Foundation of Scientific Computing

Prof. T. K. Sengupta

Department of Aerospace Engineering

Indian Institute of Technology, Kanpur

Module No. # 01

Lecture No. # 38

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On lecture 38, we restart our discussion on finite element method and its basis functions. Mostly people have started on using linear basis functions; these are the least possible ordered polynomials.

Although it is a basically a global method, but its application wise it is local because its footprint is limited and we talk about various properties of Lagrange and hermite interpolations. So, how they are different and which are used here in FEM? In FEM, we actually use the Lagrange method while we have talked about Hermite interpolation method when we talked about the compact schemes. So, we have seen that.

So, let us try to figure out how we can improve the accuracy of Lagrange interpolation? One of the ways is the hp-element method, h refers to the grids or element size and p is

the order of the basis function, functions. And if we migrate from linear basis function to quadratic basis function, one would expect that we would gain in some accuracy.

We notice some of the properties of the basis functions for this Lagrange interpolation, which are translated in various coordinate systems. Here, we will be stating them in the local coordinate system, then we will be transferring them into a global coordinate system because we need to work out the discrete equations that we will be eventually solving; and we pick up some examples of Bubnov-Galerkin method, which basically is nothing but a weak form of this governing differential equation.

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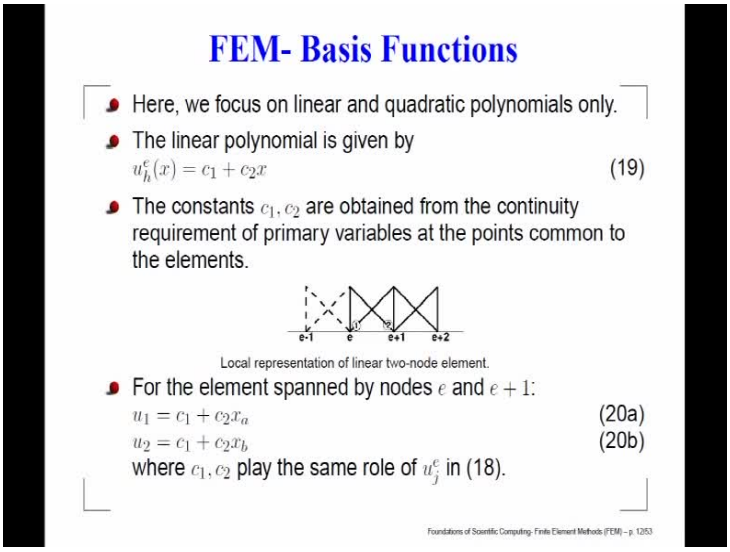

We are just discussing about finite element method and we do not have addict time to go deep into the subject, this itself deserves probably more than one semester of treatment on this topic; along that term, as I promise that we will try to connect it to the other topic that we have done.

We (( )) -cally are on the (( )) if we say that in finite element method we do is we take the domain and then (( )) it into smaller domains.

Say for example, a typical (( )) insulated between the node e and e plus 1, then we try to fit it a linear function between these two points - 1 and 2 by this linear function here. So, what you read this, find out c 1 and c 2 and then for the sake of ensuring continuity of

that across clouds so we already done that this polynomials mean such a way that at the node e and e plus 1 there would of continuity of (( )).

For example, for the point we are seeing here, the value is u 1 and (( )), I call it as x a and for the node e plus 1 the value of the function is u 2 and the independent variable x a value of x b; so, x a and x b are in a sense a global internet system with some origin somewhere else fixed.

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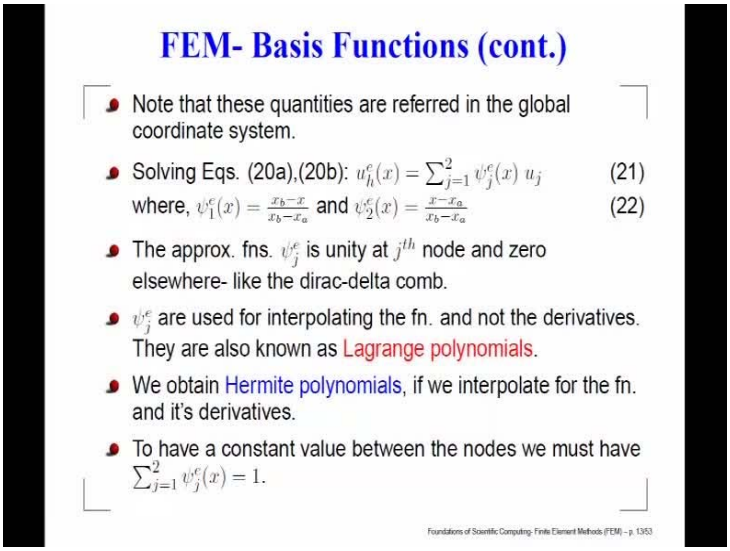

Now, we need to find out c 1 and c 2 and that should give us the representation of this code element; so it is easy, you call 20 a and 20 b and you get this equation 21, where you find out that local representation of the unknown given by the functional, is some of the (( )) functions, linear functions which we are calling as psi 1 e and psi 2 e times those nodal functions.

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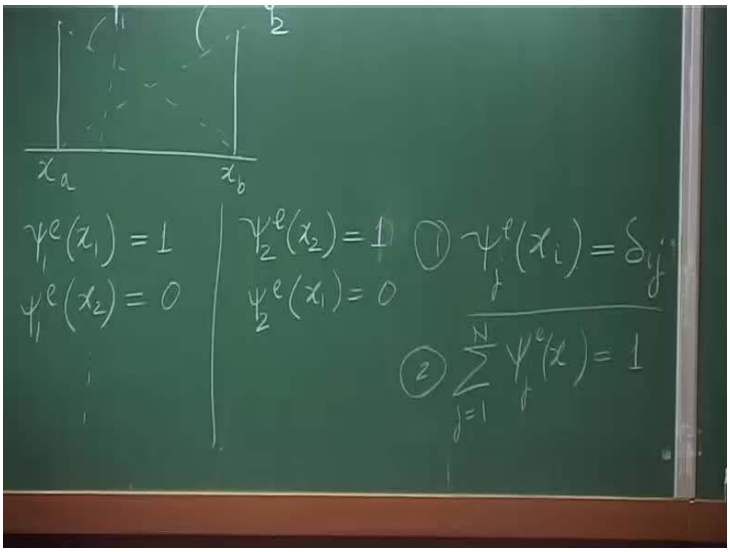

So, basically keeping this figure in front of us would help. So, this is now our x a and this is our x b and the function where at point is u 1 and at this point u 2.

So, what we are doing? We are writing the thing function here in terms of the two values at the node so that those are u 1 and u 2. Then what happens is the linear functional will give you two possible linear variations, so one is drawn like this and the other one is this. So, it is a basically linear combination of these two sets of functions - one decays from 1 to 0, so that function is this. So, this is your psi 1 and this is our psi 2; so this is the way that you write.

What you notice that there are certain properties that you can ascribe to these psi functions. The psi function is such that when I am writing psi 1 e evaluated at let us say, x 1 or x a, this is equal to 1 and psi 1 e at x 2 is 0 and you can write down the other point, they also will be 0. In contrast, you will see that psi 2 function evaluated at x 2 that would be equal to 1, whereas psi 2 e evaluated at x 1 is equal to 0.

So, this basically tells you that this if I wrote to write psi j e and if I write x i, this is like your delta function. When i is equal to j, it is 1 and when i is not equal to j it is 0.

Although you can see, in-between it is not like a delta function; let it goes here 1 and then it is 0 here, it is a kind of a linear functional, but still at the discrete nodes we can do this.

When actually you interpolate a function by the function values at the node, such interpolations are called Lagrange interpolation; Lagrange interpolation. And this kind of functions that we generate is psi j e, we will call them as the Lagrange polynomial.

