Foundation of Scientific Computing

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Module No. # 01

Lecture No. # 39

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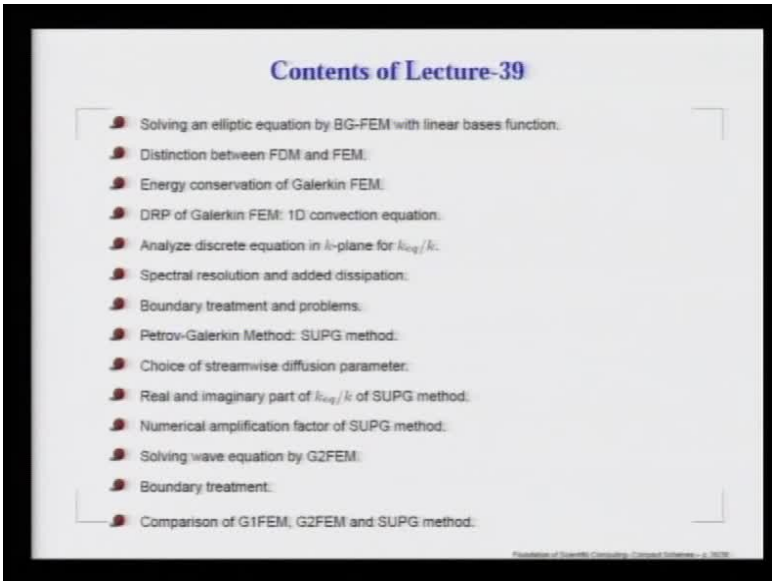

We start lecture 39 with our discussion on Bubnov-Galerkin finite element method in solving an elliptic equation with linear basis function, and when we go through this weak formulation, the special derivative term appears like a C D 2 expansion, and this is a erroneously been pointed out by various people. But we want to state unequivocally that this Bubnov-Galerkin method provides a much better dispersion relation than the equivalent FDM.

We also noticed that this Bubnov-Galerkin method has the energy conservation property and, we can, as we mentioned that we can look at the 1-D convection equation and workout the dispersion relation property of this. And as usual, we can take the FEM method in the wave number plane and find out k equivalent by k for this, and find out what is the spectral resolution achieved by this finite element method and what is the

added dissipation. We note that for the internal elements, Bubnov-Galerkin method is non-dissipative.

However, near the boundary we will have to have one sided elements and that essentially leads to the problem, exactly similar problems that we faced in the compact schemes; and this has been variously attempted in various versions of Petrov-Galerkin method. SUPG or stream line upwind Petrov-Galerkin method of Hughes and his colleagues fall in this category.

One of the aspects of this SUPG method is in choosing the stream wise diffusion parameter that is built-in into the method and this has been done following some classic work for, once again, a 1-D convection equation. But unfortunately, this diffusion parameter has been designed for a method for which one assumes that there is no error in time discretization.

Using those values of stream wise diffusion parameter, we can obtain the real and imaginary part of k equivalent by k of SUPG method and work out the numerical amplification factor. We can notice that this is a very dissipative method and which will not be able to solve 1-D convection equation at all.

So, that is why, we investigate the quadratic basis function based Galerkin method, look at its boundary treatment and compare various results of the linear basis function based FEM, the quadratic basis function FEM, and the SUPG method and with this will conclude this lecture.

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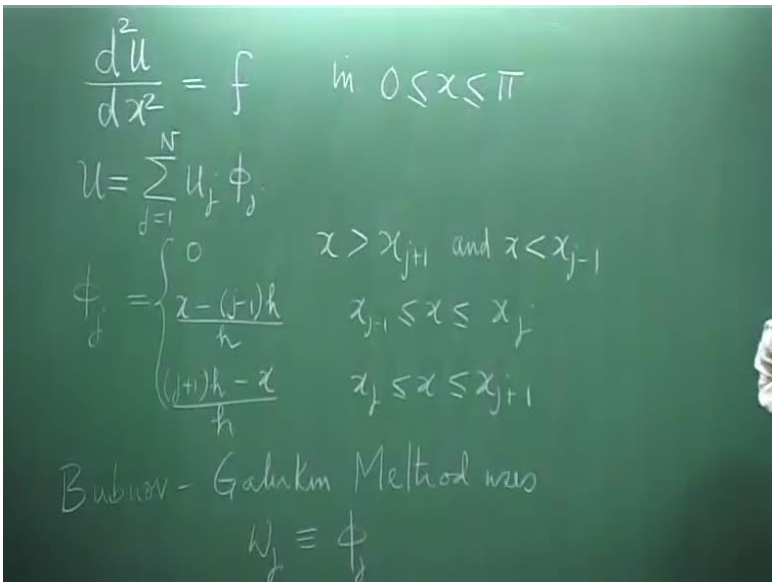

So, let us begin.

Recall that in the last class we started looking at an example of this type; how we apply a linear Galerkin type of a method is displayed here. So, what you write is u as a linear combination of the basis functions, as given here and the nodal values.

And if you recall, in the weighted residual method we had to choose the weight function which we called as w j; in Bubnov-Galerkin approach that w j is nothing but pi j itself.

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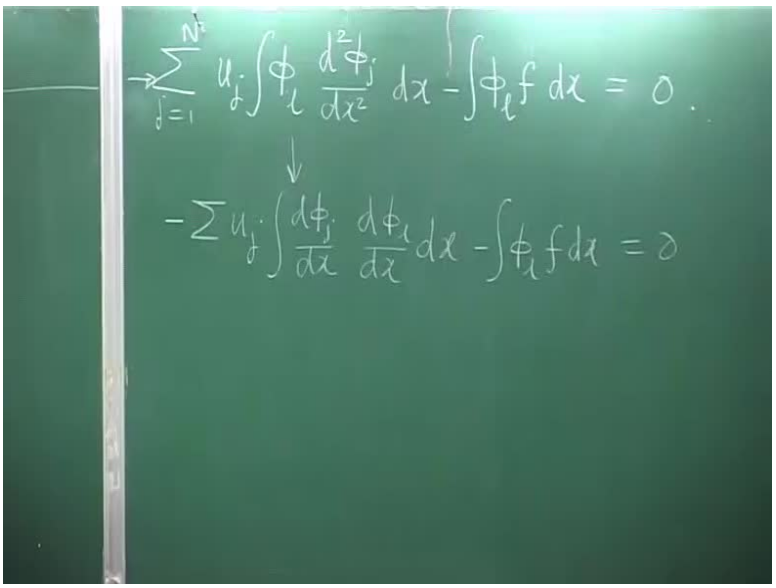

So, what then happens is then you substitute this into this, then multiply by the weight and then integrate and this is what you get; this is what we were doing in the last class.

So, we are not satisfying the differential equation, as it is, we are satisfying its weighted residual to be equal to zero, that is what is the weak form is. Now, what happens is we also noted that this basis function pi of j that we have taken here, they are linear.

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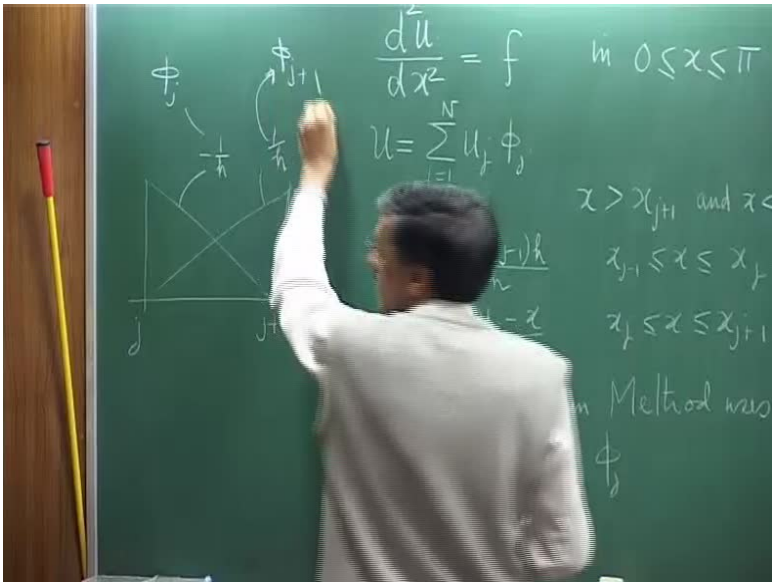

If they are linear, then, of course, you will not be able to support a second derivative; that is where integration by path comes to our rescue. You can spread this requirement on second derivative by doing integration by parts and this is what you get. And the first derivatives are obtained if you look at your basis functions like this, so one of which is basically minus h and this is 1 over plus h and what this was the jth node, this is j plus 1th node, so this curve belongs to pi j, whereas this curve belongs to pi j plus 1; that is the way we had done.

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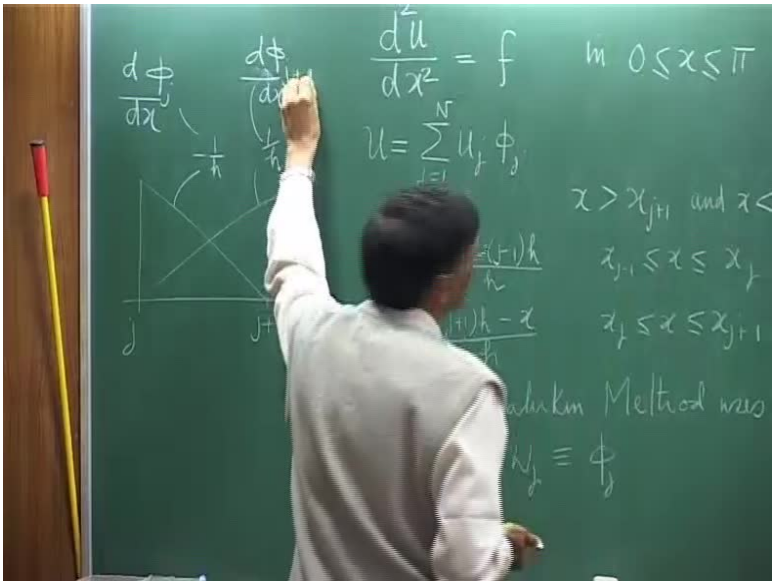

And then…, So we substitute it in here and then we will see that this j goes from 1 to n, but you will find the contribution will come from j is equal to l minus 1 l and l plus 1. And using this derivative information as we have noted here, the derivatives are given like this; so, this is essentially d pi j d x as this is the d pi j d x j plus 1, so that is what we are doing.

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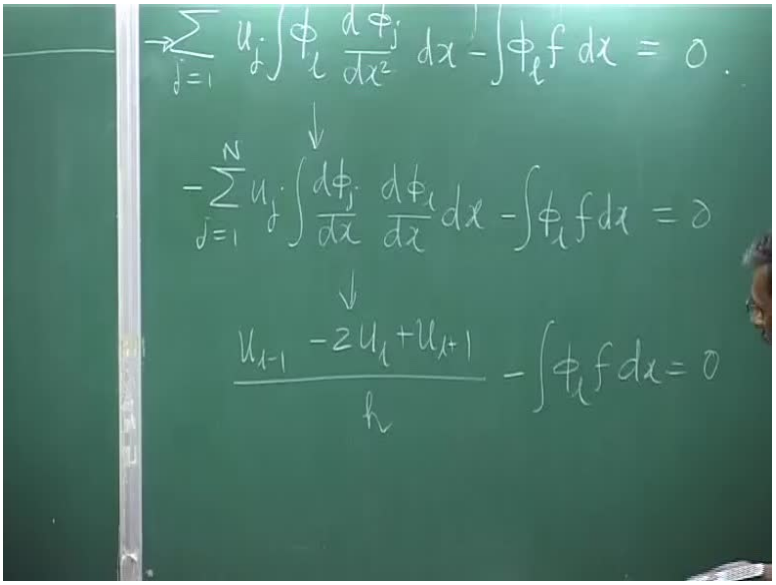

So, substituting that this part, this part comes down to u l minus 1 minus 2 u l plus u l plus 1 by h and then we have this quantity pi. So, before closing yesterday I mentioned to you that the second derivative, as it is here, this term contributes this term.

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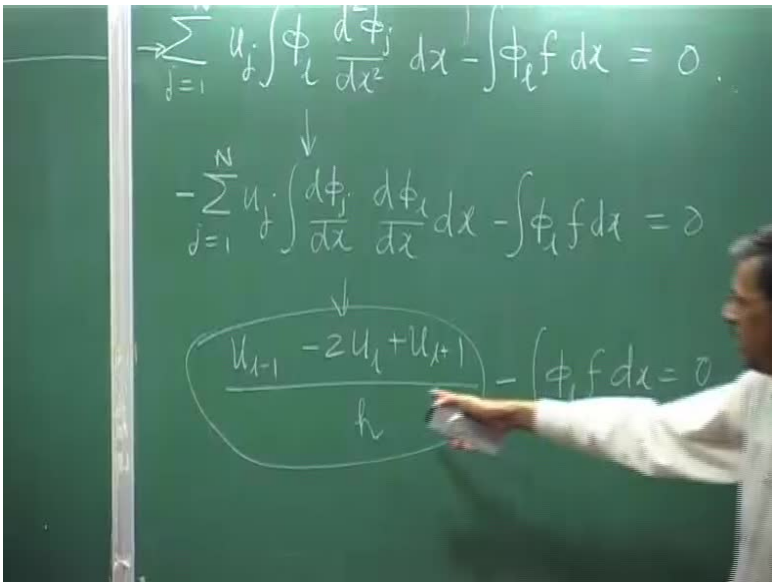

And you can see that if I divide it by h square, that is, something like your C D 2 kind of representation for the second derivative.

