derivative terms are linear. For example: talk about a test case and example where u and v are the dependent variables, and independent variables are x and y as given by these two equations. And please note that according to our definition of quasi-linearity, all these coefficients that we have here - a1, b1, c1, d1, f1 and a2, b2, c2, d2 and f2 - would be of course, in general function of x and y, but because of quasi-linearity they could be also function of v and u.

In a sense it is non-linear, but in the highest derivative terms that you see here - u x, u y, v x, v y - they appear linearly. That is the appellation that we call it quasi-linear PDEs. Now very simple urge for us to know, that if we have this governing equation, we prescribe some starting conditions and we prescribe some boundaries over which the solution is defined and the solution is progressed. And let us say we have reached the solution up to this curve gamma; that could be some parametric curve which I call t equal to gamma.

Now the solution is known in this. Well, the question that we ask is - can we advance the solution beyond gamma? That is one of the aim for this, if we want to really find out. What we have seen is that u and v are functions of x and y.

(Refer Slide Time: 05:41)

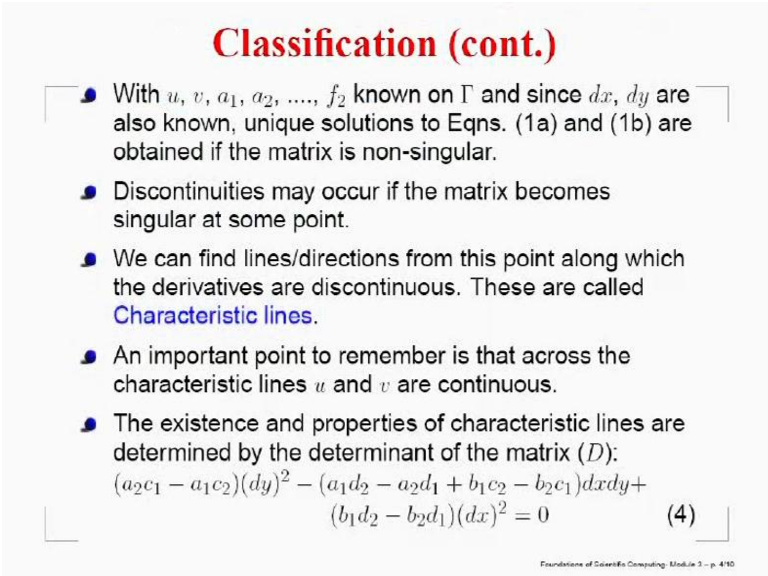


What we could do is knowing the solution up to gamma. We could go beyond it by just simply writing out a Taylor series, where the first derivatives could be obtained from - let us say the governing equation that you are given the higher derivatives; what you could do is you could keep on differentiating these equations and generate auxiliary equations for higher derivative. If I want to get u, xx would perhaps differentiate these ones and then go about trying to obtain those.

Now going back to this figure again, let us say - focus our attention on this point P, where we know the solution and along P, Q also we know the solution. If I migrate from p to q taking this increment dx and dy, then along that line P, Q what we could do is we know what is du and dv, because we know the solution at those two points - those we could relate to increments due to dx and dy via this partial derivatives here.

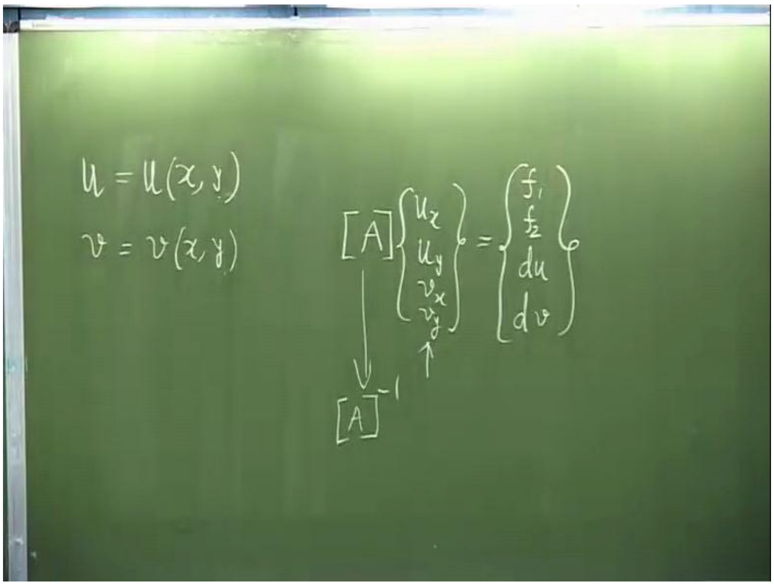
Now what we have? We are trying to figure out all these four derivatives, which we stack them up here and this push to rows are the governing differential equations that we have. And these two are the auxiliary equations that we just now concocted. These are the two last equations.

(Refer Slide Time: 07:50)



What happens is - trying to figure out those partial derivatives, you just simply have to solve those linear algebraic equations, that is fairly an easy thing to do. What we need to do is we have to observe the equation that we have written down, which had this particular form.

(Refer Slide Time: 08:24)