We are already familiar with the hermite polynomials or hermite functions, where did you use them? In your compact schemes or in the hermite polynomial, what you do? Instead, you try to interpolate the derivatives in terms of the function. So, note the difference - in Lagrange interpolation you interpolate the function in terms of the function values itself only; so interpolate functions. In Hermite polynomial interpolation, you interpolate the derivative. So, this, this is the essential difference between the two and they are all pervading in computing. So, you would most of time get one of the two kinds, of course, there are other things where we can do combination of the two.

So, what happens is, there is another property that you also notice, that this psi j is, that we have at any low x location that we note, this plus this will always add up to 1. That you can very clearly see from 22, if I add this psi 1, psi 1 e plus psi 2 e, you can see this x part will cancel and you will get the value is equal to 1. So, this is the essential of property that you have for the Lagrange interpolation.

So, these are the few essential properties, that we see, that number one is that it behaves like a Dirac delta and number two is sum over all psi j at any x, they depending on whatever you have, so this will always add up to 1.

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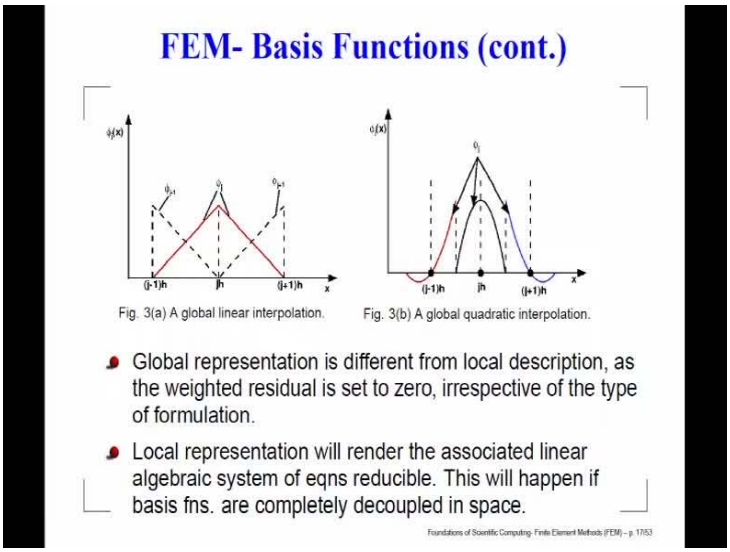

And what we are seeing here is for a linear functional, but you can see that it also works for any other types of interpolation, that you can think of.

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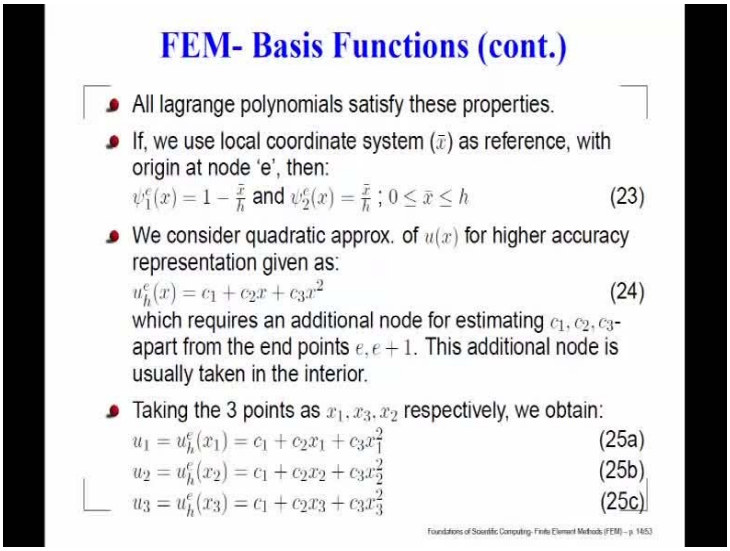

For example, we are going to shortly see this quadratic interpolation, but before I do that let me also mention, that instead of writing this interpolating functions in global coordinate, we can also talk about a local coordinate system.

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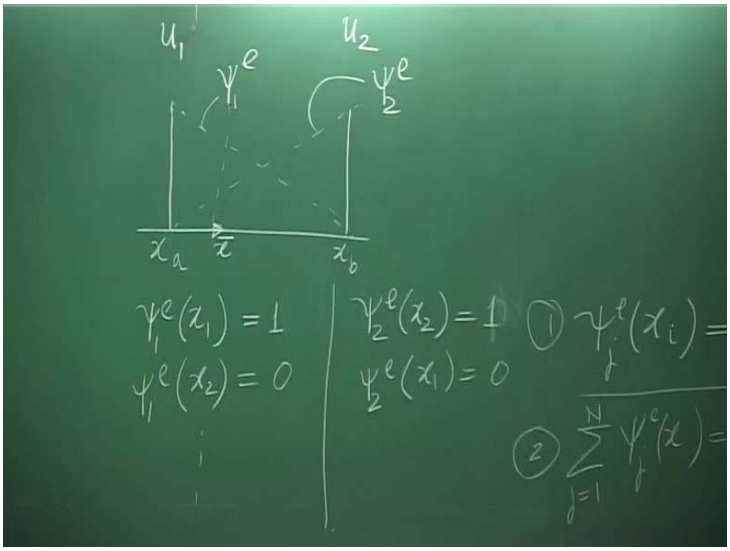

So, for each element I can fix an origin. For example, here I could just simply start a origin here and define a coordinate system, I will call it as say x bar and if I do that, so x

bar is fixed with the origin at the node e, then this is easy, that psi 1 e in this local representation would be 1 minus x bar by a.

So, it starts off with 1 and it decays with that slope of minus 1 over e and the psi 2 is, starts off from 0 and reaches the value of 1 at x equal to x bar equal to h; that is what your equation 23 represents.

Now, let us take a look at quadratic approximation for u x, and what we anticipate here that such investment of extra work should give up some kind of an extra accuracy, higher accuracy and let us see whether it does so.

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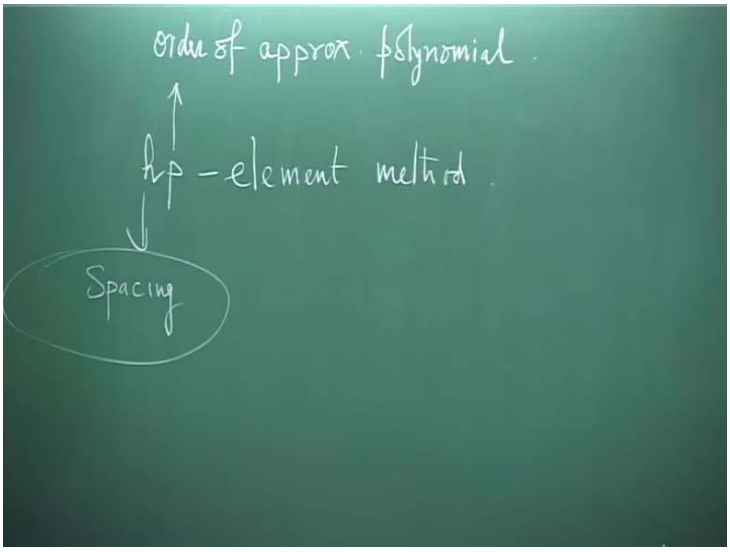

In fact, for many decades now, the finite element practitioners have coined this term called hp-element method. So, this has been claimed to be the more accurate finite element version where h represents precise of the element, like what we are talking about, in this say, delta x will be equivalent to your h and p is this order of the polynomial; so h represents the spacing and p is the order of approximating polynomial. So, with the idea that if you reduce h, your accuracy will increase; that is what you always expect for any discrete computing that if I keep on reducing the spacing, then I should get back to my continuing limit and I should rediscover my original differential equation.