So, quite often, the knack you will find in many books, they will say that it is a second order accurate representation, which is misleading because you see although this looks like a C D 2 stencil, but this is not simply f of l.

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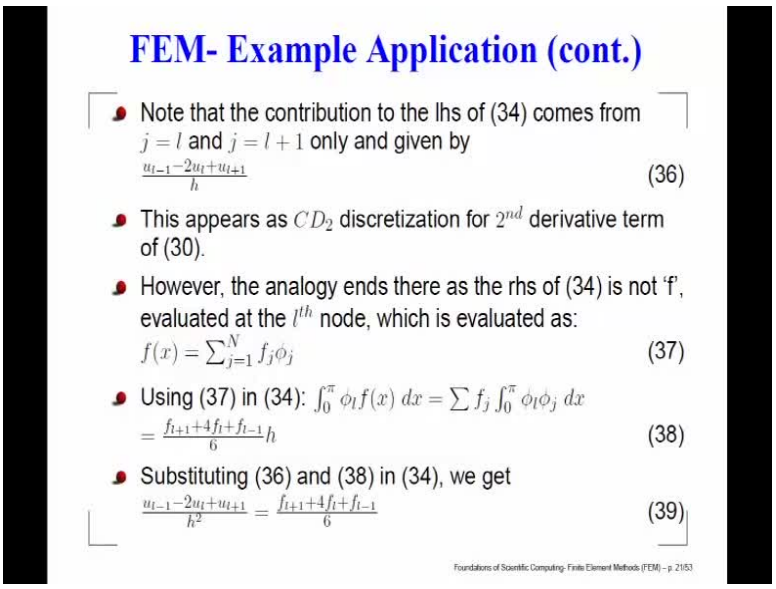

So, what you do actually to obtain this term? You also express f of x in terms of its various Galerkin nodes, so those are those nodal values of f j times pi j you do. And then

you multiply by f of l and integrate overall possible j's and once again you will see that it is only that the lth node and the neighbor on either side, l minus 1 and l plus 1 will contribute and that contribution is given here.

So, in the end what you are getting is, this is the discrete equation that you have. So, if you are doing finite difference type of calculation, let us say, with the second order central scheme, then left hand side would be same, but on the right hand side you would have gotten simply f of l.

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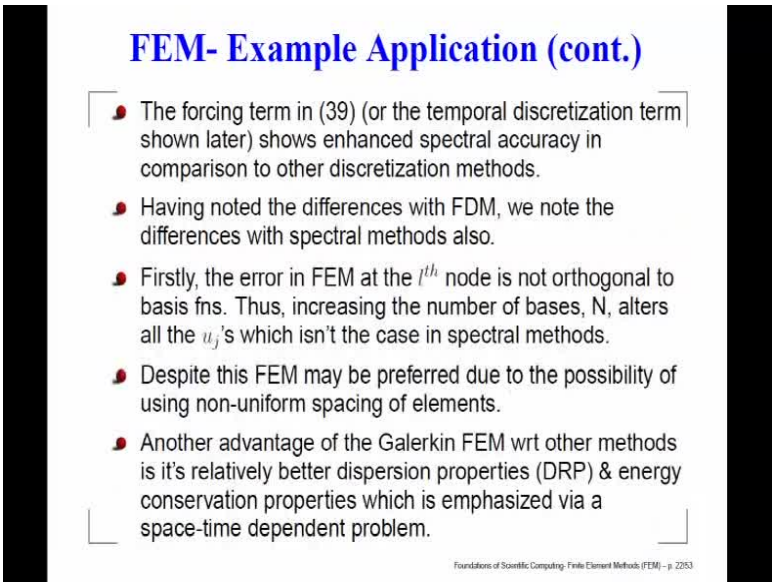

But, in this Bubnov-Galerkin approach, you are getting some kind of a weighted average of f on the right hand side also. Well, this actually leads to enhanced spectral accuracy in comparison to any other discretization method, and that we should be very easily be able to see because the way we have developed our analysis tool, we can figure that all.

But we should also note some differences why we are getting this spectral accuracy, although it is not exactly like spectral method? Because the error in FEM at the lth node is not really orthogonal to basis functions, that is what we have seen. What happens is increasing the number of basis functions n alters all the u j's, which is the case for spectral method, but FEM, it is a kind of a local adjustment.

Despite this non orthogonality of the basis function with the residue, we do use FEM because of its ease of coding and getting the solutions and because of its local nature of the solution.

And what is important for us - to realize that if you are solving some problem of the kind which we have been dealing in this course, kind of a disturbance propagation, then we require dispersion relation preservation property, and we will be able to show shortly that FEM actually give you a better DRP property than many other method, which is not always appreciated.

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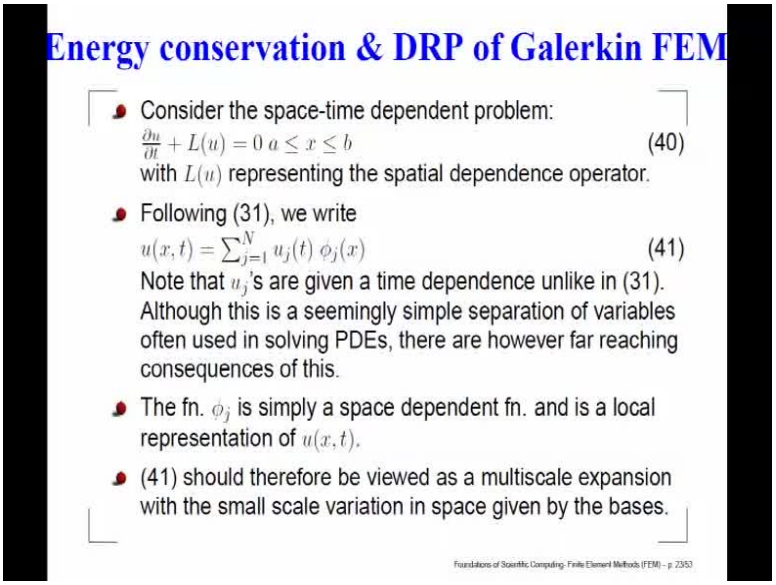

I will just skip this part, it just tells you that this Galerkin FEM has this unique property of preserving energy, it has been shown here by a representation of evaluation equation of this kind; so, whatever may be the equation after special discretization, you may be able to write it like this.

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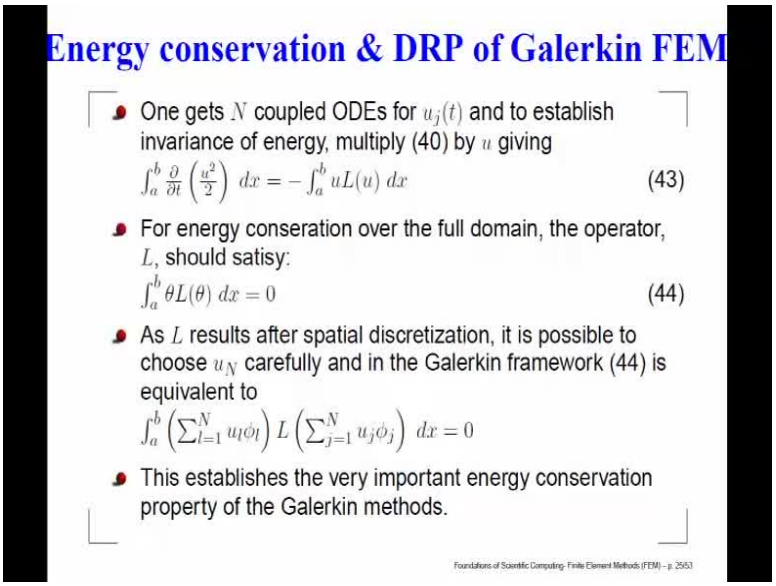

And if you are solving in a domain with x varying from a to b and you do this kind of, FEM kind of expansion in terms of the nodal values of u j of t, now what it amounts to is basically, if you multiply that equation by u, this is what you are going to get.

Now, if you integrate it over the whole domain, left hand side gives you a kind of an estimate for the energy - half u square is the energy. So, if you can show that your u is represented by some kind of a Galerkin expansion in such a way that the right hand side is identically equal to 0, then you have performed the energy conservation; so, that is essentially this.

So, basically after your spatial discretization, if any function theta could be written like this, then you have achieved the energy conservation and this is what is actually done in Galerkin method. So, that is why Galerkin method continues to be a method of choice for the very accurate calculations.

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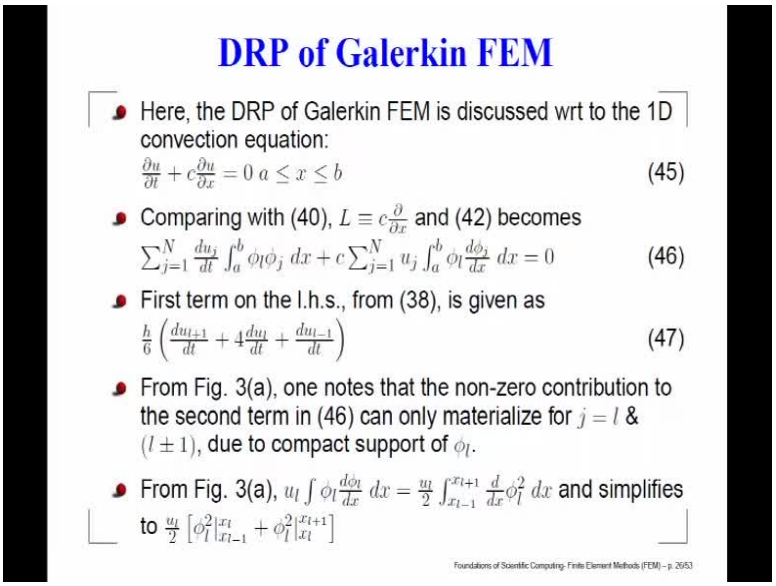

Now, I mention that, that we do get better dispersion relation preservation property of this Galerkin FEM. So, let us get back to our 1-D convection equation and then we go through this expansion of u like this, as we have indicated here and substitute it in the governing equation and then integrate over the whole domain from a to b and let us say, we have n such basis node, then the first term would give us this. So, you see, all time derivatives would be built-in in this because this is strictly a function of x, whereas this is a function of time.

So, basically, that del u del t, that partial would result in the ordinary derivative d d t of u j here, and once again you multiply by pi l, that would come from this term; the first term and the second term is written over here.

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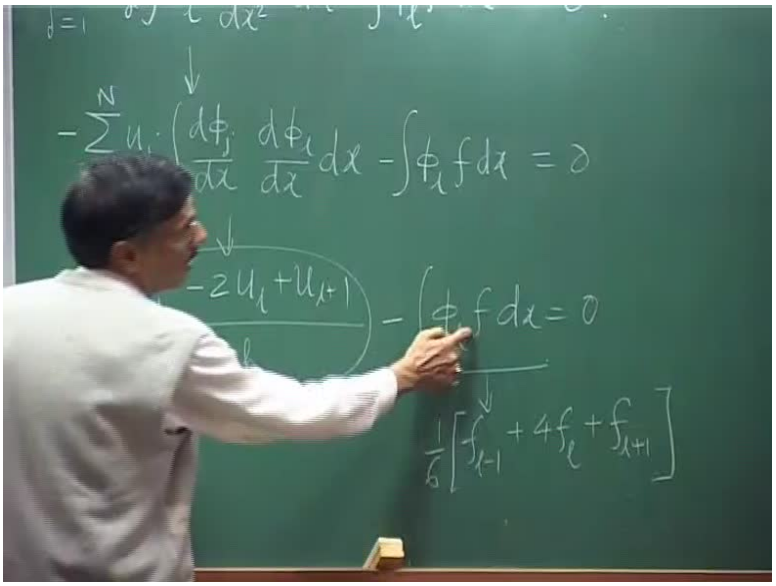

Now, this term, we have already seen this is like what we have just now done. See this term, we saw that this was nothing but one-sixth of f of l minus 1 plus 4 f of l and this is f l plus 1. So, this, this term is exactly like this term; think of f is nothing but d u j d t and then you basically get this term as this.

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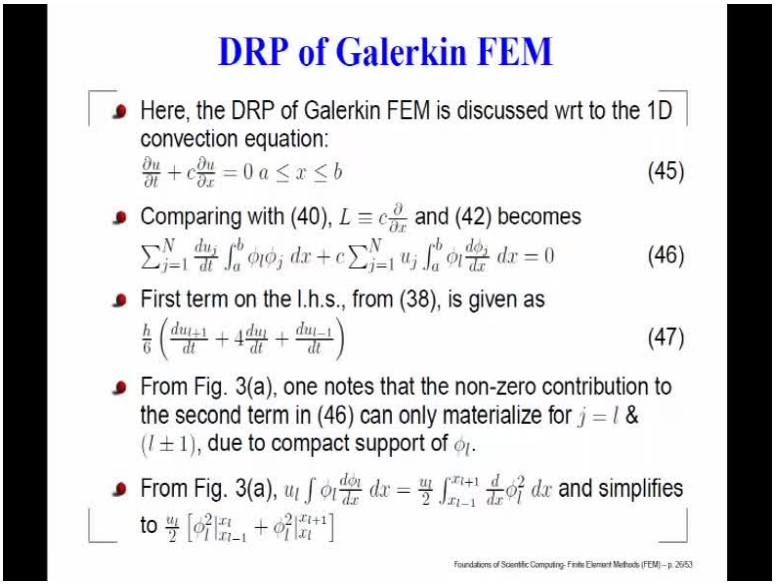

So, this is something that you notice that in solving that 1-D convection equation, the time derivative is treated differently in FEM as compared to other methods because you

are now taking a weighted average of the time derivative from the lth node and at l plus minus 1th node; so, this is a kind of a difference that we should have.