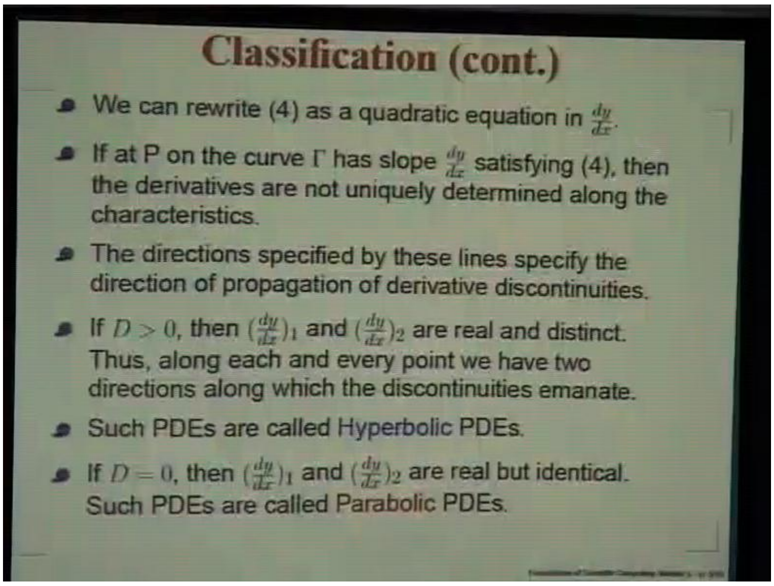


We have ux, uy, vx and vy. And on this side, we had f1, f2, du and dv. This could be solved for this - provided, we can get inverse, that is what solving that linear algebraic equation means. However, this solution process will break down if this matrix A is singular. We demand that this matrix B nonsingular, then only we can get those solutions for those partial derivatives. So, stated another way that we talk about the scenario where the matrix is singular at some point or some part of the domain. Then what happens? We cannot uniquely determine this quantity and that would amount to acknowledge in that we have some kind of discontinuities. The discontinuities will occur whenever your matrix is singular.

If that is happening at a point in the domain, from that point onwards we could workout lines along which this discontinuity can propagate. When such lines exist we call those lines as characteristic lines. We have to remember though, that we are talking about discontinuity in terms of the derivatives, but the function by themselves - the dependent variables are continuous. Please do not misconstrue about this point that we have making about discontinuity. This discontinuity is about the derivatives and not about the function itself.

We would - we did talk about little bit about shock waves, hydraulic jumps. There are many such examples where you actually see discontinuities in the physical variable itself. But these characteristic lines are not those discontinuities - these are discontinuities of the derivatives. Essentially then the existence on the properties of this characteristic line can be evaluated by looking at the determinant of that matrix and this is what we get - as you can very clearly see that we could write this equation 4 as a quadratic in dy by dx.

(Refer Slide Time: 10:54)

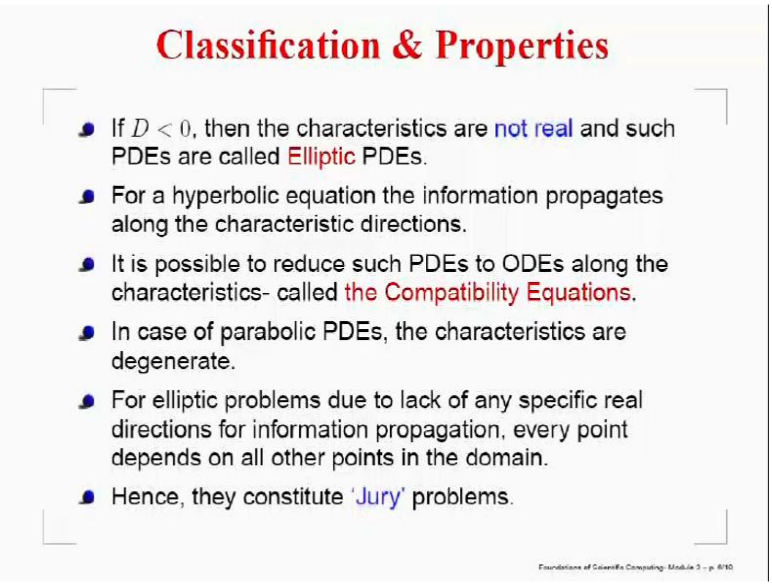


If you write it as a quadratic in dy by dx, then what happens is - starting from the point P you can work out this quantity dy, dx. It is quadratic - means you will have two of them. You can figure out from the point P if the derivatives are uniquely determined or not. If those slopes are real and exist, they do exist; then of course, you say that these lines specify the directions of propagation of derivative discontinuities. This may sound in a long drawn phase, but it actually means the following: what is your differential equation telling you? When we write out a governing equation, what do we actually mean by the differential equation? Differential equation basically tells you how the information or the signal is propagating.

If I have line of discontinuity in the domain along those characteristic line - what is happening? Along those lines, my information is propagating so that I have one set of value to the left and another set of value to the right. Characteristic lines are those lines along which this discontinuities are propagating. We can now look at various sub cases. If this determinant is positive, then you have couple of real and distinct slopes emanating from that point. You have two directions along which this discontinuity propagate and when you have such a scenario, you call those PDEs as hyperbolic partial differential equations. This is something which you probably have done it, but let us look at it from computing point of view once again.

Of course, the characteristics determinant is equal to 0 - that is the quantity under the radical sign. Then of course, these two slopes are going to be coincident and such a PDE will be called a parabolic PDE. This is also quite known to us.

(Refer Slide Time: 13:17)



The third possibility, of course D is negative and then the characteristics that you are going to get are going to be complex; they are not real. So, you do not have discontinuities propagating along real lines. What does that mean? This means that you do not have - suppose I have a domain like this. If I am now looking at a hyperbolic or a parabolic problem, I would have information propagating along characteristic directions and when I do not have them, what happens then? The solution here or anywhere would not be realistically dependent on its neighbors; instead, what it would do? It would depend upon all the points in the domain.

It is not only this point it is determining. This point is determining this but this point is determined by all the points in the neighborhood and subsequently, each one of those points are determined by their own neighbors in the continuum of the points. And in the process, you would find that eventually the solution anywhere depends on what you prescribe on the boundary. You can see elliptic PDEs are those we already have called them as a boundary value problem, where the solution depends on the boundary conditions.

That is why these are also called the Jury problem. I may like to tell you a little bit of a story here - when originally this elliptic problem has to be solved and in the absences of computers, they use to do it with the help of human assistants. These assistants use to position themselves in a room in a sort of a lattice network. What will happen is each one of them sitting at one point would get the information from the neighbor like what I just now said that it would be dependent from information coming from all directions.

One person setting in a network gets information from the neighbors and then use them on a particular algorithm. We will find out how these algorithms are developed in their future and then that person will evaluate what the value would be for the next step. And when everybody is done and everybody is agreed upon that we have arrived at the next time step, then the conductor - the professor who is to do - that is one such person in Cambridge is to do that and I will you an interesting story he is to penalize those people who is to make mistake in the calculations.