So, reduction of h leading to higher accuracy is one of the attribute of discrete computing, so there is no quarrel about it, we will all readily agree with it. However, to claim that by increasing the order of approximating polynomial, we will also increase the accuracy as a subject of further investigation, and one of the reason that I bring this subject up is basically to highlight from what we have done in the fourth semester.

So, how do we define accuracy? We have already exploded that higher order does not always mean higher accuracy in the context of other discrete method. We have seen that what we really need to worry about is look at the sources of error and what helps us in reducing that error.

So, here also we start with a discordant note; however, this sold by this claim that hp-element methods are higher accurate method compared to traditional finite element method, so we will invest some time in that; we will take a look.

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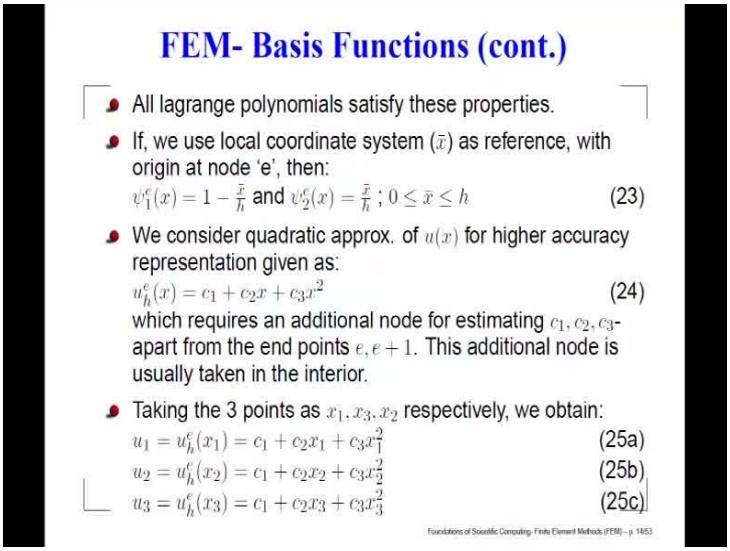

So, for that purpose we will look at this quadratic representation of the elemental functions and that would require us to solve for this three constants - c 1, c 2 and c 3. However, we are still restricting our self in the node; the element is still stand by e and e plus 1.

But we will have to fix three unknown constants - c 1, c 2, c 3. So, we will have to introduce some additional node and it has to be ostensibly inside, inside the domain; we will take it somewhere inside.

Usually, it could be anywhere, it could be anywhere, but we will be talking about here as the additional node to be, let us say, the middle of the element.

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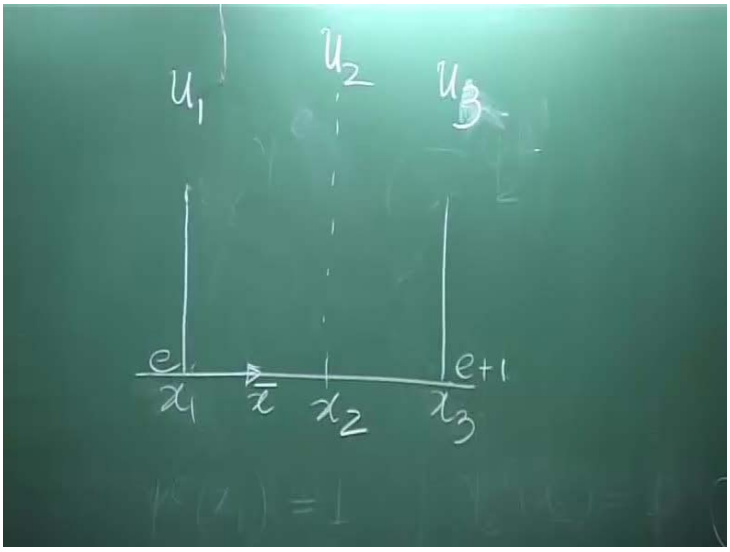

So I will call this…, So, let us remove this of linear functional witness here. So, we still have those two functions which we, I have call them as u 1 and u 2 and let us say this one, I will now call it as in the global coordinate system - this is x 1 and this is x 3 and we have introduced an additional node called x 2.

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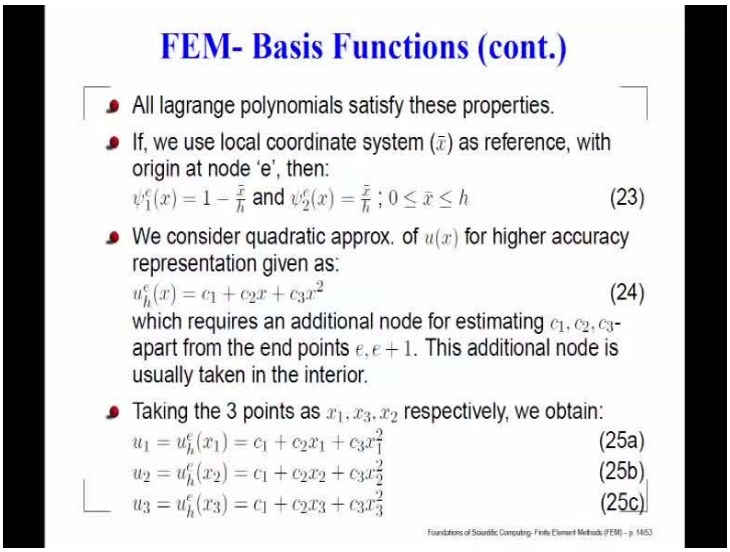

And then, we will have some value here which will call as u 2 and let us call this as u 3. So, that is what you have, this three relations written down here - u 1 is evaluated at x 1, u 2 at the midpoint of the element and u 3 at the right end of the element.

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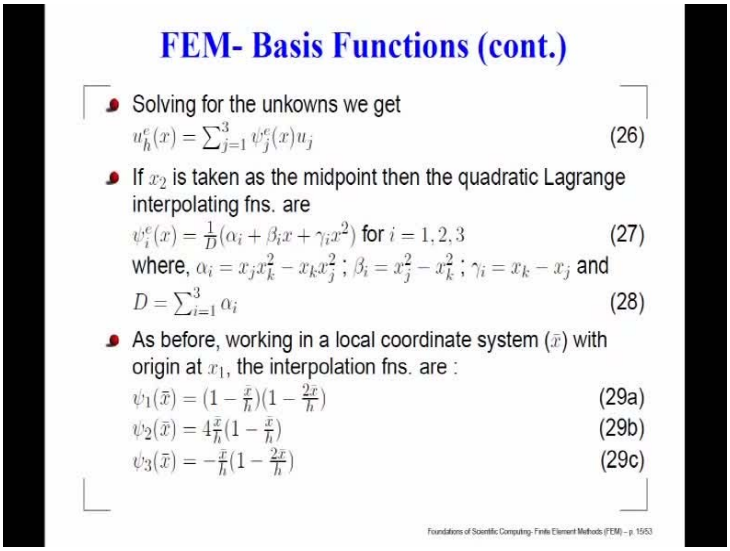

So, once again you can obtain the value of these three constants - c 1 c 2 c 3 and substitute it back and what you will find that the elemental representation is as given in this top equation 26. That will be a sort of a combination of the nodal values times some

space dependent functions and that space dependent function would be a quadratic as we started off with.

I will just not go through this algebra, which I am sure you can do it yourself. By using Thomas rule you can find out those c 1, c 2, c 3 and then put it back and order them neatly and you would find that these are those functions; you would get now three functions.

For the linear interpolations we have two functions because we have two unknowns, c 1 and c 2; here we will have three such functions, so we have psi 1, psi 2 and psi 3 and they are written there in terms of this quadratic expression.