Now, we can estimate this term slowly, say for example, we are integrating over all possible j, so that contribution would come from j equal to l and l plus minus 1. For example, if I take j is equal to l, then I will get this term as phi l d d x of phi l in this.

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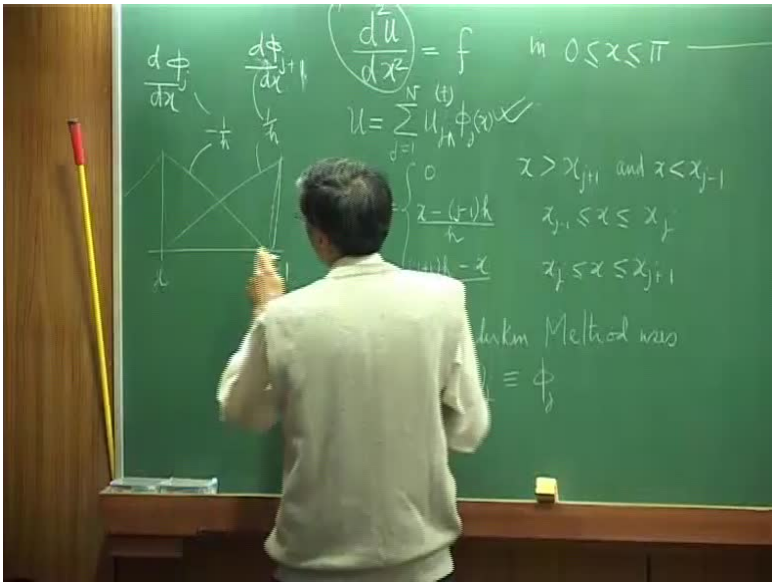

So, that is like u l by 2 d d x of phi l square d x and what do we get? We get this because the lth node is…, If this is my, say the lth node, so then what we are getting this is the function. So, this goes from l minus 1 to l plus 1; so that is what where we have the footprint of the node, is from x l minus 1 to x l plus 1 and if we substitute it there, we will see that there would be a total cancellation and there is no contribution coming from l.

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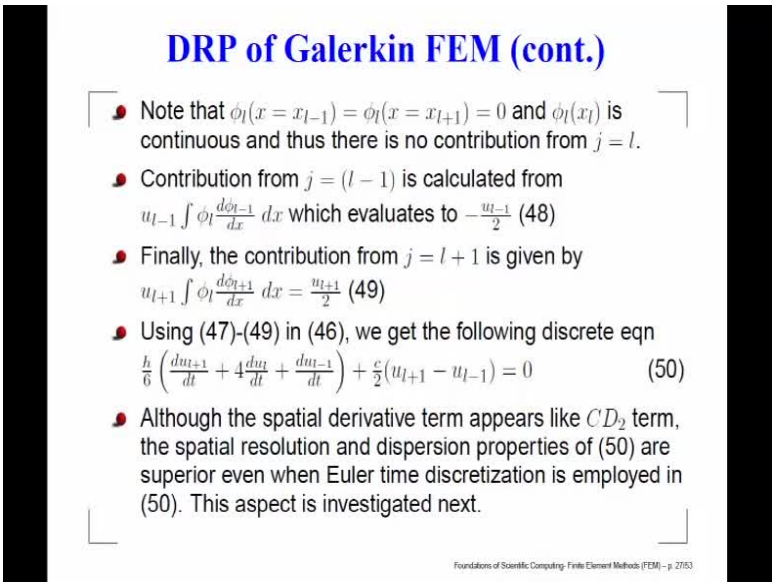

So, that is why, we find that j equal to l does not contribute to for that integral by any amount. Next, look at the interaction of the lth node with the left node by taking j is equal to l minus 1 and we can perform this integral because we know what this derivative is. Suppose, we are looking at say the lth node, so this lth node is this and l minus 1 is this.

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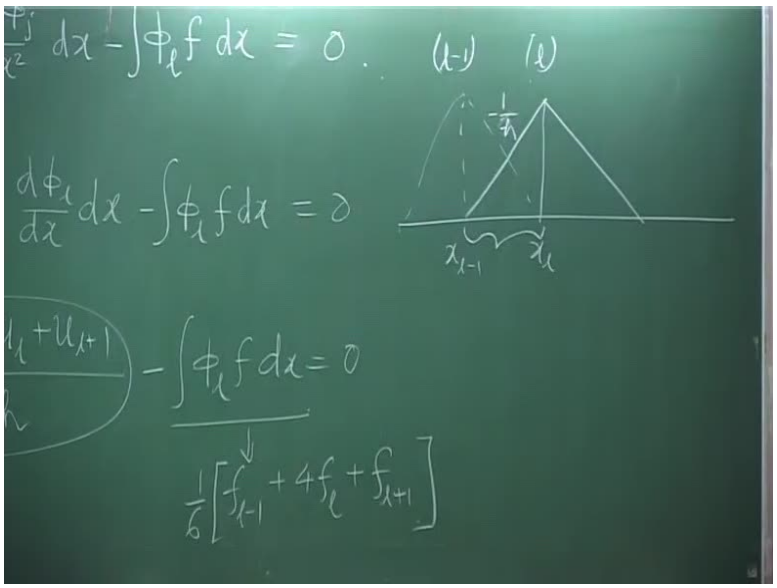

So, this integral that we are performing, here l minus 1 interacting with lth node, so this is your l minus 1th node and this is your lth node; so, the interaction would come only

from this region. So, that would be nothing but integrating from this will be x l minus 1 and this is x l; so, that is what we do.

And what is d phi l minus 1 d x? That is this slope, so this slope is what we know is minus one over h.

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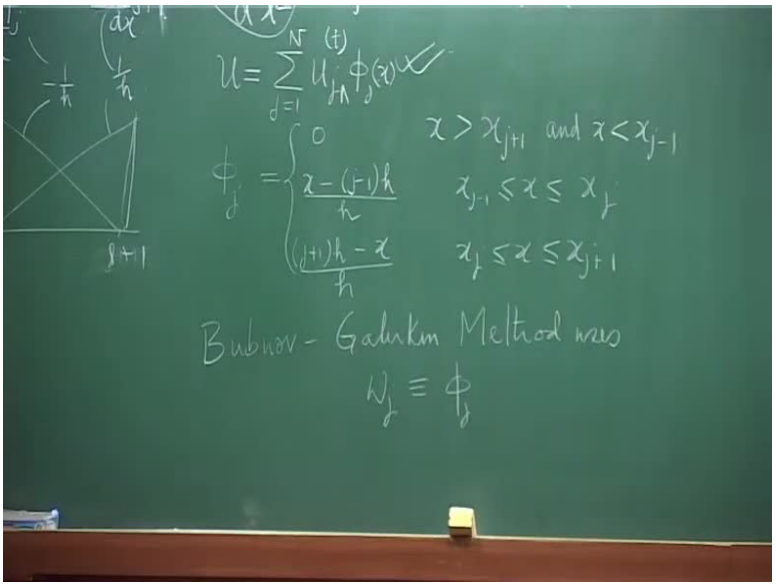

So, I will just simply replace this 1 by minus 1 over h and phi l expression we have written here and we will see that would be this part, phi l minus 1 would come from this part; so, that would give you this quantity.

So, I have omitted those steps, but you can do it yourself, that would work out to u l minus 1 by 2. By the same way, you can find out the contribution coming from l plus 1th node; so, that would be this path.

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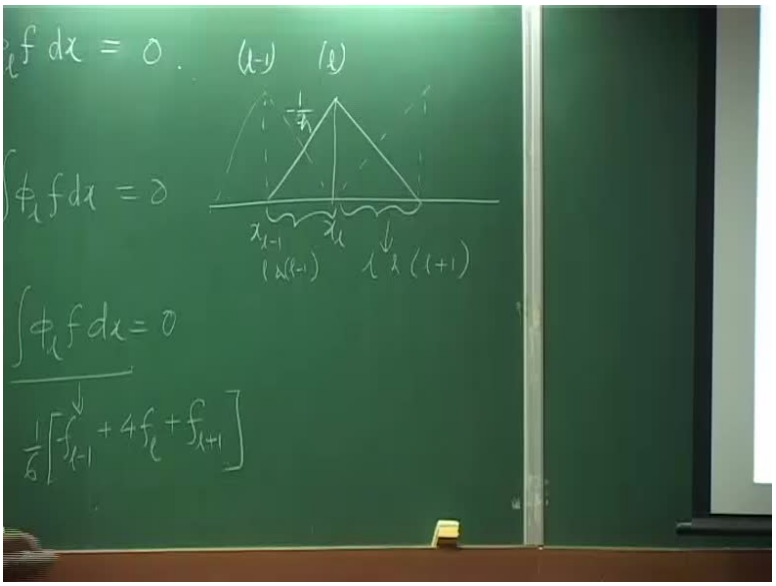

So, I am talking about interaction that will come from this, so this is the interaction between l and l minus 1 and this 1 would be l and l plus 1, that is what we are going to get and that would give us u l plus 1 by 2.

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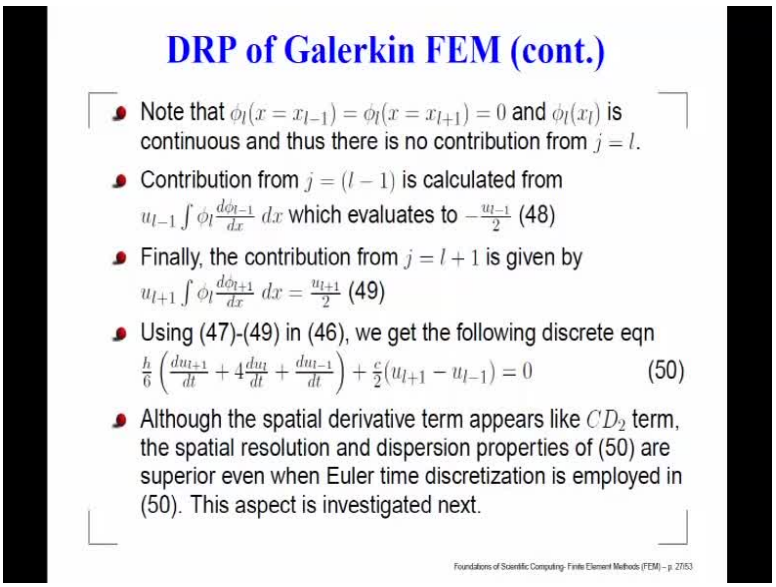

So, what really happens is, then your discrete equation for the convection equation comes out like this. Once again you notice, that this is that familiar C D 2 kind of representation of del u del x term, but look at the time derivative, it is a kind of a weighted average of u l obtained from its two of neighbors as well.

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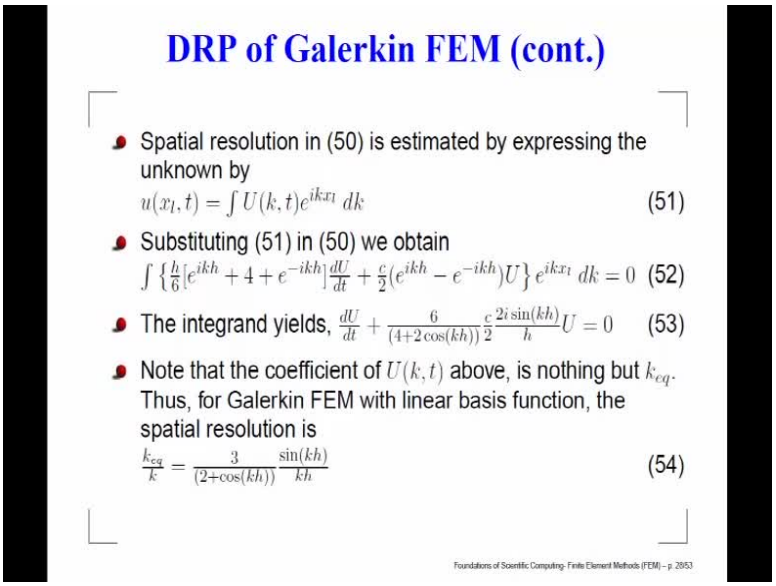

So, what happens is let us look at how we can analyze it by representing this again in this kind of a Fourier representation? Fourier Laplace representation at x l, I would just simply put the phase at i k x l and t dependence kept in U and substitute it in the equation that we have it here.

So, we are going to write these as the…, that in this form, and then wherever I have x l plus 1, I will write it as i k x l plus i k h; so, those ones will contribute here. So, this first term that you have seen, that contribution is coming from u l plus 1, the derivative, time derivative; so, that is it that is equal to nothing but d u d t times c to the power i k h. And the middle term is simply 4 d u l d t. So, that is what we are getting here.

And the, this term is of course, the u l minus 1 d t; that term will give you e to the power minus i k h and whereas, the spatial derivative terms give you again u l plus 1 and u l minus 1. So, they, those give you this e to the power i k h minus e to the power minus i k h.