At the end of the day, there are lots of those assistants who would collect negative salary over the day for making mistakes. Anyway, that is what this jury problem comes about. Everyone sitting in the room is a jury. When they decide together that we have achieved our task for one time step or one iteration, then the jury agrees and the problem is settled. That is why historically elliptic PDEs also called Jury problem; that is a legacy of history. In contrast, actually if you have hyperbolic PDEs, there is a possibility that you can reduce those PDEs into set of ordinary differential equations along those characteristics and these equations are called compatibility equations. There is a vast literature and the subject called method of characteristics which actually depend on this methodology. We would not be talking about it, because they also have their limitations because of following: in real life any problem that you encounter, you would hardly ever see that you have only one type of PDEs governing the equation.

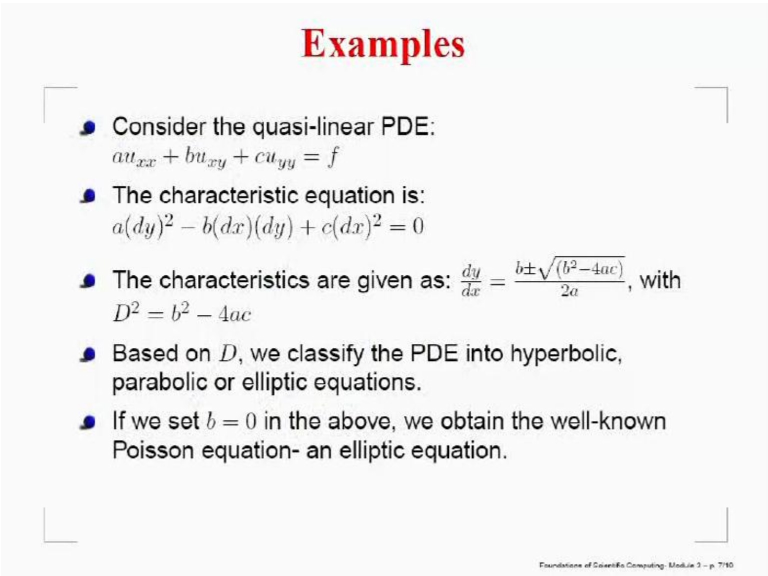
What I mean is- your governing equation may be the same. But in different part of the domain you are going to see that - some part it might actually behave like a parabolic PDE, another part is going to be elliptic, some part it could be hyperbolic and so and so forth And if such a thing happens, then you realize the mixed character of the problem would not allow you to come out with a generic solution procedure for such a mixed problem. In fact, one of the most spectacular developments has come about in the field - a Fluid Mechanics. Believe me, till 1971 people did not know how to solve a mixed flow problem.

That was starting point where people could handle flow in a domain, where part of the domain you have hyperbolic equation and part of the domain you have elliptic equation.

Basically, that is what I was saying. That method of characteristics being very well developed, but it only works in the hyperbolic part of the solution. It requires your knowledge where it actually starts. That is part of the solution. In many cases it is not so easy to workout, although in some cases people do have some advantages and they do start from specific region of the domain, where you would know for sure that from this line onwards you would have hyperbolic nature of the PDE.

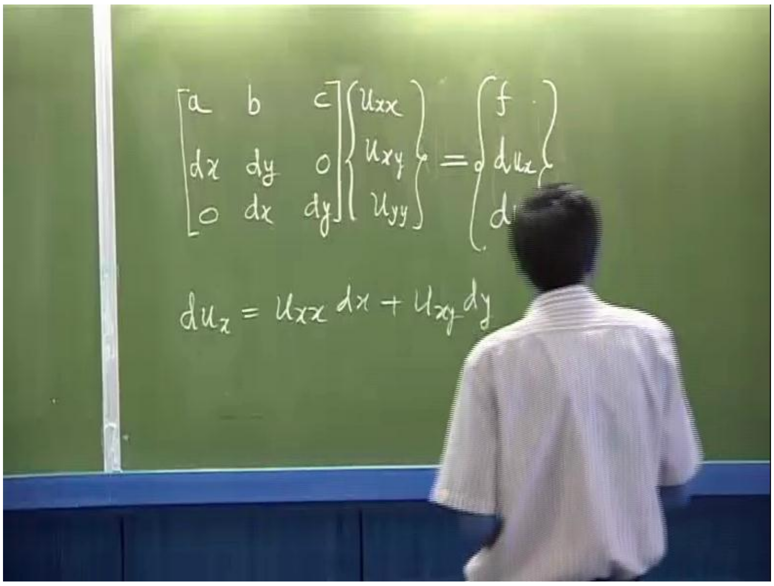
Well, let us work out some examples this is probably you have been talked time and again in your course on PDE, that this was a second order PDE on a single variable u. You have the pure derivatives and the mixed derivatives appearing together and what you would be doing once again? First and foremost, I think we may like to take a little relook at the problem.

(Refer Slide Time: 19:09)



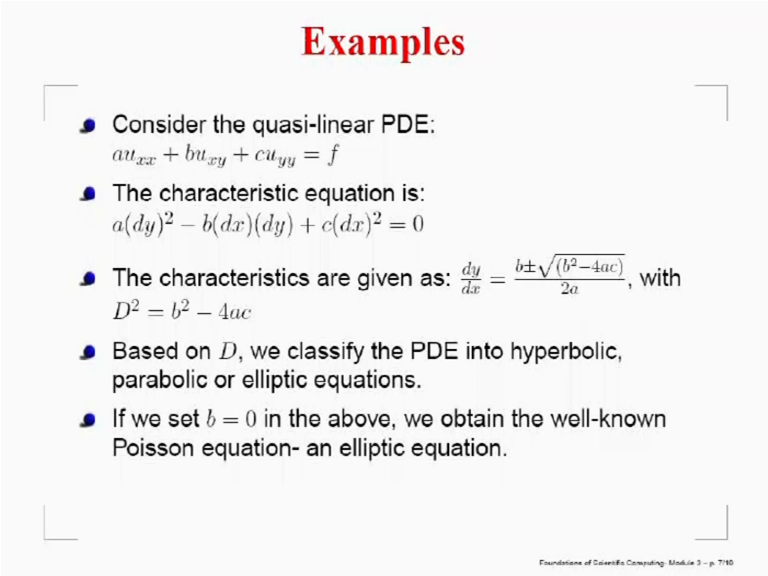
When I mean is that we call that diagram we drew that in this let us say, we have a solution like this. When Say I have solution up to this point, what exactly do I know? Say for this equation - if I say I know the solution, what exactly I know at P? Do I just simply know u? I also know the derivatives - which derivatives? We know those derivatives which are one degree lower than that paring in the differential equations. That means when I say I know the solution, I know not only u but also know ux as well as uy. Please do remember, how do we define what we mean by a solution? In the previous case, if you recall that we obtained two auxiliary relations here, what would we do here? In this case, for this particular problem, then we would be writing it down in the following manner: Let us first identify what are unknowns.