For each value of i 1 and 2 these coefficients…, You know these x alpha i beta i gamma i, they are nothing but the coordinates like this x 1, x 2, x 3, and this form a pattern, cyclical pattern, so that you can exploit this and you would be able to do that.

And there is this denominator d which is nothing but the sum of is x independent part, the sum of all the alpha i's would give you this. Now, what we would like to do is as before, we would switch over to a local coordinate system and set the origin at x 1; so, then call that as x bar and then we are going to get three such functions.

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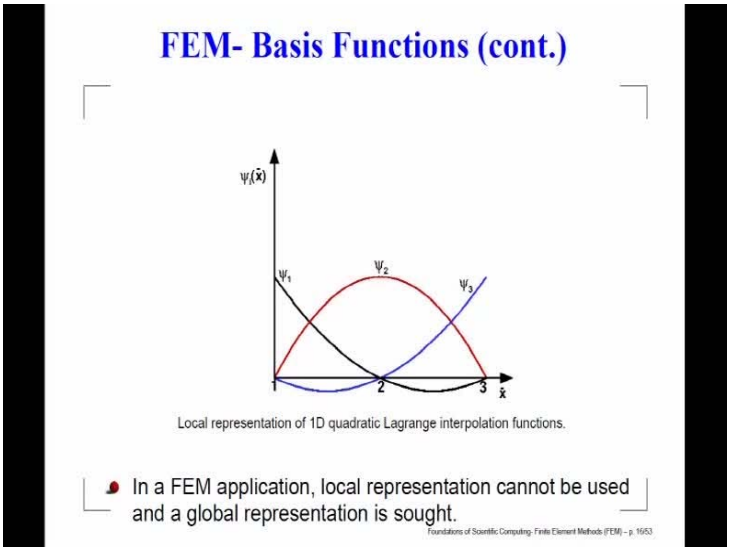

Going through this exercise, you will find out that you are going to get three such functions, they look like this. psi 1 is one that starts off with the value 1 at x equal to x 1

or x bar equal to 0 and then the property of Lagrange interpolation, that it should be like a delta function. So, at the node concerned, this is going to be 1, everywhere else it should be 0; that is what you are seeing. So, at x 1 psi 1 is 1, at x 2 and x 3 psi 1 is 0 and that is what exactly we are (( )), that it has a quadratic dependence, but it process through 0 at 2 and 3.

Now, the same way or if I look at the psi 2 function, this is also a quadratic function, which has to be equal to 0 at 1 and 3 and which will be equal to 1 at x equal to x 2. And the same way, psi 3 would be here that would be 0 at x 1 and x 2 and at x 3 it will be 1.

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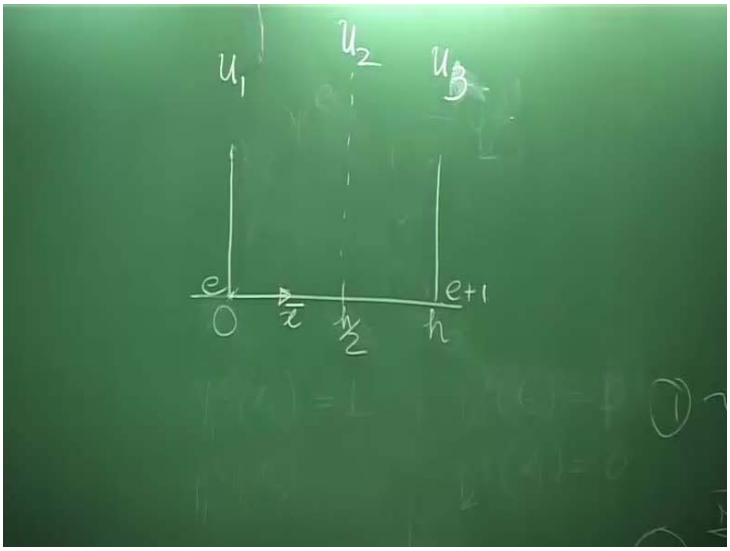

So now it is not very difficult for you to understand that in this x bar coordinate system, in x bar coordinate system what this is going to be? This is going to be 0 and this is going to be h by two and this is going to be h; so, in this x bar coordinate system these are the three nodes.

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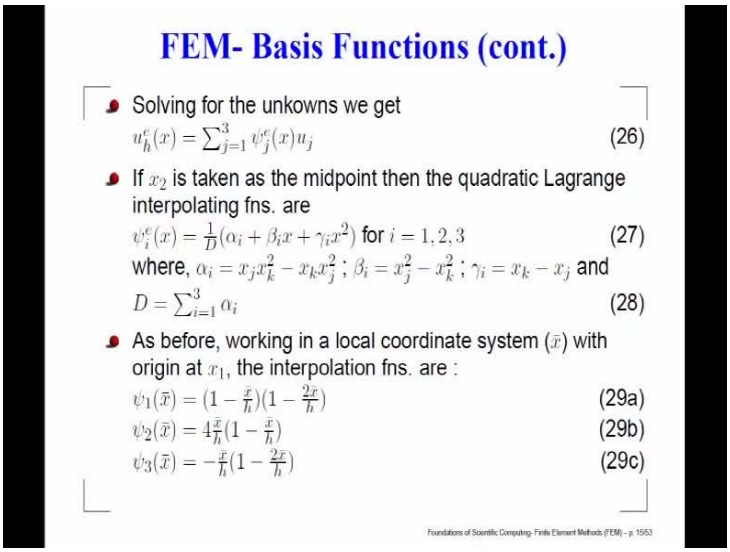

Now, if you go back it should be very, very easy for you to construct that polynomial. For example, psi 1, what do we expect that it should be at x bar equal to 0? It should be 1 and you can very easily see, the very fact, that this psi 1 is going to be 0 at h by 2 and h gives you these two factors. So, that is very easily constructed.

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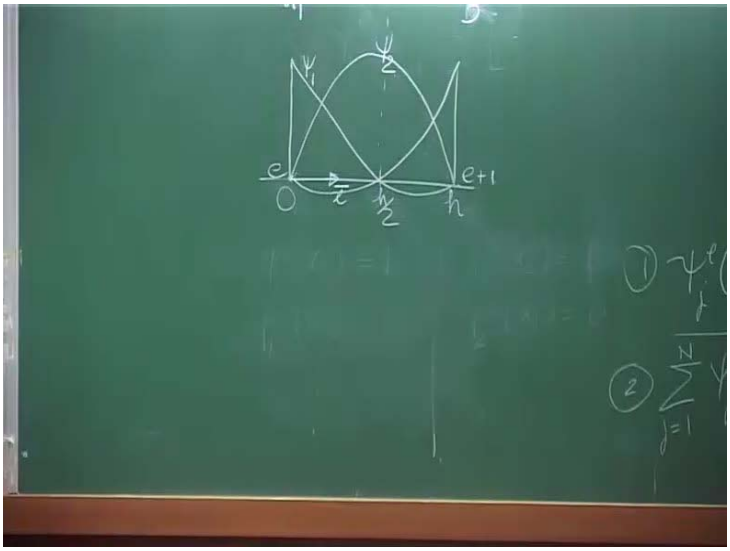

The first factor ensures that x bar equal to h, it will be 0 and the second factor ensures that at x bar equal to h by 2, this will be zero. So, that is the reason that I just directly wrote down the local coordinate system. It is easier for you to see and that should

convince you that it goes off there and it is a quadratic function. So, it is a…, So, it would be something like this; that is your psi 1.

The psi 2 function is symmetric about h by 2, it is 0 here and it 0 there and that is ensured by, you can note that when x bar equal to 0 or x bar equal to h, these two product of h; whereas, it gives…, it is maximum when x bar is equal to h by 2 and that is what you are getting. Your psi 2 would be a function, it will be like this; so, that is your psi 2.