So, basically then rearrangement of this would lead you to this equation and what you notice is then, that it is equivalent to as if you have evaluated the derivative at the lth node itself, which is given here, and the rest of the term, the product is nothing but what it is your like i k equivalent, is not it?

So, what happens is, you are noticing that i k equivalent form FEM is not what you would have gotten form C D 2. C D 2 would have given you simply sine k h by h, 2 sin k h by h, but here because of this weighted averaging of this term, you get this additional term in the front here 6 by 4 plus 2 cos k h.

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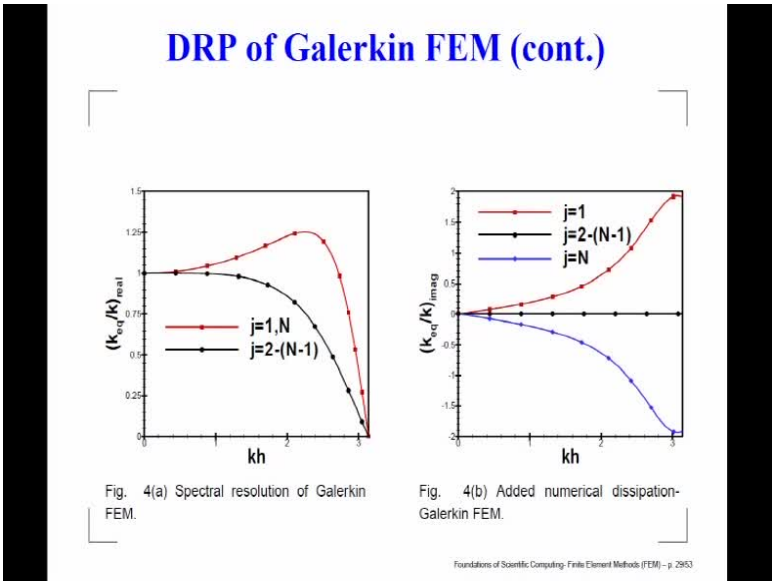

So, basically then this quantity that we have here is nothing but our k equivalent by k. So, what happens is, as a consequence you will find that in the interior nodes, that what we have just now shown, we get k equivalent by k, the real part goes like this. And let me just point it out to you, that this is far more accurate than your C D 2 representation.

If you would have taken a C D 2 representation, that would have started from 1 here, but that k equivalent by k curve would have been part below this; whereas, you are preserving that k equivalent by k very far into this k h range, it is more than about 1.3, 1.4 as compared to 0.25 to 0.34 C D 2.

Whereas, of course, the imaginary part you will not get anything because it is a kind of a central representation and you do not have that; only thing is you will have to do something more for the nodes, which are at the end of the domain.

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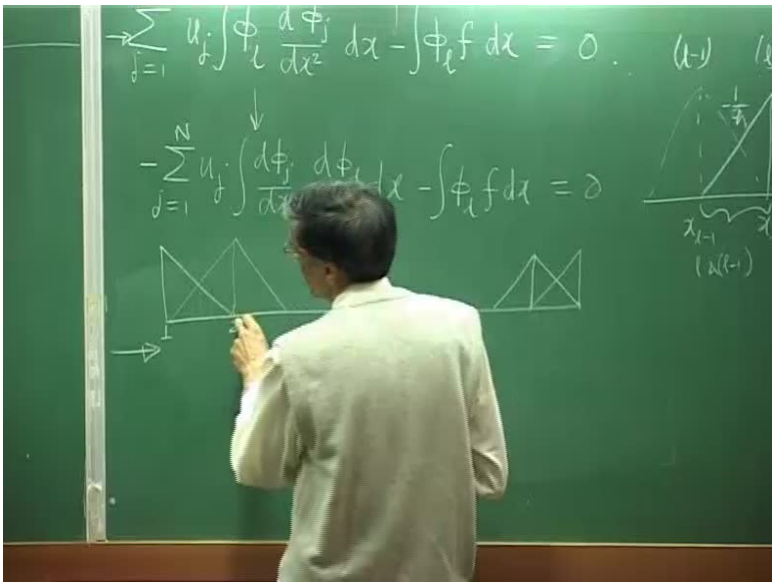

Let us try to just explain it a bit. Suppose, this is my domain and this is my first element, first and last element; we integrate it like this, the second element is, of course, like this. So, the second element is a regular element with the two paths, but the first element only has this path because what you want that in this range, you want that everywhere, this plus this should add up to 1 and that is what you get by having this.

So, the same way for n minus 1th node also, you would get this kind of a representation and this makes it somewhat of a one sided kind and that is what we are seeing here.

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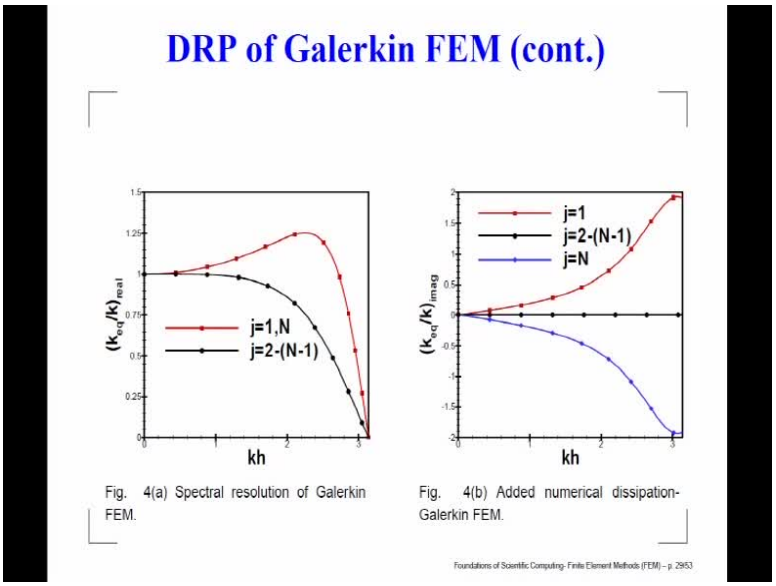

If we evaluate that differential equation for j equal to 1 or n with the stencil, the discrete equation different and that would make this real path obtain a overshoot, whereas the imaginary path is going to be of the following kind.

Now, for this wave equation, as we have seen, is going from left to right and because…, So, basically your convection direction is this and for this, the first node, what you are getting the information? You are getting it from inside and that leads to kind of a numerical instability and that is what is indicated by this imaginary path taking a very high value.

So, at the inflow you have instability problem for FEM kind of representation, whereas if you look at the last node, it is just the opposite; so, these two are basically nothing but mirror image of each other.

That is because of the central nature of the thing, but you understand that this is one of the drawbacks of FEM, like even in compact scheme that if you are not careful in handling the near boundaries or points correctly, we may get into physical instabilities.

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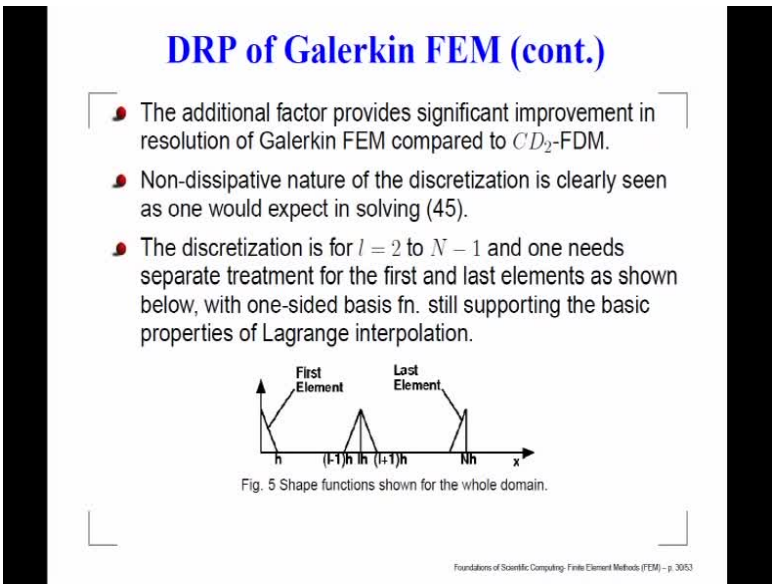

However, what we could do is that is what I just now shown you, how you handle the first and last element and then you can actually discretize, perform the discretization from l equal to 2 to n minus 1, but you have to have a special treatment like this.

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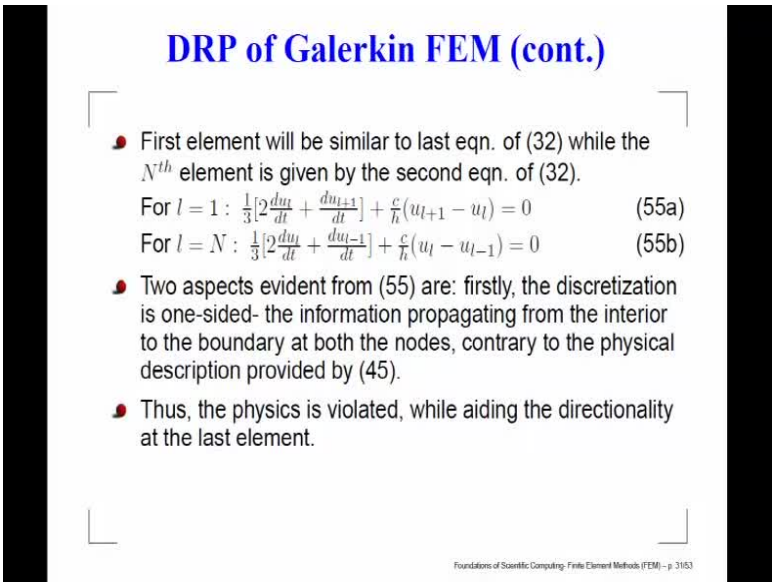

For example, if I do that what I just now explained - for l equal to 1, this is the discrete equation I would get and for l equal to n, I get this, and this is not a symmetric representation. You can see, this is u l plus 1 minus u l and this is what I was saying that information is propagating in the wrong direction because you are sitting at l equal to 1; but the information is coming from l equal to 2 and which is a wrong attribute physically, whereas for l equal to n, this is o.k. because you are going from inside the domain towards the outside; so, that does not cause any problem. In fact, that causes extra dissipation.

So, thus, we can conclude that the physics is somewhat violated while aiding the directionality at the last element.

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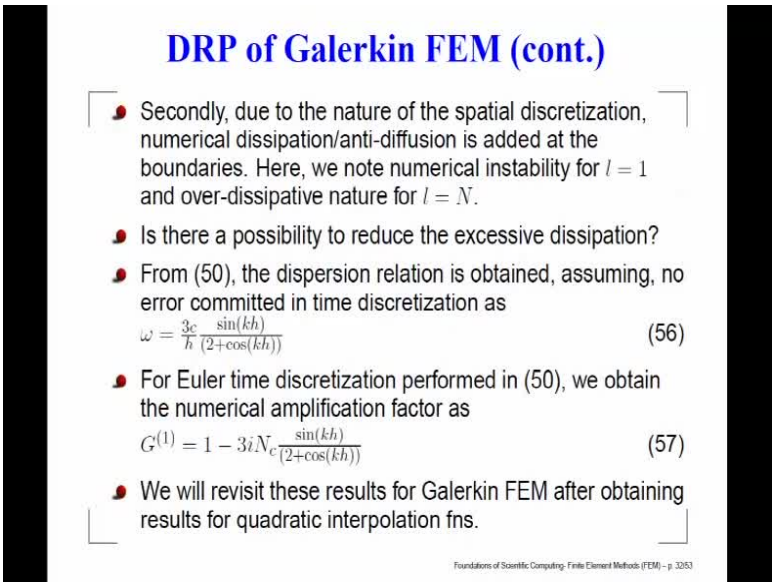

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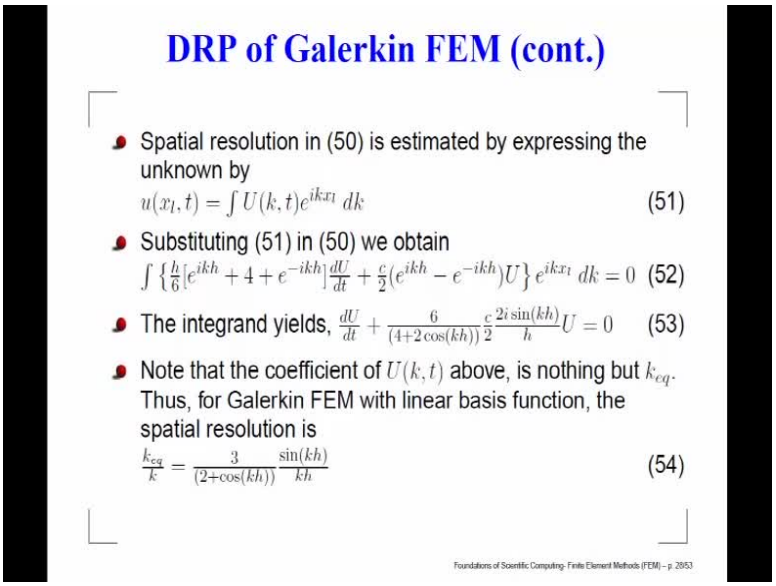

Now, let us get back to this discussion of the dispersion property. We have noted, we have noted here this is this, so even if I write d u d t is equal to something like your i omega and then transform, so this equation would give you an expression for omega. That is what we are saying here that the dispersion relation is obtained, let us say, by assuming that the time integration is the exact. So, d u d t i will just simply write as i omega you had. So, that means that omega is nothing but this factor; that is what we obtained in that expression 50.