(Refer Slide Time: 21:03)



We are trying to evaluate - these are unknowns. One of them given here is a, b and c. On the right hand side we have f. Now, what we are saying about the solution? We also know this du x. What is du x? du x would be u xx dx plus u xy dy. The second equation of course will have a co-efficient here, dx; here we will get dy and here we will get 0. In the same way the third equation, we will write it down is dy and that will have 0 here, dx here and the dy here. Now you can investigate the determinant of this matrix and that gives you this equation. Once again you come out with that quadratic for dy, dx.

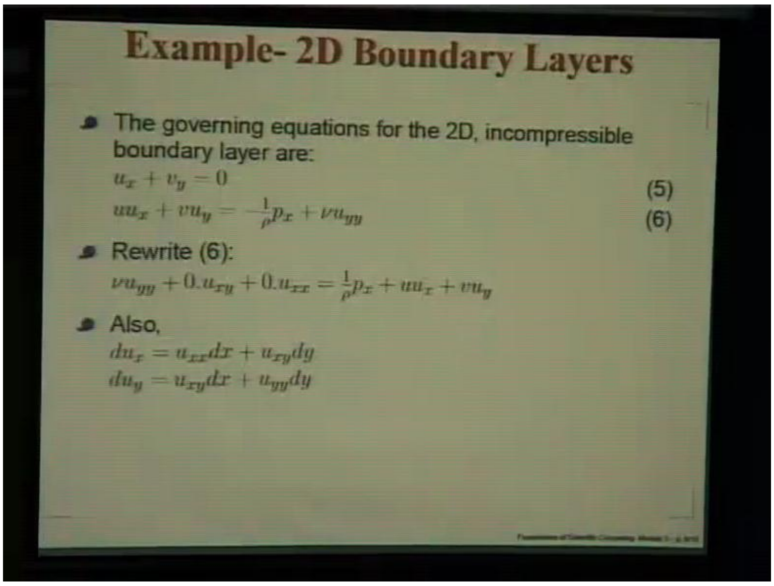
(Refer Slide Time: 22:23)



The characteristics are then given by this - b plus or minus square root of b square minus 4ac by 2a. I have noticed among the student have this tendency to show them a partial differential equation, the first thing they try to do is find out b square minus 4ac and if it is not in that form, they are full of panic. You please do understand that this a very specific example, where this particular form comes in and again you can classify based on whether this b square minus 4ac is positive, 0 or negative.

You notice one thing, though that if we have b equal to 0, the equation that we get is the well-known Poisson equation. Poisson equation is a very prototypical elliptic equation appearing in many branches of Science and Engineering and we have to know how to solve.

(Refer Slide Time: 23:27)

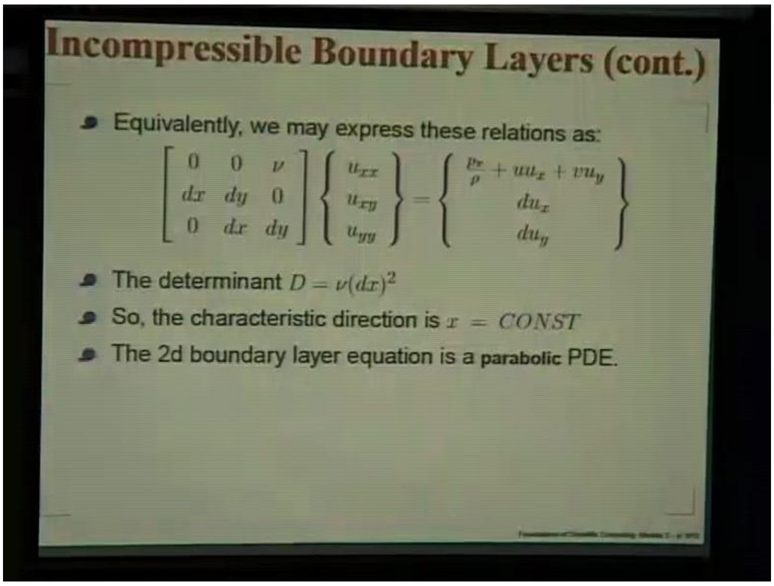


Now, this is something we have done already and I made that remark that the work done by Prandtl was revolutionary - why? We will shortly see. Remember that the boundary layer that we have obtained have two equations - one originating from mass conservation that gave us ux and vy here, and the x momentum equation we wrote in terms of this. What we now see when we are trying to solve this equation? In this equation 6 of course, you have the highest derivative - again it is a quasi-linear equation.

Left hand side – these are those non-linear terms, but they are the lower order. The highest order is the second derivatives. That is why we call it as a quasi-linear PDE and we rewrite it for convenience in this form nu u yy 0 into u xy and 0 into u xx. These are the things that we will assume known, because that is what our definition of solution is.

Now in addition, we could also write this couple of auxiliary equations du x and du y. We again get a 3 by 3 matrix equation and what do we get? We get this and as you can very clearly see, that you can expand it about the first row and then you get this nu into dx square.