And the last one is the psi 3, which will be given here which will be equal to 1 at x bar equal to h, at x bar equal to h I get here minus 1, and here I get a minus 1, so that becomes, make it plus 1 and at x bar equal to 0 it is 0 and at x bar equal to h by 2 it is 0; so that is what will happen.

So, that is the way the function looks like and that is the way we plotted.

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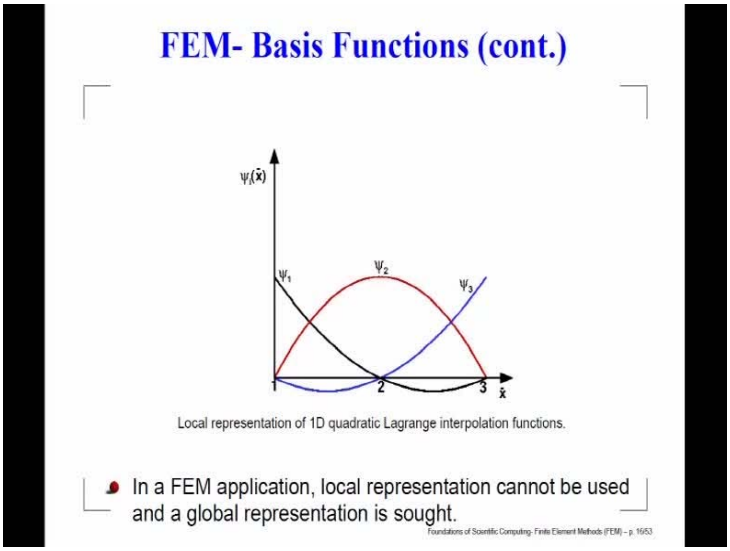

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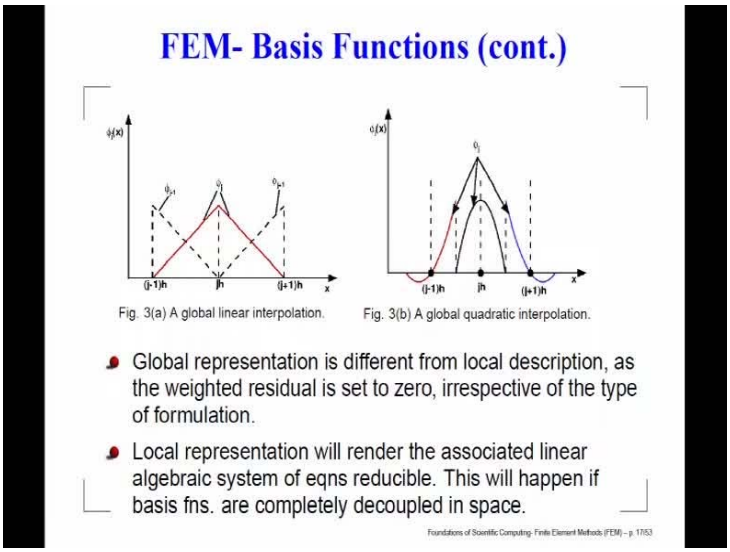

However, we need to really write the equation down, not in this local representation, we need a global representation, why? The reasons are the following, that if I keep it like this and then I try to plug this representation interpolation form into the differential equation, what we are going to get is that everything for this node will be related to this, only these three nodes, only, whereas we want to bring in the implicitness of the method. I mean, what we want? That the various elements, the various of domain should interact with each other.

That is why what we are going to do is for example, if I look at this - the linear basis functions, then I said that my jth node starts off from j h to j plus 1 into h, then what happens is if I would have taken the basis functions as this phi j as this and phi j minus 1 as this, then what will happen? If I plugged them back in the differential equation, it will just give you something like your explicit representation of the function.

But what we need is something like this, with the Lagrange interpolation property that phi j would be equal to 1 at x equal to x j and everywhere else it should be equal to 0.

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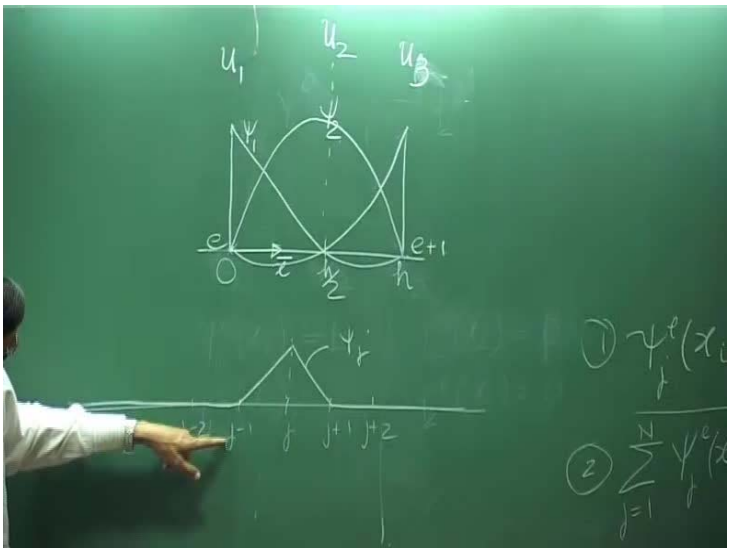

So, what is happening here is that if I take the node - this is the jth node, this is j plus 1th node - then what will happen is I would have a function which would be like this, this is my psi j in a global system; this will be my psi j, so, it will be 1. So, when x equal to x j, everywhere else it is 0. It is only that in between these two nodes - j minus 1 to j plus 1 - it linearly falls up to 0; that is the property of this. So, that is what we have shown.

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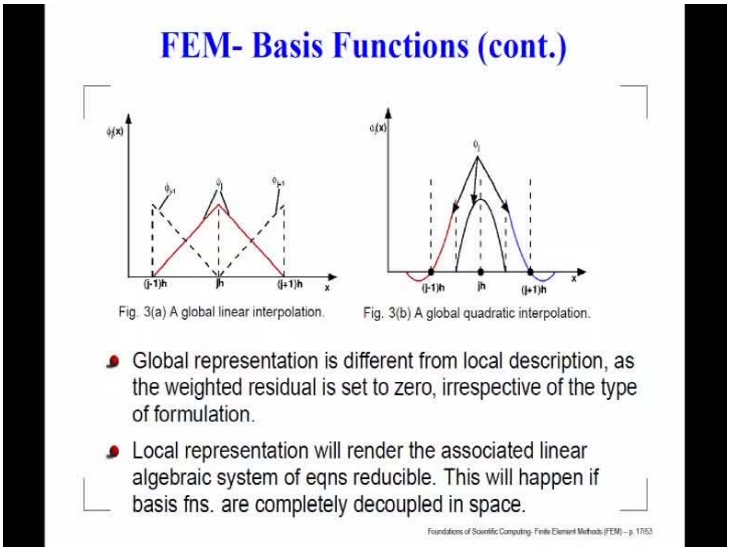

The global representation is given by this red line whereas the local representations were what we earlier talked about. It is something like encapsulated between j and j plus 1th node itself.

Of course, as you can see the quadratic interpolation function, the global representation is somewhat little more complex because what we have is… Well, it is not written very clearly here, but you can see this is the center of the node and this is actually the element; the element goes from here to here and this is the midpoint.

Now, what happens to psi 1 and psi 2? We do not keep the psi 1 inside here, we just shift it by one element on this side and psi 3 which was here like this, this is shifted to the left one. What we are trying to do (( ))? You are trying to couple the elements in the representation.