So, basically once you have the expression for omega, you can calculate the corresponding group velocity by taking d omega d k, but that would be somewhat artificial because we are taking the time discretization to be exact; that is how we get it.

However, what you could do is you could decide having obtained, let us say, discrete equation of this kind, what we have written here 50.

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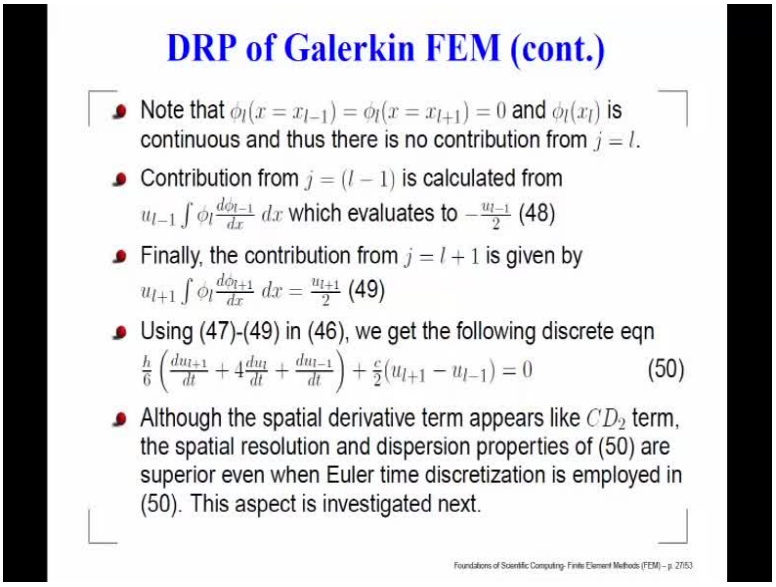

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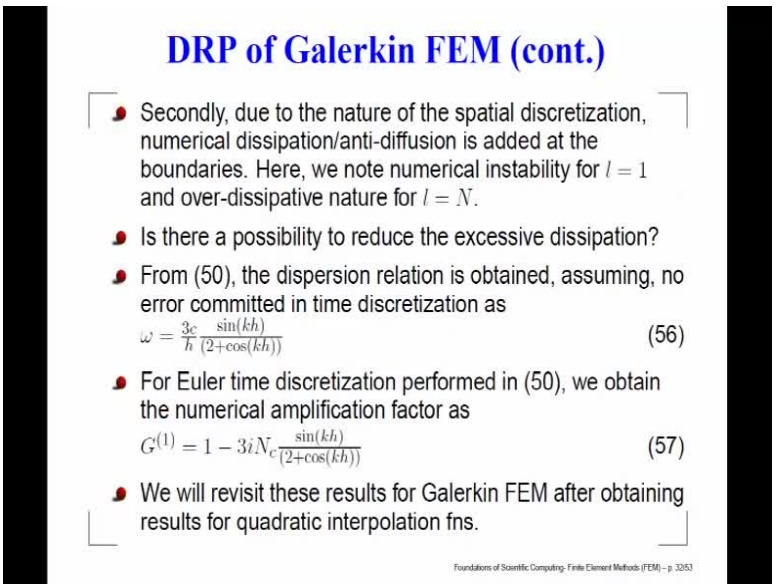

Suppose, we perform a Euler time integration, so in each of this term I will write it as u l plus 1 at n plus 1th time level minus u l plus 1 at nth time level and then apply that

Fourier-Laplace transform, then we will get this G of1, G of 1. What, why did I put it as superscript within bracket 1 is because this is a p equal to 1 polynomial we have taken; we have taken a liner basis function.

So, we will call that as G of 1 and substitute it there and do this analysis the usual way that we are familiar with; now, this G of 1 comes out like this.

So what does this G of 1 represent? You can very clearly see what modulus of this will be greater than 1, so this is going to be numerically unstable; so, that is one of the issue of Galerkin method.

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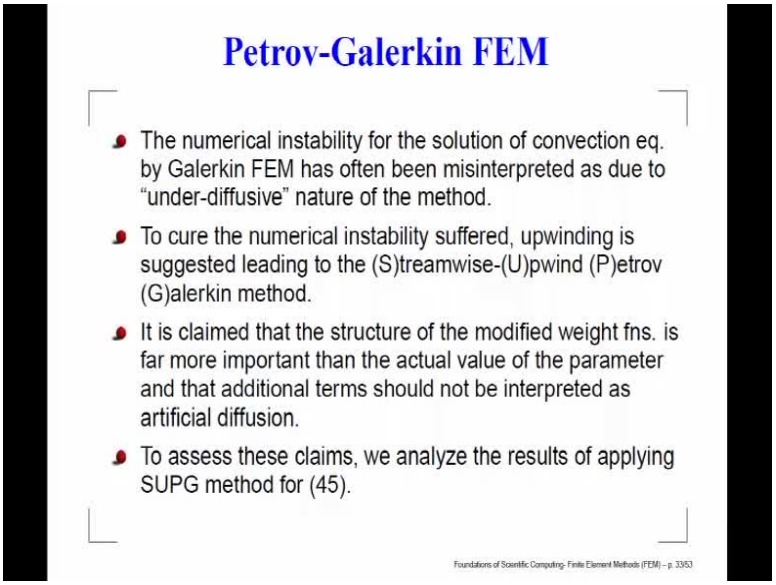

So, this had created lot of problems for FEM development when you look at unsteady problems. If you are doing, let us say, structures structural analysis where you are looking at steady state, that is where Galerkin methods seems to have no problem because there is no time dependence coming is…; the problem comes when you have coupled space-time dependence.

And that is what prompted the early practitioners who wanted to use finite element method in fluid dynamics scenario. They realized that you will have to develop a methodology which will work when you have simultaneous space and time dependence and this is where they gave up on Bubnov-Galerkin approach by not taking this.

So, what (( )) did was they switched over to Petrov-Galerkin method and in Petrov-Galerkin method, what you do is you do not use phi j equal to, oh sorry, w j equal to phi j, you just give up that option.

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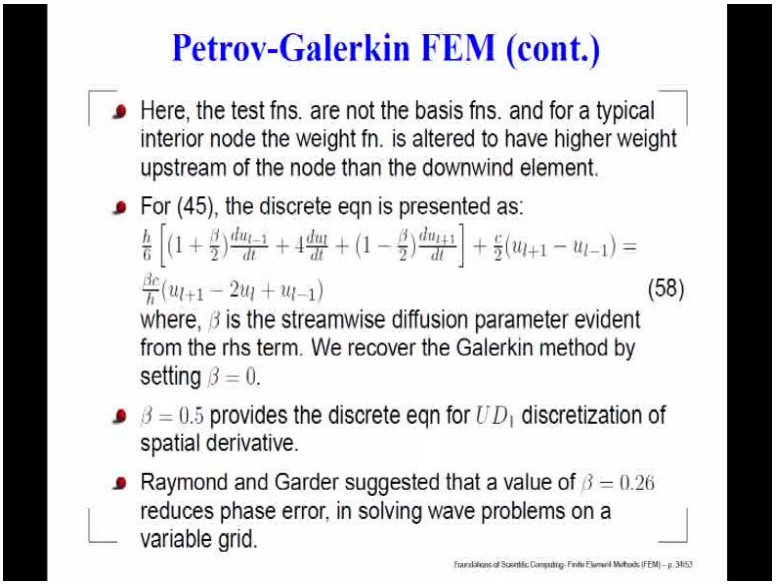

So, anyway we will talk about this Petrov-Galerkin method a bit now. One of the most used method is that SUPG, it is called stream wise upwind Petrov-Galerkin method. This seems to be used by many, many people, let us (( )) what this is, I will just put some results.

What do you do is basically you take weight functions, which is now not like your Bubnov-Galerkin method. In the Bubnov Galerkin method we took w j as phi j, so if I am looking at this, I have equal weightage coming from both the side because this function is symmetric.

In Petrov-Galerkin method, what they instead say, that you actually look at the weight function which should have higher weight upstream of the node than in the downstream. So, if, let us say, the information is propagating in this way, then I would perhaps take a function which will be skewed like this and which may be weighted more heavily on the upwind side.

So, this is your Petrov-Galerkin kind of an approach and if you do some such thing, which was done by, once again, by weather prediction people begin with; so, they found out that there is another parameter that comes in, that actually gives this bias.

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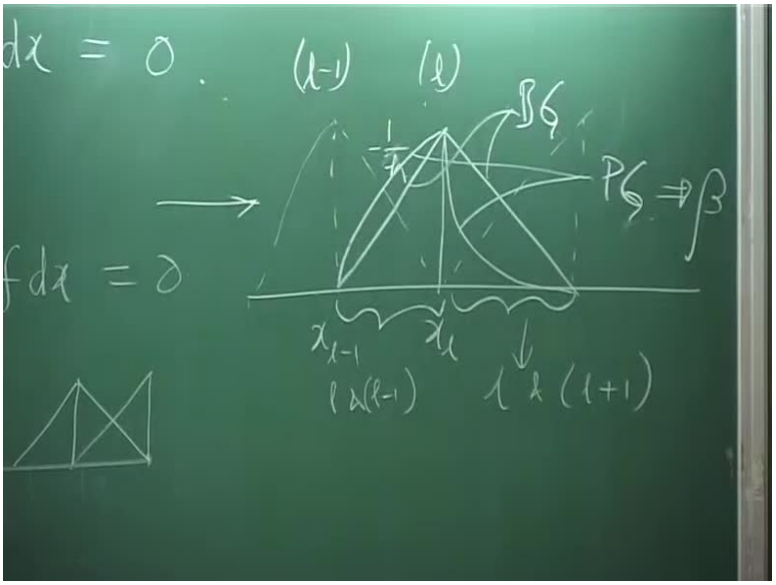

So, this is your Bubnov-Galerkin method and this is your Petrov-Galerkin method; so, these two are Petrov-Galerkin method and this is the Bubnov-Galerkin method and this biasing in Petrov-Galerkin method comes through this factor, parameter called beta, which is called the stream-wise diffusion parameter.

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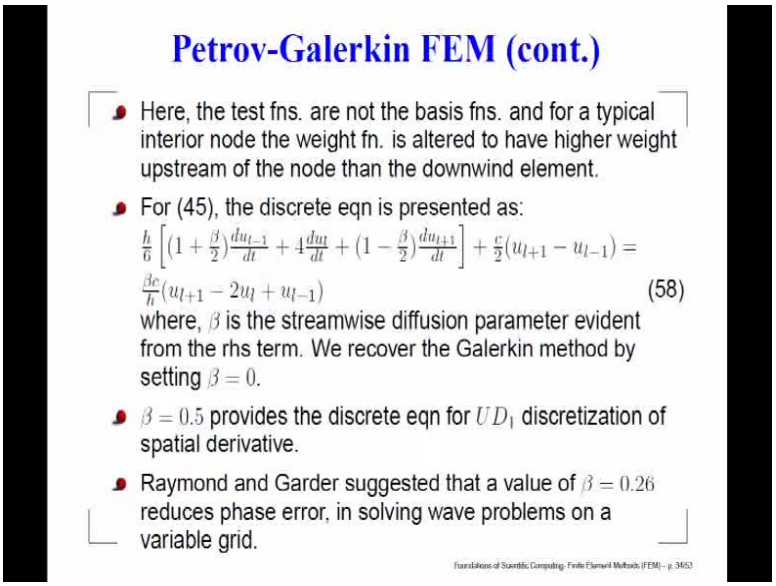

Now, I am not going to go through this, it is a quite elaborate area. There are thick volumes speaking about this method itself.

So, we would not go about it, but what happens is we get almost a similar thing like what we had obtained for Bubnov-Galerkin when it comes to c times del u del x term; that is exactly there. But you also get this additional term, it is proportional to beta - the stream wise diffusion parameter.

What you find, this is nothing but your second derivative. So, in the name of up winding in this Petrov-Galerkin method through this parameter, beta is equivalent to actually adding dissipation to the background; that is very evident from this term.

This is nothing but your second derivative of u with respect to x that comes out and the time derivative has the same structure as before, except that you have 1 plus beta by 2 and here you have 1 minus beta by 2, but still you get the weighting similar.

So, what you find that if you substitute beta equal to 0 in this equation, you get back to your Galerkin method. We also can check it out that if we put beta equal to half, then that would be like what we had done for finite difference first order up-winding case.

That is, that is, what we are referring to here that the first order up-winding via finite difference gives us something like u l plus 1 u l by h.

So, that is, that would be written equivalent to u l plus 1 minus u l minus 1 by 2 and then you will have to add this part up, and when you put beta equal to half, you will see that will exactly map.

So, you can see that beta gives you an additional degree of freedom to actually use Petrov-Galerkin method. And these two gentlemen from weather prediction side, they did some analysis on wave propagation problem and they suggested that you choose beta equal to 0.26 and you get pretty good results.