(Refer Slide Time: 24:55)



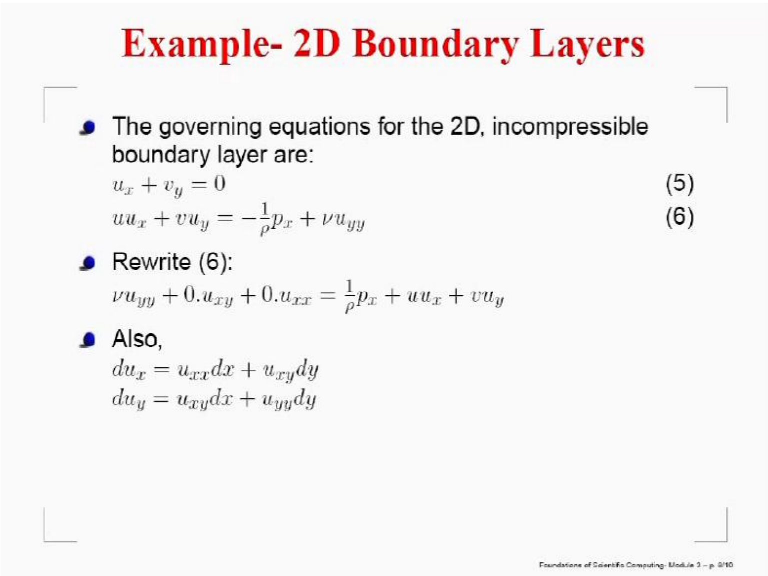
And very clearly you can see now the characteristic directions are coincident, because both the roots are same. So, dy, dx will be infinite or dx is equal to 0, that means x equal to constant is the characteristic direction; that is what I told you - that if I had this problem of a flow in a domain, I could just simply march. Go from 1x station to the next, because the information is propagating along constant x line.

The moment I realize that I would actually solve this first equation along x equal to constant. And once I have that solution at a particular x, from there I go to the next station. Why do I have to go to the next station? Because in the next station I would require this derivative - that is why I need the solution of the previous step, but remember, the information is propagating along x equal to constant, but not across.

That is why wherever I would have this kind of derivatives ux, I will be actually discretizing thinking of the following that look at it this way. Suppose I have the problem like this, then this is my x direction. I would get the solution at x equal to constant. So, this is my y axis. If this is the region I would solve it here at all those points and then we will migrate to the next line wherever this quantities are involved. If I call this line as 1, this as 2 and if I have to obtain ux, I will just simply write it as u2 minus u1 by delta x, because that is what this is telling us. You see that is why using the knowledge of the nature of the solution that we should not try to get information from here, that would be violated because this is a parabolic equation, we can only get information from what happened before and not what is going to happen in upfront downstream. This is something that you realized that what happened.

Now, supposedly what we have obtained in boundary layer approximation was, if you recall that we actually dropped one single term here, only that was nu times u xx; that was our boundary layer assumption. Boundary layer theory gave us that we can safely drop u xx in comparison to u yy.

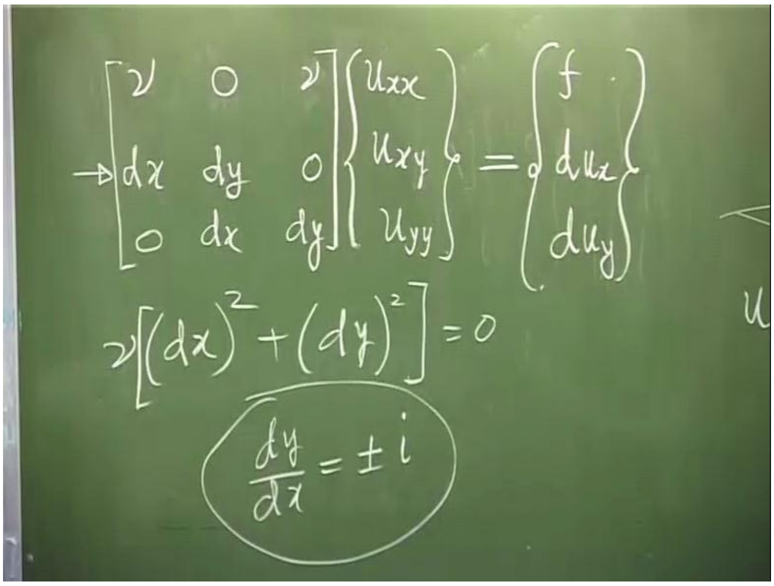
(Refer Slide Time: 27:44)



Suppose I would have retained that u xx here, then what would you have obtained? You would have obtained here a nu in the 1, 1 location. If I look at that what I have for boundary layer approximation, this is nu. If I would not have obtained - made that boundary layer assumption, then I would have also add a nu here. Now you try to work out the characteristic determinant what you would find.

This will give you. So, what you are seeing there? What comes out from this if you would not have used the boundary layer approximation? Then our characteristics would have been complex; it is basically a boundary value problem.

(Refer Slide Time: 28:57)



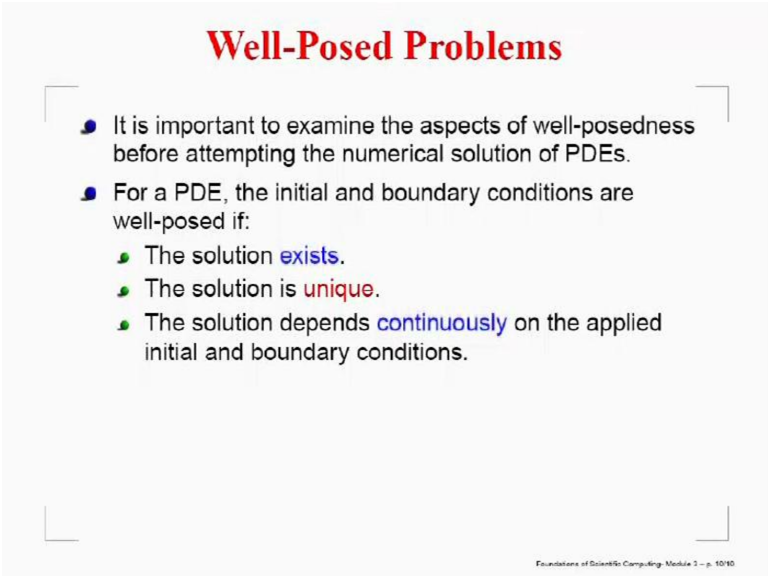
We would have not to solve the whole domain together, but the moment we make that boundary layer assumption, we are going to march. So, this is what happens that from a boundary value problem, you end up in getting a marching problem. This is the marching that you are going to undertake along x equal to constant line.