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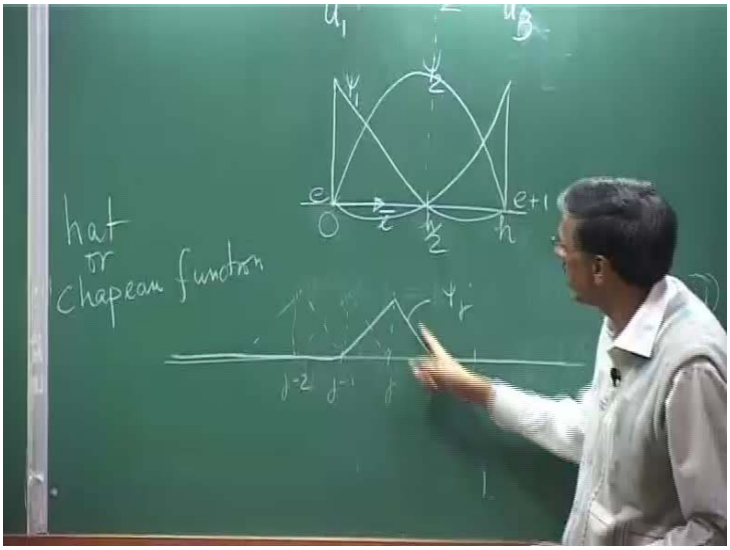

And what happens? What does it mean? That despite having done that, say for example, the next element for center of j 1 would be like this, so what we are doing? We are actually superposing this with the hat function or what is called as a chapeau function. So, what happens is those properties are always maintained.

Whatever we talked about individually, they are like Dirac corm, and that at any x, you add up all the contributions of this elemental interpolation function, adds up to 1; so that is, that is fine.

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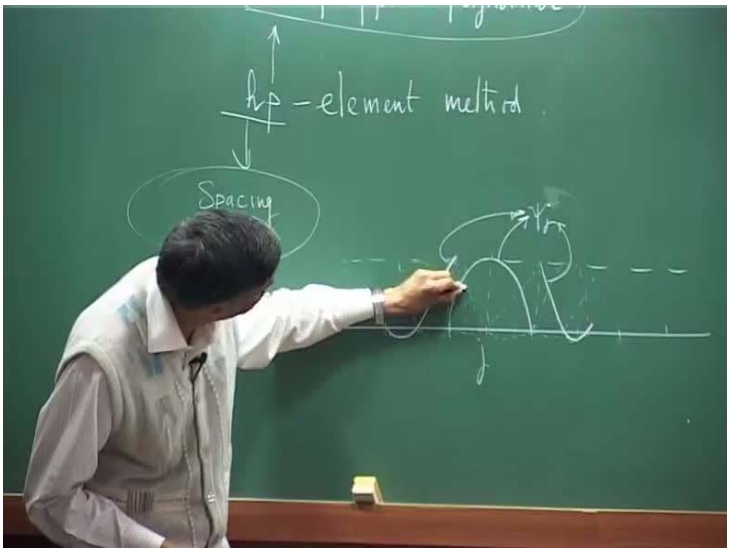

So, same thing that what we are doing; over there when I look at the quadratic element you know, the figure has to be treated with somewhat care, so this is my actual one element and this is a center point of that element.

So, the next point would be…, next element would be like this and there would be another, let us say, there is this next element which is like this. So, if I superpose this, let us say, these are the…,

So, I have this function present there, this function also present there, but here when I am doing this, I also have this and we have a function like this.

Now, the same way we are doing a functional representation like this. So, what we are doing is basically identifying what constitute psi j. So, suppose this is the jth element, then this is going to be one of the elements that we called it at psi 2 j and to that we add up this part and this part. So these three constitute our psi j; so these three elements constitute psi j. That is what I have shown here with this function on the right.

At the same time, you would have the interpolation functions for different nodes superposing on each other. Those satisfy those basic properties of the Lagrange interpolation, but they still constitute the same thing that at any x i add up, I will have three components – one is a negative contribution coming from here, positive contribution coming from there and another positive contribution coming from there.

You add all three, still get equal to 1; this property two that we talked about. So, this is going to be always satisfied.

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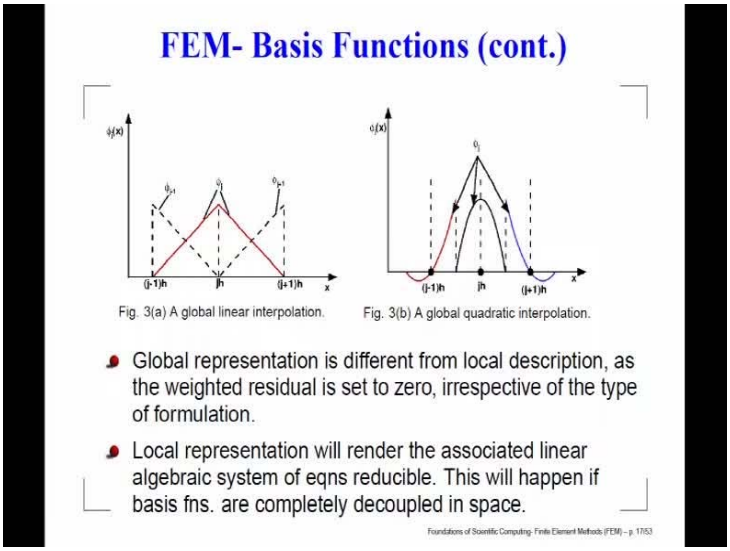

So, do not think that we have this kind of a representation in isolation here. We will have superposing functions and when we add them up, they will still satisfy this. So, that is the jump from the local to global representation.

And why we do that? That is what we are saying that global representation would always set the weighted residual to 0 irrespective of the type of formulation, whether I do a linear or quadratic, I should have this property satisfied. I could have p equal to 3, either could have p equal to 4, we could keep on ratcheting up the level of the order of the polynomial.

What is also important that why we give up the local representation in favor of global representation, is that when I substitute this representation back into my differential equation, what will it do? If it is a PDE, time dependent PDE, it will convert it into ODE or if it is a study state problem, it will convert it to a linear algebraic equation.

So, whatever we do, suppose it is a study problem then the corresponding linear algebraic equation that we will obtain as the discrete equation, if I keep it like this, then the ith element only resides within the ith element itself. All those psi 1, psi 2, psi 3 are

defined only in here, so what will happen? It will…, The different elements will not communicate with each other.

That is what we are essentially saying that if it does so, then I will get sort of a disjointed, I mean, difference equation, discrete equation for the elemental level and then what will happen? Well, I mean, they are all independent of each other, so this so called linear algebraic equation; even if I do, they will turn out to be reducible because they are all decoupled; so, that will not work.

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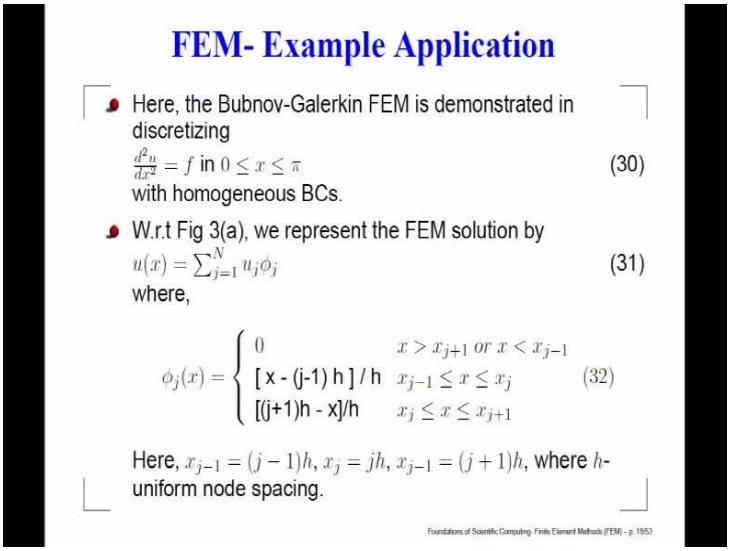

So, to couple the system we will have to go through this global base. Now, let us take a, take an example and this is the classic Bubnov-Galerkin finite element method and let us say, we are just simply trying to solve it; very, very simple equation. The second derivative is given as f and let us also make our life simple by considering the variation of x between 0 and phi and we will see, what will it do for us?