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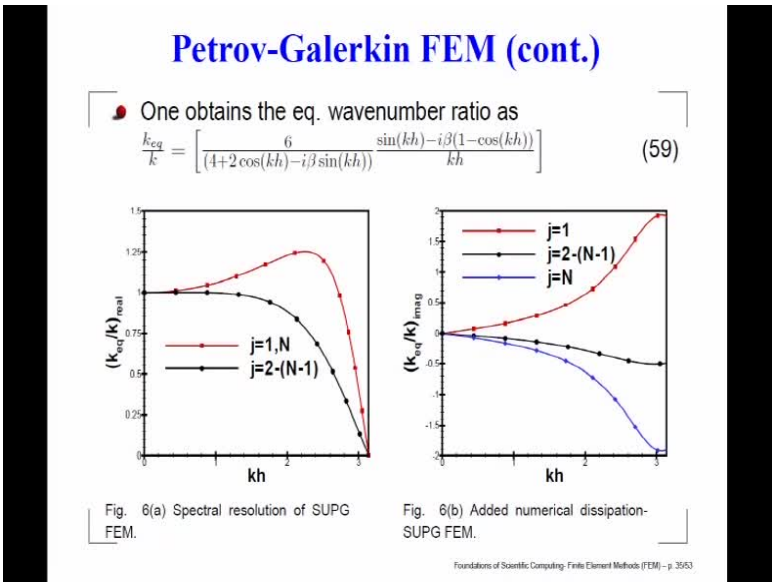

We will see such results shortly, but before we do that we can find out that this Petrov-Galerkin method that we are talking about, gives us a k equivalent by k given by this function.

Previously, what you saw that this function was purely real, but because now we have resorted to up-winding, you will see it will become a complex quantity and that is what you are seeing, sign k h minus i beta times 1 minus cos t.

Similarly, here also there is imaginary part involved and what does that do is that if you plot the real and imaginary part, the real part looks exactly like what we have gotten in Bubnov-Galerkin method; there is absolutely no difference.

So, this figure is exactly like what you would have gotten with the Bubnov-Galerkin method. What differs now is the imaginary part. Because of up-winding, what you would find that you would get a negative value of k equivalent by k and that is what you are seeing in this black line, this is for the interior nodes; you find that the significant amount of numerical dissipation is added.

And please do understand this value of k equivalent by k imaginary. Even at the Nyquist limit you can see this value is something like about 0.3 or they are about, and if you compare with what we had done in compact scheme, this is hugely dissipated scheme; this is very, very dissipated scheme.

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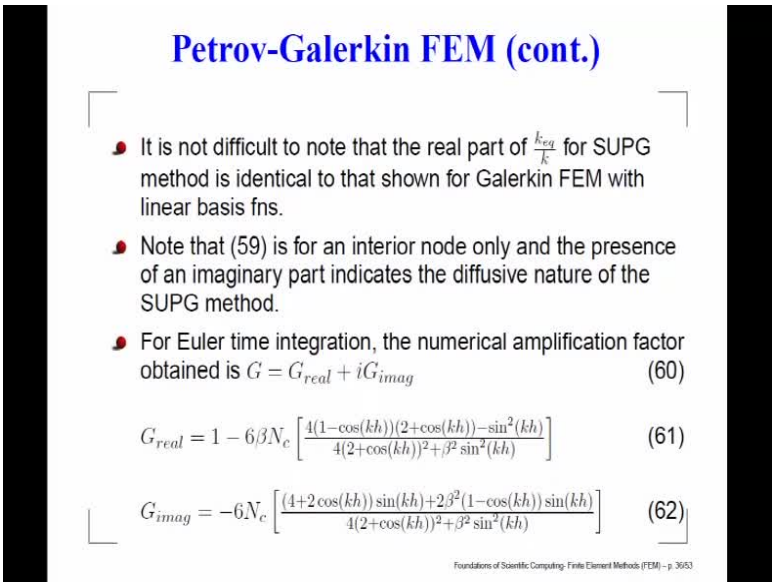

So, what happens is that should remind you that when we were talking about error propagation equation, I compared all four methods; I compared a C D 2 method with a compact scheme, then this SUPG method, and that method that is used in fluent, that quick. And what we found, of course, that the fluent and SUPG actually removed the signal and that was the reason.

Because of this if you look at the G path, you will find that depending on beta, G real itself will bring down the value from your ideal value of work; so if beta equal to 0, then you would not have gotten the second part.

So, here in SUPG that attenuation comes through this path where u deviate severely from its ideal value of 1 via the second term and of course, you have this imaginary part as well. And having obtained the real and imaginary part, we can calculate that beta, if you recall; and from beta we can calculate the c n, the numerical phase speed and from there we can calculate the group velocity.

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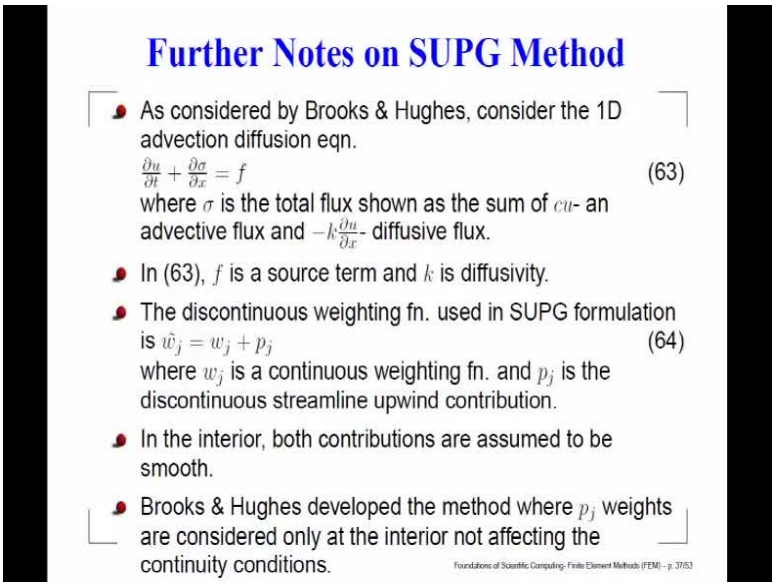

And doing all these analysis, Raymond and Garder did talk about a, did talk about a better wave propagation property for beta equal to 0.26.

I am again going to skip this, you can read it leisurely. This is little more explanation of SUPG as given by Professor Brooks and sorry, Professor Hughes and his student Brook and it is, it is basically trying to sanctify the usage of up-winding by claiming that you are adding some kind of a cancer stress term; a stress term.

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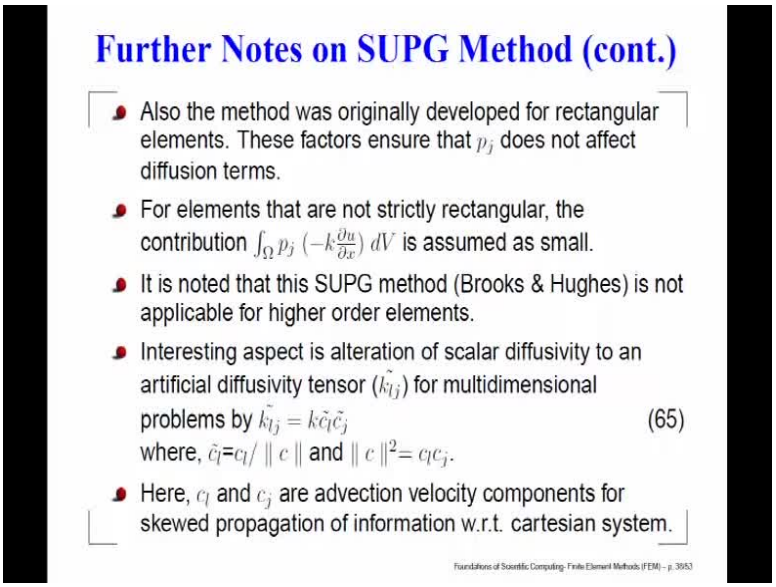

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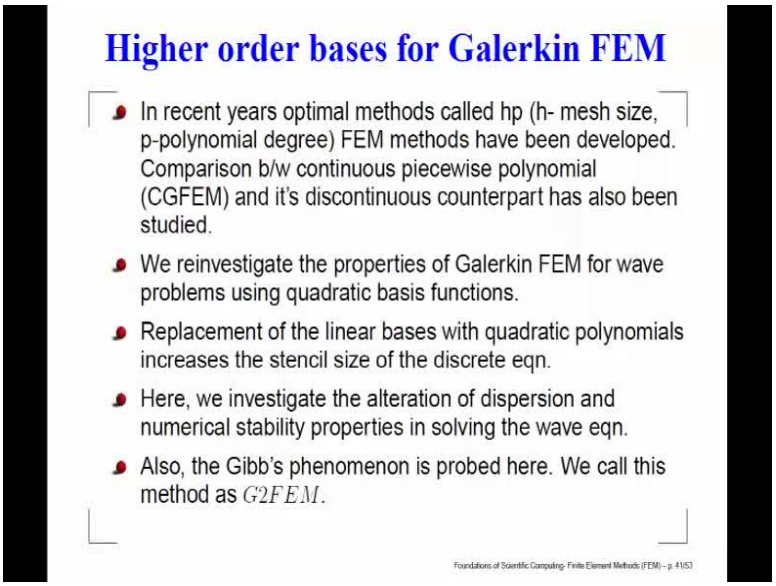

Let us omit that part where we will not talk about that. However, let us come back to the dispersion that we were having, namely that what is now called as the hp element method, which depends on simultaneous refinement of the mesh, that is, reducing h and increase in the order of the polynomial p.

So, if we look at this, the first candidate would be, what we have done is trying to take the basis function in terms of a quadratic polynomial and this we will be calling as G 2 FEM. So, this is the second order polynomial - the Galerkin expansion second order polynomial finite element method.

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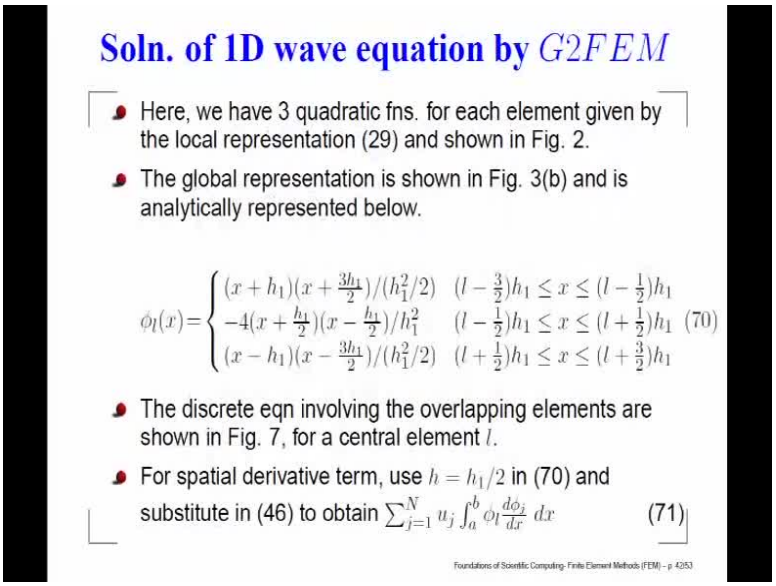

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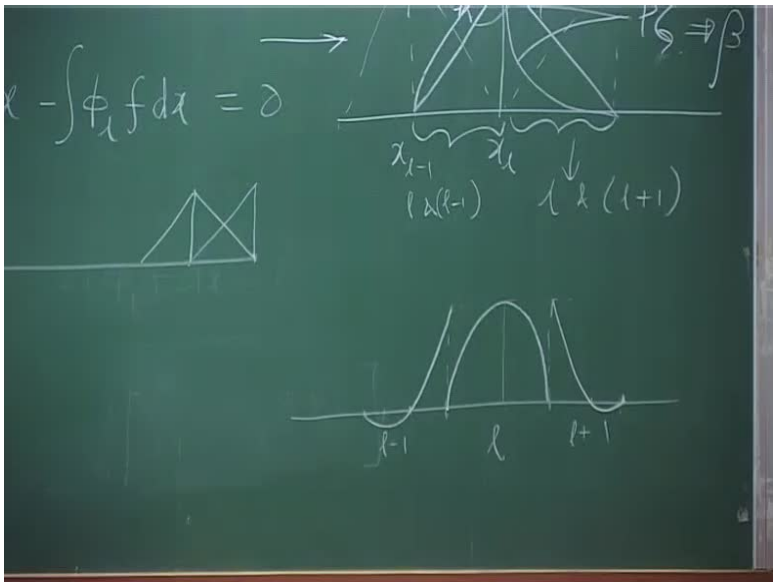

So, this we have already seen the basis functions look like this. So, we have those three sets of functions, if I do that again, so this is my point; so, the, this is one node, so this is let us say the lth node and I will have a midpoint.

So, this is my l plus 1th node and this is the midpoint and this is my l minus 1; so, what happens is lth node is defined into 3 parts - one is this part, that is given by this middle equation here, this is this path that I just now drew and the other path would come from

here, this will be like this and here we will have the third part which will come like this, that would be like this.

So, basically, that the top function is the left one, the middle one is this parabolic curve that we see here symmetric and this is that right hand element; so, this is what you get from the quadratic polynomial used in this.

Now, what we are going to do is we again look at the wave equation that is our perennial test bed. We will use that and note that we have added a midpoint as the additional node.