That is what we are saying- that discontinuity is propagating along this line x equal to constant, that is what characteristic lines imply.

What exactly you mean by 2 solutions. They are same, they are coincident is not it.

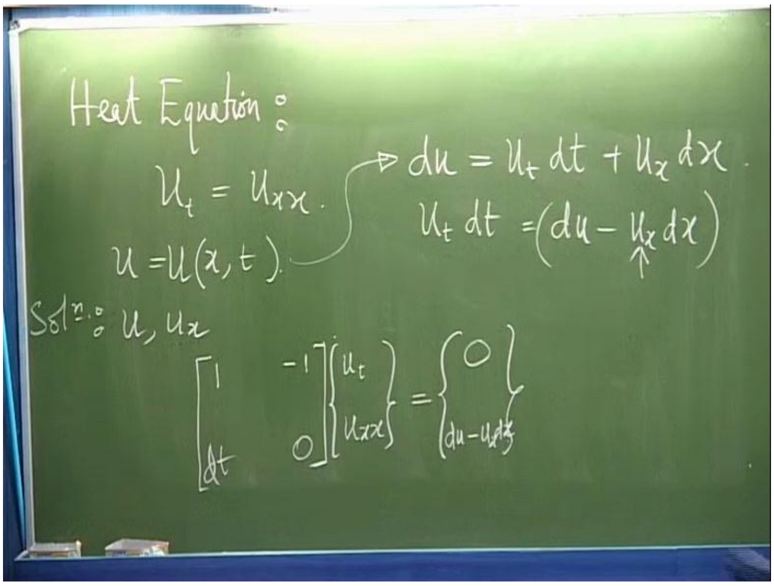
They are coincident - that is why you have a single line. So, you see some of these classifications help quite a bit.

(Refer Slide Time: 30:22)



Now having said all that, why we are not that enthusiastic about this classification? I will just spend a little time and I will explain to you.

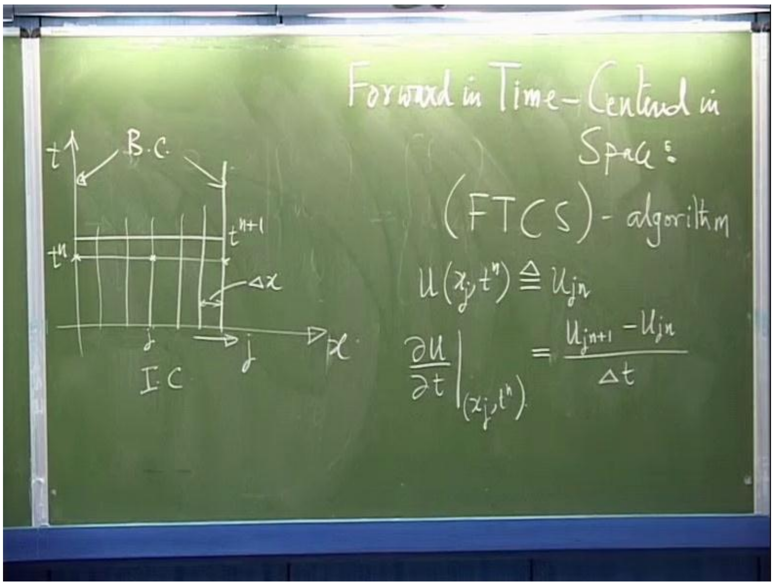
(Refer Slide Time: 30:47)



Let us say, we want to solve this heat equation. The heat equation is simply given as ut equal to u xx and so if I see the solution, then we are talking about the solution being dependent over x and t and if I say, I have the solution up to some point in xt plane that would involve the knowledge of u and ux. So, we know this; this is your solution.

If I have to write a similar equation, first we should write down the unknowns. Unknowns are ut and u xx and if I look at the governing equation of course, that here is 1 and there is minus 1; and this is 0. What would be our auxiliary equation then? We will come from here. We will write du - du would be ut dt, plus ux dx. How would I write it? I will write it as ut dt should be equal to du minus ux dx, because this is known. Right inside is essentially known to you. What you would do? You would write this equation as here dt and here of course is 0. There is no. On this right hand side, we will have du minus ux dx and it is very easy for you to tell me that the characteristic direction is given by dt equal to 0 and that corresponds to t equals to constant**.**

(Refer Slide Time: 33:46)

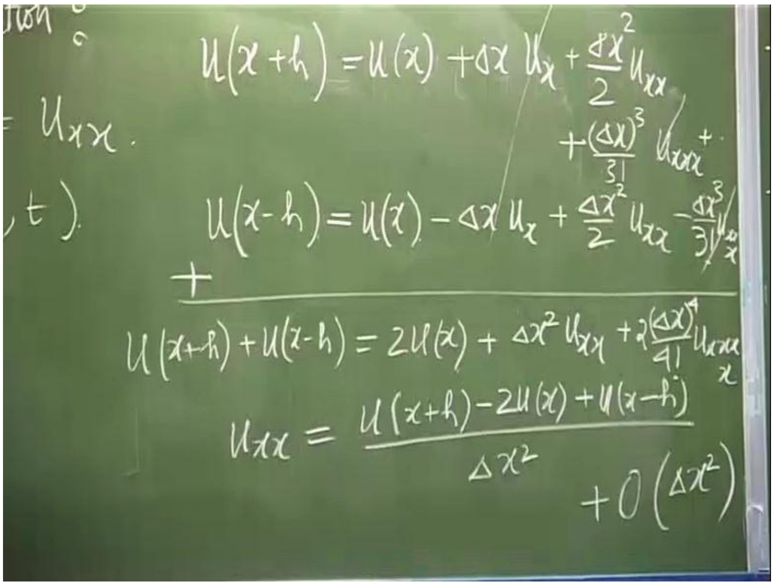


Basically done. The solution that we look at would we plotted in the xt plane. I will plot x in this direction and t in this direction. As you know, in most of the heat transfer problems that you would be probably know. The simplest possible problem is this one dimensional - I mean - you have a rod, you heat from one end and then you try to get the solution.