What we are going to do is that we have drawn a figure like this, which is called as figure 3 a. There we will represent the unknown u of x in terms of their nodal values times this. So, as you can see this phi j, as we have drawn here by the solid line, that is zero; all phi this j minus 1 and j plus 1, that is the first relation, tells you that everywhere else it is identically equal to 0.

Only within this span going from j minus 1 to j plus 1, we have two segments of a discontinuous function - one is that increases with x, this path, this path that is your middle equation and the last one is this path, that is the decaying function; so, that is what we are doing here.

You can now see what is the essential idea if we do it with a linear basis functions? And you can identify this nodes, x j minus 1 is simply nothing but j minus 1 into h.

We, for the sake of understanding, I am taking the all the element phi at state, it will make life simpler and we will understand what is going on, but it is not necessary or it is never practiced that you use uniform space. That is one of the strongest point of finite elements in finite volume method that you can do it with local spacing, which keeps varying from element to element, volume to volume.

Now, let us observe that the linear basis functions here at lot of coming on. We have two functions, psi 1 and psi 2, they are not orthogonal. So, that is why, although in some books you will see that spectral method and finite element method, especially the Galerkin kind are clubbed together, but one of the difference is - in the spectral method the basis functions are all orthogonal to each other, whereas here in FEM we gave up that condition in favor of low order polynomial. So, the resulting monomial or the polynomial that we have, they are not orthogonal to each other.

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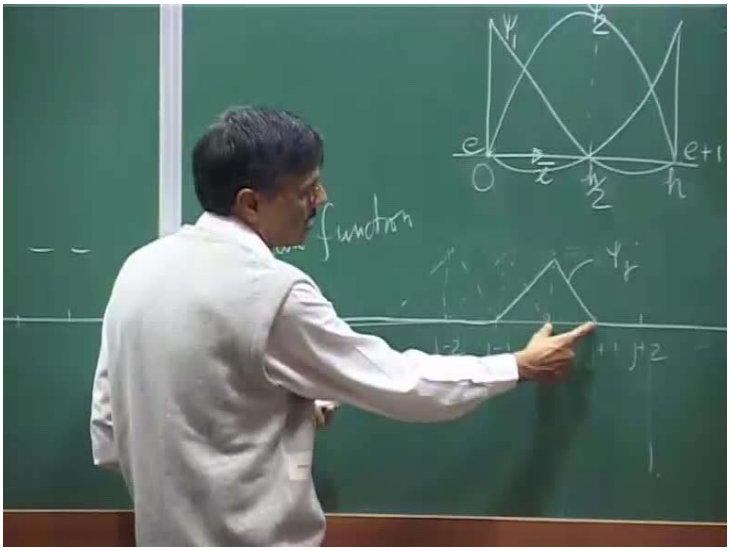

And why do we do it? Because the discrete equation will be simpler and only the neighbouring elements will interact with each other. So, if I am looking at jth point, if I am looking at this element, you can see that at the elemental level, the interaction will happen with this element and this element, this from linear basis function; for quadratic, it will be little more, we will see, we will see that.

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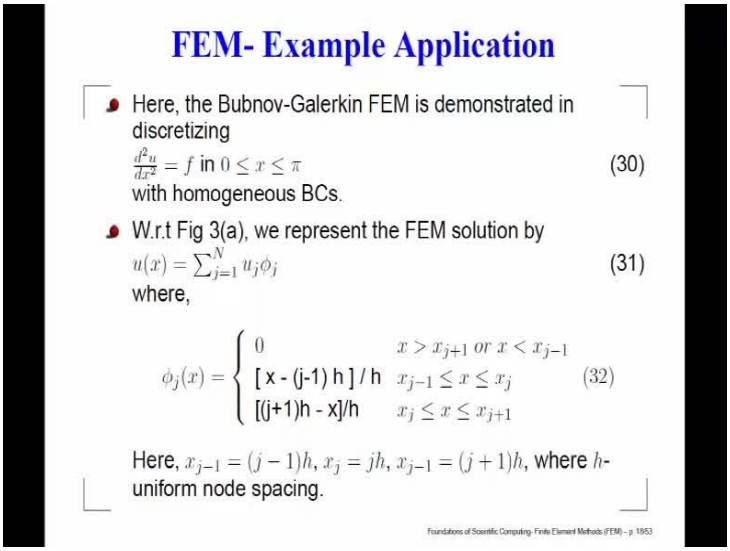

Now, one of the problem that we are going to face, that we are trying to solve a problem, which requires evaluation of the second derivative. Now, a question will automatically come to your mind that if my space dependence is given as a linear function, how is it going to satisfy a second derivative? And this is a legitimate concern.

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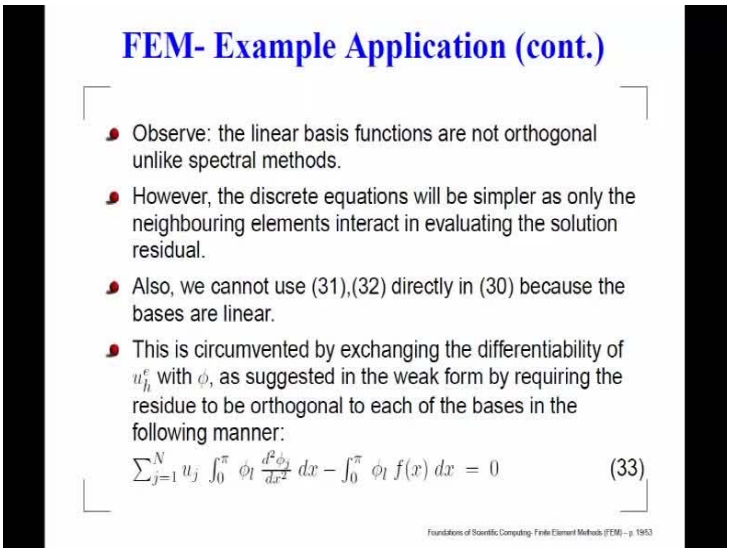

This is where the help of weak form as oppose to strong form comes into picture. How? Let us see that. What we do actually?

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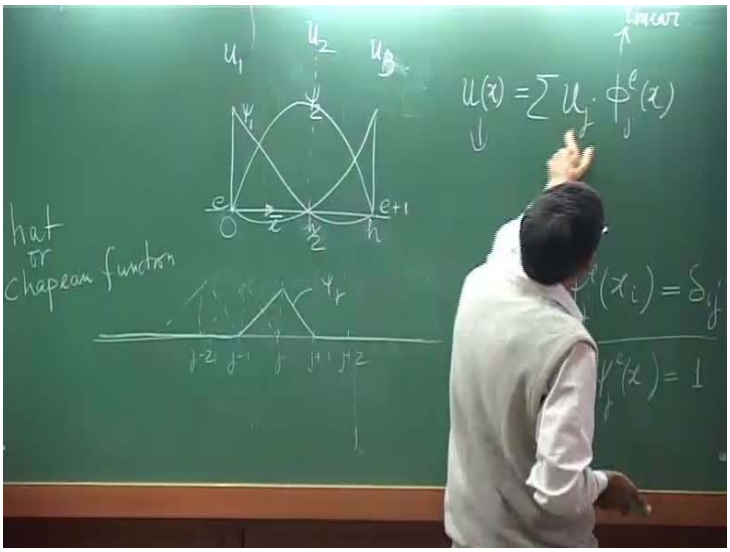

We try to take…, See, basically what we have? This is our function and that we are writing it as the nodal values times this. So, this x dependence is built-in here and this is the original function, but differential equation says that it should be at least be having a continuous second derivative, that is what this thing says. But whereas, this one we have taken as linear function, so there seems to be a conflict. What we could do is, we could

exchange, we could exchange some of this differentiability of this functions from here to here, such that we are going to look at it like this. So look, I mean, substitution of this, this expression in their differential equation will give us this kind of a form. And what have we done?