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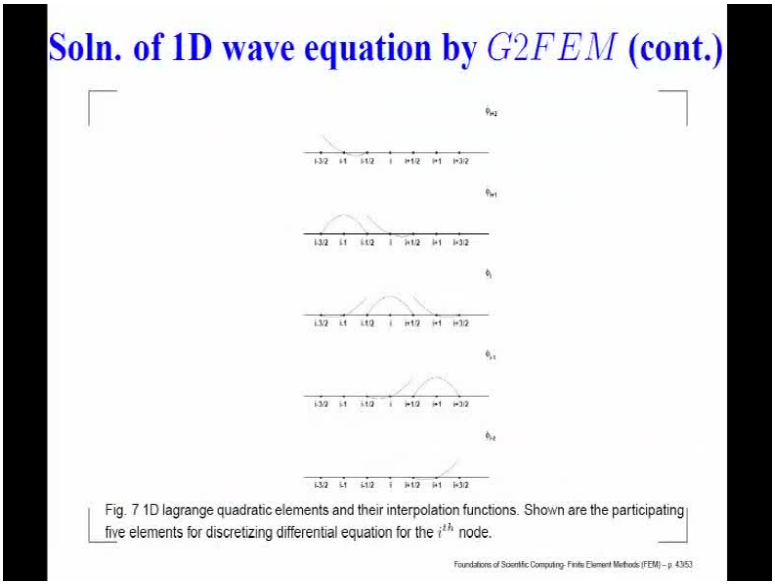

So, that is what we do that we define a h which will be nothing but h l by 2 and then we go through this exercise and this exercise is not trivial, but this is a figure that would actually explain to you how you evaluate the terms.

So, let us say, I am trying to evaluate the terms for the lth node, so these are my basis functions, this is the middle path, this is the left path and this is the right path. Now, we will have to see, which are the nodes that are going to interact with it to produce nontrivial contribution?

For example, for phi l, I will have contribution coming from l minus 1 and that is what it is. So, I will have a function which is a parabolic function, which is there and plus the right element, that is there. It will also have contribution coming from f of l minus 2, that

would come from this path only, the right hand path; the other part is, of course, here and they do not have common interface with the lth node.

The same way, if you look at the contribution coming from the right hand neighbors, you will find that contribution will come from l plus 1 and l plus 2, and this is the way that we will have to find out that there are actually seven Lagrange quadratic elements. And this is the way the functions look like and you would be seeing that if you are looking at the lth node, the contribution is going to come from l minus 2 to l plus 2.

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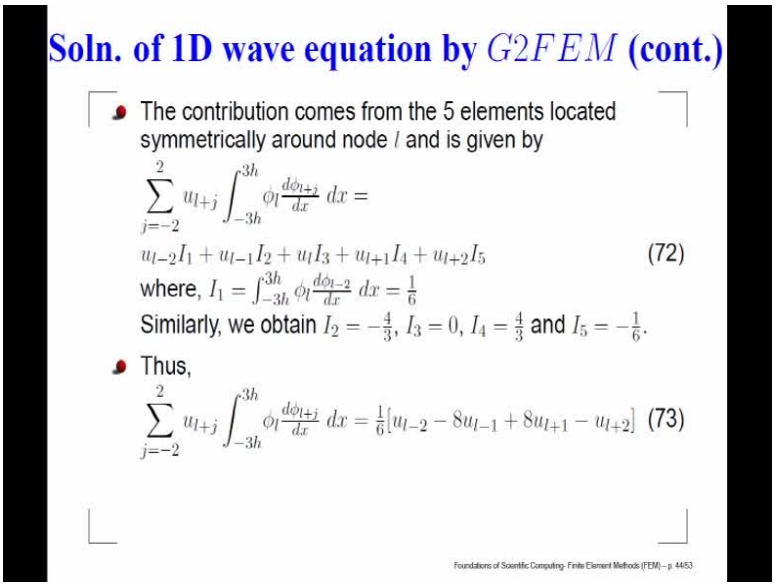

So, this is like your C D 4 kind of an expansion, so that is what (( )), you go through all those. And I have given you the detailed step; please check it that you will find the contribution that would come from the convection term would be like this - U l plus j, we are multiplying by phi l, then we will find out that the contribution would come from l plus j, where j will go from minus 2 to plus 2; and integral non-zero contribution would come from minus 3 h 23 plus 3 h.

That is, that is the only thing that we are going to have a non-zero contribution coming from, rest of it everywhere else, it will be zero. And once you work it out, you have the expression for those phi l of, or phi j's, then you can calculate the derivatives.

Now, the derivatives are no more like constant in the linear basis function. So, it would need little bit of calculations, go through it and then you will find that u l minus 2 will be multiplied by this integral I 1, evaluate that integral and you will get 1 6.

Then you will get contribution coming from u l minus 1, the multiplying integral works out to equal to minus four-third and you will find that lth node on itself will not have anything because it is a symmetric function; symmetric function means you will never have the contribution coming from the central node itself.

So, that is why I 3 should come out to be equal to 0 and I 4 will be plus four-third and that also comforting because you will see that if you are taking a symmetric function, then it would have a anti-symmetric contribution coming.

So, if I 2 is minus four-third, I 4 will be plus four-third, and if I 1 is plus 1 6 I 5 will be minus 1 6. So, basically then you will find out that this will give you this and once again now you can clearly identify, this is like your C D 4 expansion of del u del x infinite difference method; that is what you get.

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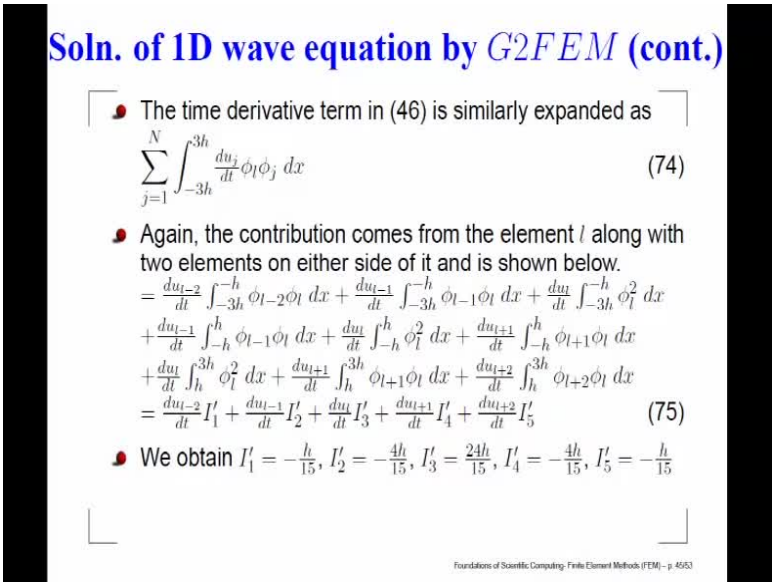

Now, like what we have seen with linear basis function, the time derivative terms will be much more complicated here because of the quadratic nature of the function, the weighting will come from these and you can see, once again this term will be getting its

contribution from the nodes at l minus 2, l minus 1, l and similarly you have this kind of representation finally.

And we can go through this exercise and we will find out I 1 prime is minus h by 15, I 2 prime is minus 4 h by 15. I will actually send you a sort of a link of a paper where all these things are worked it out; have been worked out by us years ago where you can see.

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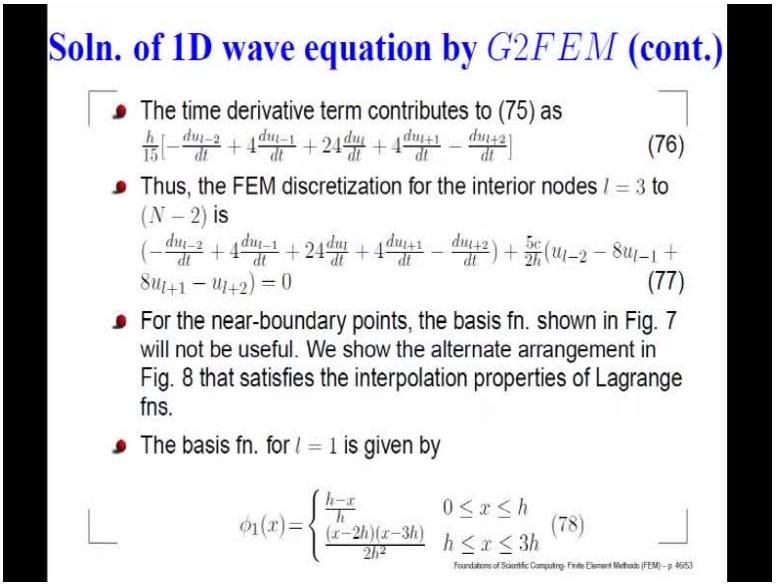

So, what happens is, for the central node the time derivative actually works out like this. It may look little complex, but that is what it is. So, your discrete equation is this taking (( )).

Now, what you can do is you will have to also do something more for the near boundary points. Near boundary points are a little trickier because you will have to mix first and second order elements, otherwise you will not be able to close it and we have shown you here, say, for example, if we are trying to see how we are going to handle the near boundary points; phi 3 is a regular point because that goes all the way up to 3 by 2 node.

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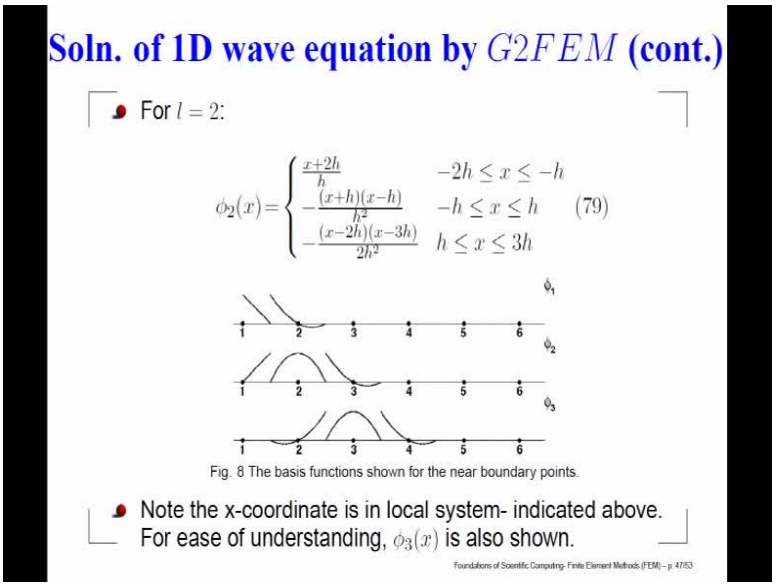

So, this middle segment would be symmetric that goes from 2.5 to 3.5 node, and this left node goes from 1.5 to 2.5, the right node goes from 3.5 to 4.5.

Now, if I try to extrapolate it on to phi 2, then what will happen? Well, I cannot have a quadratic function here, I cannot have, because that will spill out from outside; so, we cannot do that. So, what we instead do is we take a mixed element here.

So, this part comes from the quadratic side and this is the linear part. And similarly, this phi 1 will take the right hand side element coming from quadratic element, whereas this middle part is replaced by half of the linear element.

Why did you do that? Because we have to satisfy that Lagrange property, that it should all add up to 1 and that you can see is done very, very adequately.

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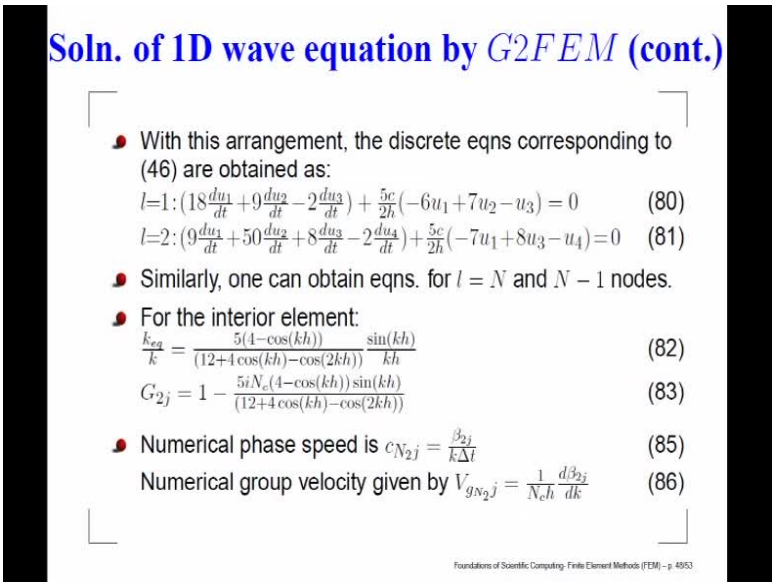

So, this is the way that you will have to do. So, for l equal to 2 we can write similarly this kind of thing that is what we have shown. And you will have to go through that exercise and it is a little bit of algebra, you go through it and you find out l equal to 1 - this is the discrete equation, l equal to 2 - this is the equation and having obtained the discrete equation, we can write out k equivalent by k.

By using that Fourier Laplace transform, we will get that and then substitute it and let us say we do Euler time integration, we will get this G, the numerical amplification factor like this. Once again you can very clearly see, it is an unstable method because the mod G is going to be greater than 1.

However, having obtained the, the G, here you have the real part and the imaginary part. You can calculate the numerical phase speed by this expression, what is beta 2 j? Beta 2 j is tan inverse of G imaginary by G real; that is what you are going to see. Since G real is 1, so this itself is going to define your beta 2 j and you can calculate the numerical group velocity.