The length of the rod will tell you the region where you are interested. You would be given the solution here at t equal to 0, that is your initial condition. And then of course, you would be prescribed your boundary conditions are here and here. So, the characteristics suggest that your information propagates like this. Now, if we are going to compute this equation, how do we do it? Let me tell you - solving heat equation, one of the earliest successful attempt was by what is called as Forward in Time Centered in Space. This is what is called also in short form as FTCS algorithm. We talk about how we implement it. If I talk about the solution at field, time level t of n and I would have discretize this problem along some network points. And then I say that I have the solution along this line and we are trying to find out what is the solution. What I would do? I would represent the solution u of x and t at- let say - x equal to xj and t equal to tn. What exactly meansis that we are identifying x also with the index j .And let us talk about these are uniformly spaced lattice. We have these are some kind of a delta x.

So, this we denoted like u jn. This is definition- this is pure definition that we lump the system in discrete points, discrete times and we associate indices subscripts like j of n. That indicates that solution at the location xj at time tn. If I am trying to evaluate this quantity del u del t. This I would like to do it - let say at that point - at j point and this time. Basically, if this is - let say my jth index. So, we are talking about writing the derivative about this point. One of the easiest thing that we have learnt just recently was to use Euler integration. Remember that is what we said, that was the easiest way of the discretization and that is what we are also saying here - Forward in Time. So, what we would be doing? We will be still restricting ourselves at the jth point that we will write this derivative like this. So, that would be that.

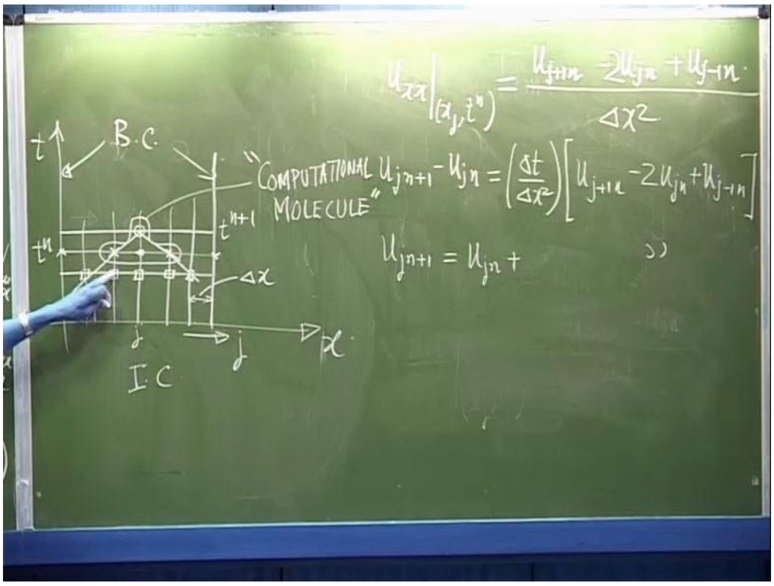
(Refer Slide Time: 38:48)



Now, we did also say about using Taylor series as a means of discretization of our derivatives - special derivative. If you remember, we wrote it down like this - u of x plus h as u of x and delta x into u x and so and so forth. We could keep on adding many more terms. Now what we could do is we could also similarly write the unknown at the left point; then what you would find that the odd derivative terms appear with an altered sign and so and so forth.

Now, one of the easiest way is just simply add it up. Then you are going to get ux plus h, plus ux minus h. On the right hand side, we will get to 2ux and notice that all these odd derivative terms will cancel. This will cancel with this and the first term that you are going to get is delta x square by 2 and couple of them will give you this. Then next set of term you will get two times; and the fourth derivative term and so and so forth. What happens is I can see that from this equation, we can write u xx would be equal to ux plus h. I pull this over on the other side and divide by delta x square, then whatever the term that we have left over that will begin with delta x square.

(Refer Slide Time: 41:20)



So what we have just figured out? We can keep this ut here and then make use of the Taylor series expansion that we have obtained just now. We will write that as u xx at xj and tn. We are only varying x, keeping t the same. I would write that down. So, what this would be? This would be u of j plus 1. This will be uj. This will be u of j minus 1. So, u of j plus 1, n is the same. So twice here, we got this now you see what we have obtained? We have now replaced the continuous differential equation in terms of a lump parameter system, where the values are lumped at those notes intersection of this lattice and that is what we are going to write jn plus 1, minus ujn. I will write this quantity here delta t by delta x square and in parenthesis we will have u j plus 1n minus 2u j n and u j minus 1. So, that is easily done now. So, what we are seeing now- u j n plus 1 is nothing but u j n plus this quantity- that we have written down.

So, basically, we are saying that the unknown quantity at this location - that is what your left hand side is - depends on the function values of previous time step at this point. That we have already identified and also it depends on the two neighboring points of the previous time steps. So, in computing, this is what is called as a finite difference method. In the finite difference method you work out what we call as the computational molecule. So, this is your computational molecule. What were the computational molecules suggest to you? How is the information propagating? We have just done this classification and we have figured out the information should have a propagated along t equal to constant line. Are we getting that here?

If I were trying to figure out the solution at this point, then my information should have come along t equal to constant line, but instead what we are getting? We are getting information from this, it is coming from a region which is given by the apex at that point and these three points. Now, you see what happened in the previous time step- this point was evaluated by getting the information from these three points.

This point was evaluated from these three points on this point was of course, evaluated from this. So, what you notice that information seems to the confined in a wench and what is that? That is your typical hyperbolic PDE kind of behavior. So, what is happening from each point, we are finding out two characteristic direction along which the information originate. So, for example, this point will influence those points which you will be in this wedge.