It is a Galerkin method; so, we are going to say we have the differential equation, we multiplied by that with the basis functions. The basis functions are files, so what we are going to do is we are going to multiply, let us say, a lth basis function phi of l and then we integrate over the whole domain. That is the whole concept of finite elements.

That is what we are doing here that the weighted residual…, it is not the collocation method, we are doing some kind of is, a sub domain method. So, what we are doing? We are multiplying by lth basis function with this form and then integrating over the whole domain.

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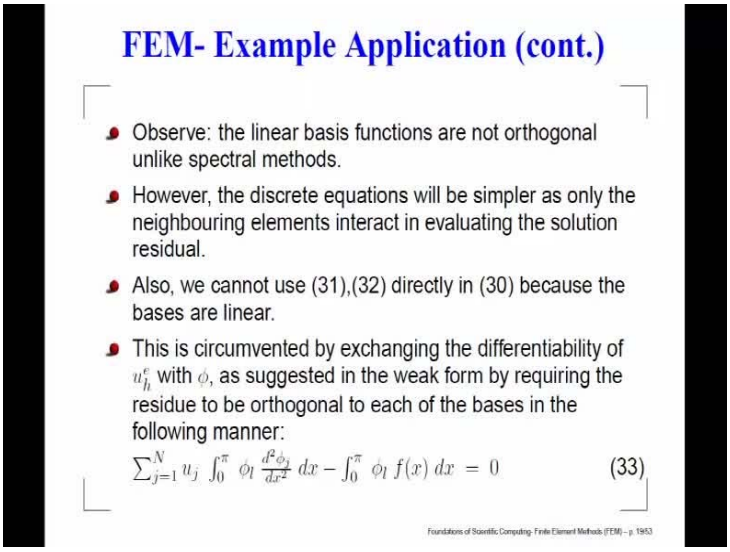

So, domain is defined from zero to phi and this is how we keep it. So, the same way the f is also multiplied by phi l and then we are integrating it and this is your discrete equation. So, 33 is the discrete equation of the equation that we noted on top here by 30. So, what happens is, as we said that phi is a linear function, so that d two phi j d x square does not exist.

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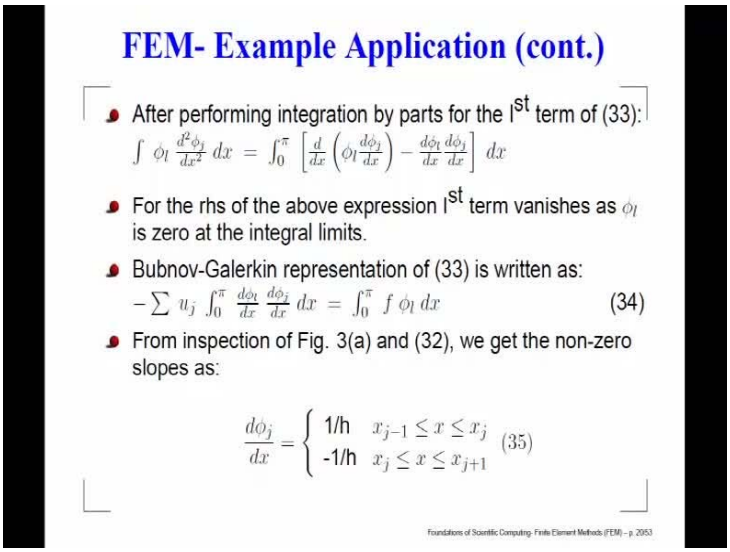

But, what we could do is we could actually perform an integration by parts and that is what is done up here.

So, if you do this, you could write it like this, that this is nothing but d d x of this times this. Now, what happens is this is the first part, is a perfect differential, so I could integrate it and I will get phi l times d phi j d x, substituted at the limit 0 and phi. What is the property of phi? phi l is, has that, this property what we have written that it is 1 only at the node, everywhere else it is 0.

So, what will happen? For any arbitrary phi l, if it is not the end element, this perfect differential will vanish because phi l is 0; that is what happens. So, this first term drops out because phi l is, are 0, is at the integral limits other than when j equal to l. See, basically what we are doing? These two are distinct j and l. Only when l will become equal to j, then we will have to think about doing something, but otherwise this part will be always contributing to 0.

At the end of the element, all the conditions would come from…, I mean, we are not going to satisfy the differential equation at the end at 0 and phi, so in all the cases what would you find at x equal to 0 or x equal to phi? So, first part will give you 0 contribution.

Then you have come to this, this is what is called as the weak form. See, what has happened that we are not satisfying the governing differential equations, we are satisfying an integrated form of it. That is what we did, multiplied by phi l and then we integrate it, that led to drop, dropping of this term and then we get this.

Now, you can see that you are no more requiring the existence of the second derivative because you have would being the problem into giving your product of two first derivative. You do not need the second derivative, you see these are the first derivative and the first derivatives are also very easy when we look at the linear basis functions, they are either one over h, the slope or minus one over h; that is what we have written right here.

So, what happens is that make our life rather simpler, that going from x j minus 1 to x j we have a positive slope and going from x j to x j plus 1, we have a negative slope; so, this is what we have.

So, we could just simply plug those on the left hand side of 34 and then perform the integral, that should make life rather comfortable, and this is what you are going to get. On left hand side we will give you only this, equation 36.

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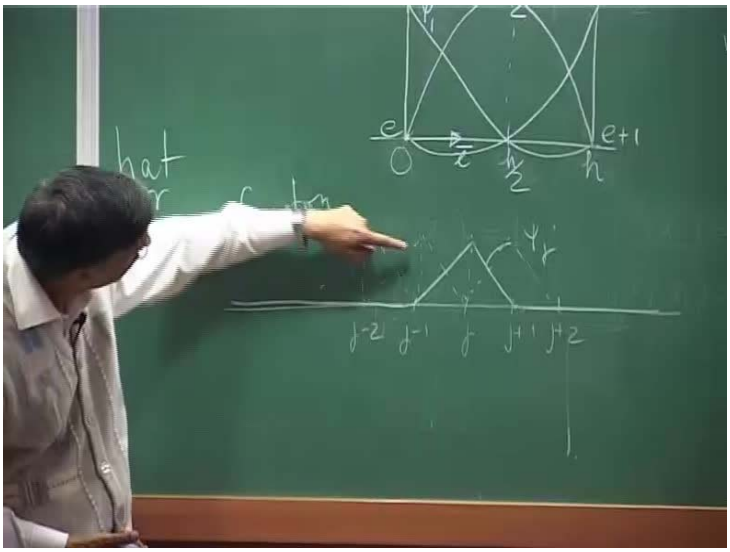

See, you realize that in the previous page when we are looking at this, which are the l and j's which can contribute? They can contribute when all, only, l is the next neighbor

of j; so, that is what we say that if I am looking at this, so l is equal to j will give me the contribution from here and l is equal to j plus 1 will give me a contribution coming from this basis function.

Then, this has a common intersection between j and j plus 1th element in this part. The same way j minus 1th element and jth element will have a common path here. So, that is what we have written there that we will come…, well, actually there should be a j plus minus l plus minus 1, and if you do that, if you do that l plus minus 1 and l, you get this. And this should convince you that it looks like your second order central difference scheme.

So, this should make us conclude, as if, finite element method spatial discretization is like second order accurate discretization, which is not true, which is not true when you look at it is wave properties, when you look at the complete differential equation and I would like to do that tomorrow.