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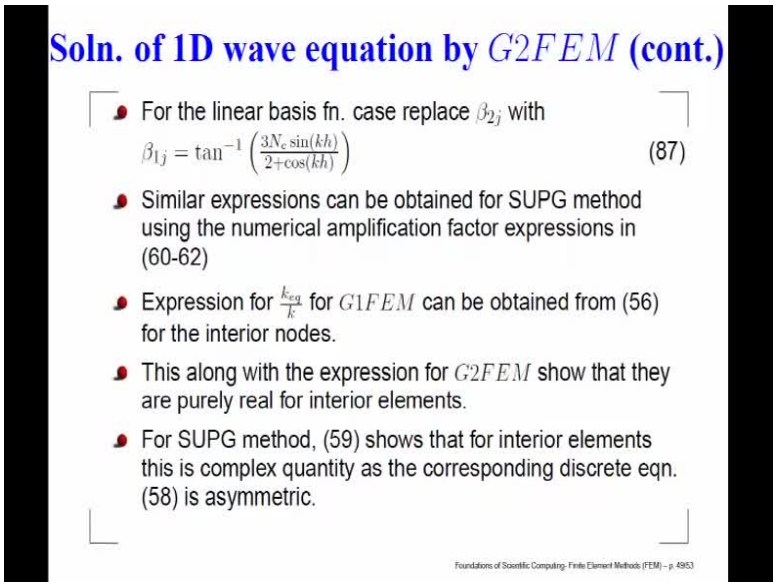

Well, that is what I wrote that for beta 1 j we had this expression, so we will replace the corresponding expansion from here, this part could be replaced by, this part will be replaced by that expression.

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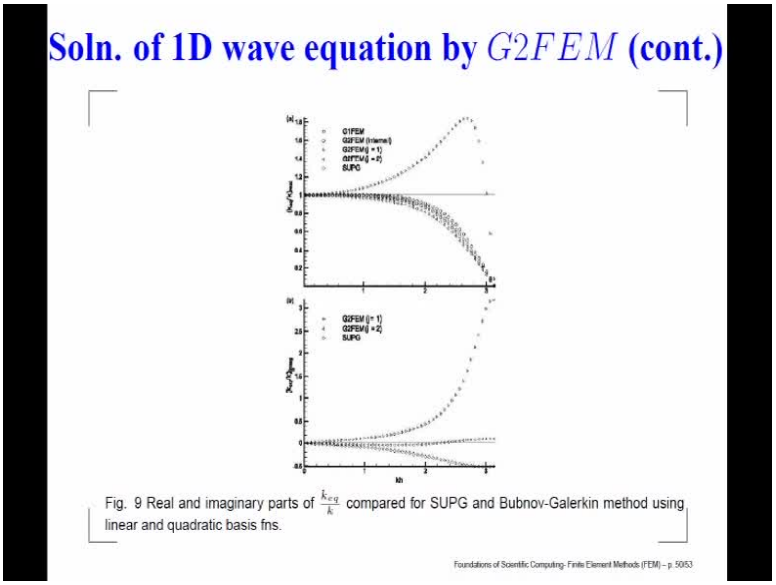

We have now basically the tool, all the elements that we needed to have; we can compare various methods and that is what you are seeing here. It is not a very good reproduction, a copy from a paper, and you can notice that k equivalent by k for the real part, various methods go like this, all of the Galerkin methods are crusted together.

What is important for us to realize though, that the difference between G 1 and G 2, the linear and quadratic elements, they are not greatly different, they are closed to each other and G 1 and G 2 are basically mutual methods.

So, in the interior nodes the imaginary part is 0, but only the near boundary points you get this. And for j equal to 1 you can (( )), for G 2 FEM you get a very, very larger instability because of the quadratic nature, whereas the second point, the instability comes down, but still it is in the larger range of k. Whereas SUPG, as I explained to you earlier also, gives rise to a massive dissipation; that is what you see here at the minus value.

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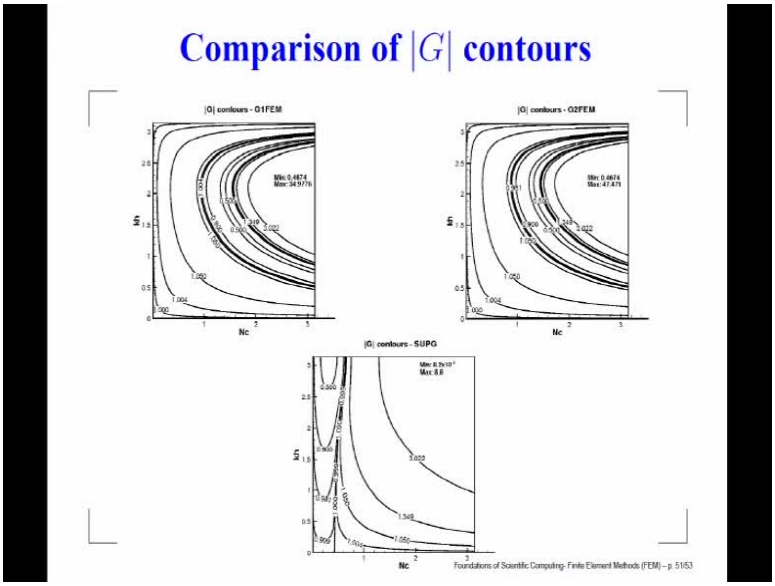

And these are the contours that (( )) did it very recently, last week only. So, we plotted all this G contours - this is for linear basis function, this for the quadratic basis function, and this is for SUPG.

This, this results are absolutely new and what you notice is that there is a very small region where you get close to 1, but nonetheless both this methods are unstable.

So, what I have shown here is 1.30 is, that does not mean it is neutral, stable; if there is a fourth decimal point involved there and if you want to go to 6 decimal and a 8 decimal, though that point will eventually shrink to the origin itself.

So, these two methods, essentially going from p equal to 1 to p equal to 2, does not give you much benefit. It is only the maximum value you will see, the instability will be stronger; here the maximum is 34, there its maximum is 47, for it is, it is just that. But anything above 1 is bad enough; it does not matter whether you have 34 or 47.

So, both these methods are unstable, that we know. This is your SUPG method, it is interesting for one to note that you are actually not allowed to take a very large value of n c because this is also unstable – 1.004, this is 1.05 and so on, so forth and it is also unstable. It is only stable in this vertical strip, but you also notice that it is not neutrally stable because of that imaginary part of k equivalent by k; this line is 0.999.

So, even if you are looking at small values of N C, the solutions are going to be attenuated and that we have seen before.

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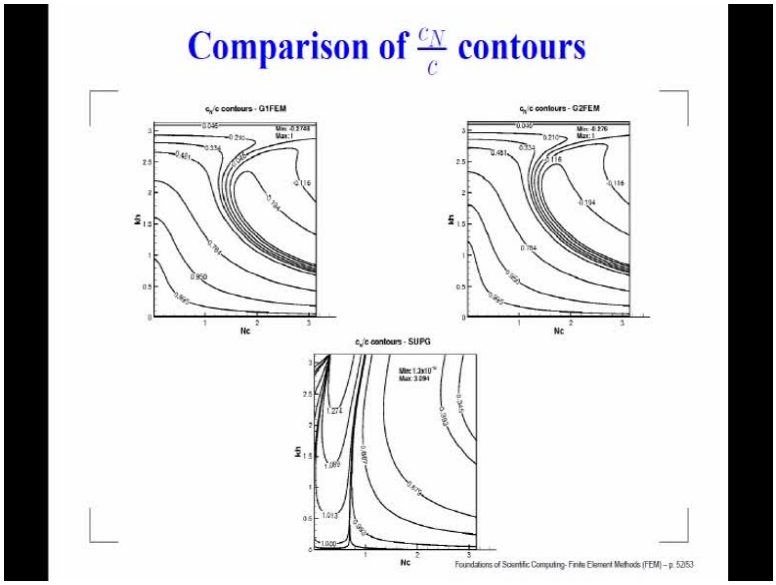

Now, we can compute the C N by C for these three methods, these two once again are almost same, so I do not see any difference between the two. It is the method that is significantly different is that C N by C contours and you can see there is a small range of k h, small range of k h and small values of N C where you can get C N equal to C.

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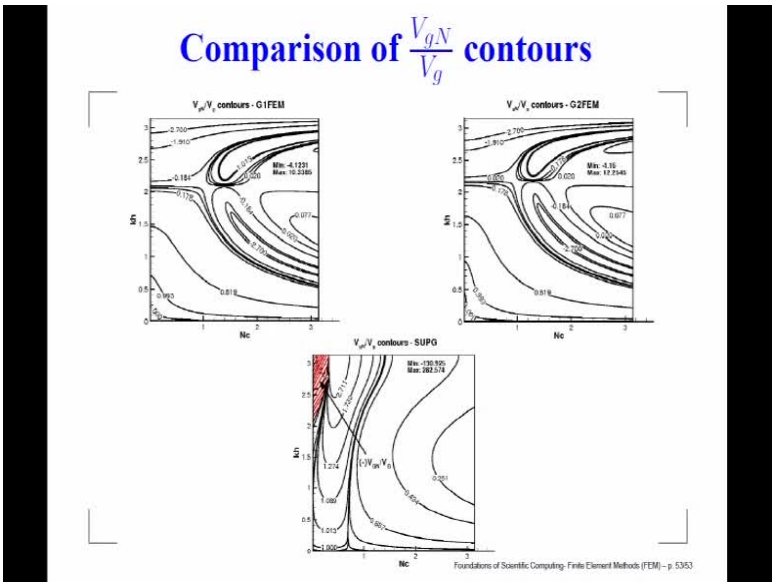

So that (( )) so you can realize the utility of SUPG also. And this is what we need to really understand, this is the group velocity features and once again you do not see anything between linear and quadratic version of Bubnov-Galerkin method. It is the SUPG method which you would be interested and you can see that is equal to 1, is a very small strip and anywhere above, you are going to get the energy - numerical energy, propagating faster.

What you also see is this region, this shaded brown region. This is the region where actually you get that upstream propagating waves, the q waves that is marked here.

So, you can realize that even if you take a very small value of N C, you may end up with a region of k h for which your wave can propagate wrongly, but then this is not an issue, why? Because this, you will have to couple it with your G contours.

So, if I look at the G contours here, these are the areas where you would see massive attenuation. So, this is the 0.5 line, so you will not get in actual computation ever, upstream propagating waves. But there is this possibility if you excite the flow forcibly in that region with a large amplitude disturbance, then you may actually get this spurious propagation.

I think, this is the, all that I wanted to tell about this. So, what you realize that despite what people have hope, the in hp element that defining h, of course helps, but not to a sort of a linear estimate.

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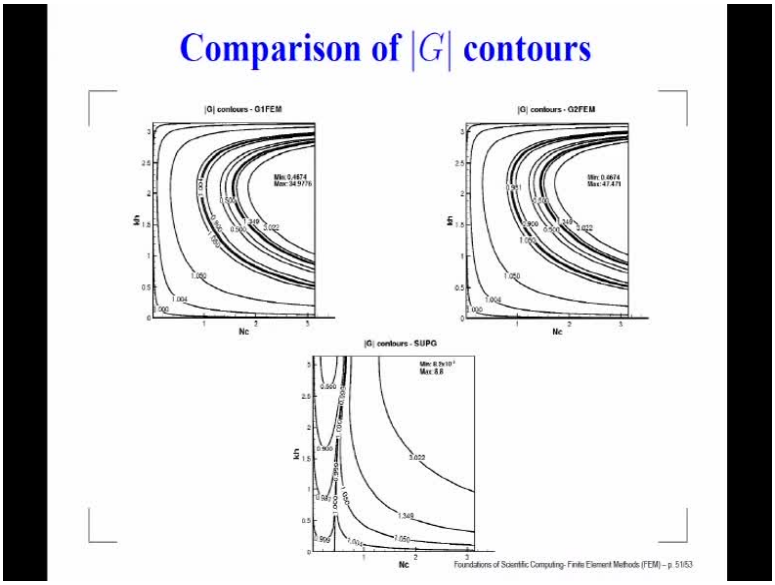

For example, so if I look at it this way that if I keep refining h, I will, for example, can get in to a range which has to be really very, very local, close to the origin to have a usable element.

So, what happens is SUPG continues to be the choice for so called engineering calculation. We do not consider it to be of use although there are lots of practitioners within this campus itself, but they are of no scientific relevance.

What happens if the Galerkin method…, We have this issue of numerical instability itself; given all those things taken together, I do not think finite element method is still a strong candidate for scientific calculations, whereas if we go to finite volume method, we may be able to recover the situation in a much better way. Finite difference method, as it stands, still happens to be the best method of choice.

If you are trying to calculate some problem, solve some problems with good accuracy, high fidelity, we still would cast our vote for finite difference method and that is why I purposely spent most of the semester talking about this high accuracy finite difference.

So, this is all that I really wanted to tell you, finite element method, this is not something what you would find in the books on FEM because they do not talk about waves, but I suppose there would be some monographs on weather prediction that people may talk about it.

But here also this picture very clearly reveals that this Petrov-Galerkin versions are a still long way off from giving us the kind of accuracy that we require. So, this is something I just wanted to show you that whatever we (( )) for the disturbance propagation problem, that was the constant theme that we had in this course.

So, we started off with defining how waves are central to any computations and then we have focused upon these essential properties, and that why are these three properties are essential, that we saw from that error propagation equation.

That was one of the key features of also this course. We wanted to show that what Neiman assumed that signal and error to be governed by the same dynamics is wrong; we found out that error propagates differently. It is more like a post excitation problem and those forcing comes through this modulus G term; that is what we see and we have seen that if modulus G is not equal to 1, whether it is greater than 1, of course that solution blows up, so you do not have to worry about; you have lost it, so you know that.

The problem comes when you actually have G less than 1, you get some solution which looks plausible are essentially wrong.