So, although we are solving an elliptic PDE, in our actual solution procedure we are remaking hyperbolic PDE and there is a price to be paid for this. Why we are willing to pay the price? You can see there are certain features coming out of this different equation.

These computational molecules written like this - is also called the difference equation. So, from differential equation, you come to a difference equation and what you also notice that the difference equations are some kind of a linear algebraic equation that is what we have obtained.

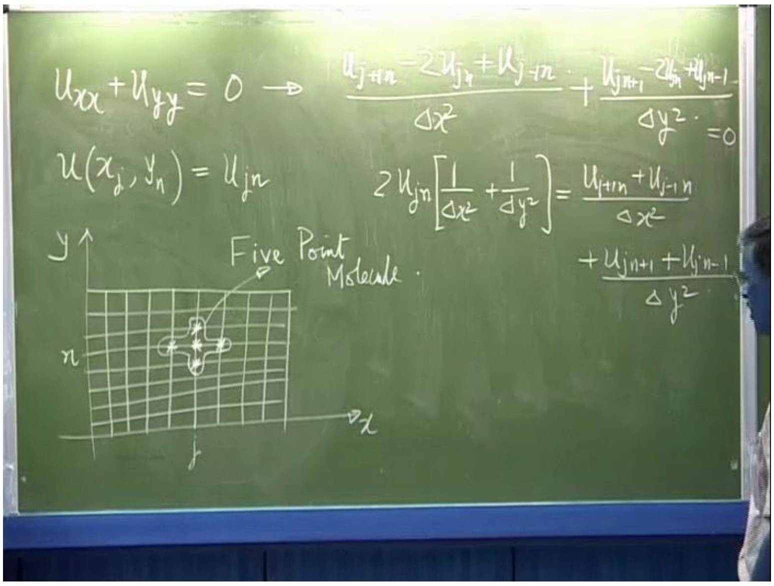
However, what you notice that this particular way of evaluating the solution at an advance time level from the information, from the previous time level is an explicit procedure. I have the information at nth level; from there I could suggest that directly take one point at a time and evaluate it explicitly by using this difference equation. This is an explicit method and that is what you are paying the price that you have try to get a solution of a parabolic PDE by an explicit method. In the process, you ended up with a hyperbolic PDE and that puts in some kind of a restriction on the time step that you can take.

What you usually like to do is you would like to take this time step as small as possible. Then what will happen? This characteristic line, these lines will open up. They will as you reduce delta t, this hyperbolic characteristic will approach their parabolic limit - you see that. So, that is the reason why you have to keep your delta t as small as possible.

So, that is the story that you try to simplify. You have to pay somewhere else and one of the things that I just now demonstrated to you is a case where although you want to solve a parabolic PDE, your method was not really taking the benefit of the classification, but you are taking the benefit by saying, I will have delta t as small as possible. That is a knowledge, but we can see also the restriction arising out of that activity.

What we could have done, of course, we will do it in a lecture- may be 4, 5 lectures downstream. We will come back to this and show you how we can discretize. You can see that what I have demonstrated here is one possibility. One particular algorithm, I have shown you, that is not the end of the world, there are many possibilities by which we can solve the differential equation. So, one of the easiest and the simplest one I showed you. I showed you its limitations and I showed you that many times in computing try not to really follow what are embedded in the governing equation and we try to do something artificial and we a have to be cautious about doing that. This is one example that I told you. Now, having obtained this expression for u xx let me drive your attention to another problem.

(Refer Slide Time: 50:32)



This problem is another very simple equation, which is prototypical of elliptical PDE and that is this, what is known as let us say the Laplace's equation, I could also keep the right hand side on having Poisson equation. So, if I try to do that now, what I could do? I could write the solution again x j and call it as y n and I will write it as u of j n like this; then, what are you now noticing, that your computational domain is in the xy plane and just for the sake of simplicity, let us say, we have identified a region like this and the first step will be, of course, deciding where how many points we would like to deploy in getting this computer distribution. So, we discretize the domain like this.

Let us look at say this is my jth point and this is let us say my nth index in the y direction. So, we are focusing our attention on a point like this - that is what we are doing here. Now, this equation, what we would be doing? We would be writing again a similar kind of a module, of course, here y is involved. So, when I am evaluating u xx, I keep y constant. So, I have omitted here, but you could write out a y in all the index. The same time, when you are trying to figure out u yy, you will keep x constant and you will write Taylor series in the y direction.

So, in the x direction when I try to do that I see that these three points are involved in u xx and when I am trying to discretize this, I am going to involve these these points. So what you are seeing here that you get a computational molecule here now like this and this is what is called as a Five Point Molecule; that is a very elementary discretization one can think of its used even now, time and again. So, that is your five point molecule and what you notice that if I write that equation down. Then, I will write it as u j plus 1n minus 2, u j n plus u j minus 1n by delta x square and will write u jn plus u jn plus u jn minus 1 by delta y square. So, that is equal to 0 and when you write it down, you find out that here u j n. If I write it down, put it on one side and 1 over delta x square, plus 1 over delta y square keep that 2 outside and the rest of this quantities you could just simply write. Look at the way this equation is written here, if I have to stack the knowns - now this elliptic problem, so, it is a boundary value problem; so, I can do any of my other tricks that we have used in the heat equation or used in the boundary layer equation.

What will happen? Each point will be influenced by the neighbors for this kind of discretization. If I write down all the unknowns, so what I will require boundary conditions here and I come back to this boundary condition, very interesting topic at a later time, but let us say we have information available on you and all the four sectors of the boundary and then what will happen? My unknowns will begin from here. This is where the unknowns will be described. So, what this would be? This would be j equal to 2 and this will be n equal to 2. So, I would have my unknowns, which will go like u 2, 2 and so and so forth. So, I will have u 2, 3; or j I am writing first; so next point would be 3, 2. So, we are following a particular pattern, we will write it like this, first row wise, then we will implement the column. This is what is called as a Lexicon graphics style, but we just run out of time. So, we will spend a little time talking about this issue and we will end this and we will move over to discussing something else.