Foundation of Scientific Computing

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Module no. # 01

Lecture no. # 37

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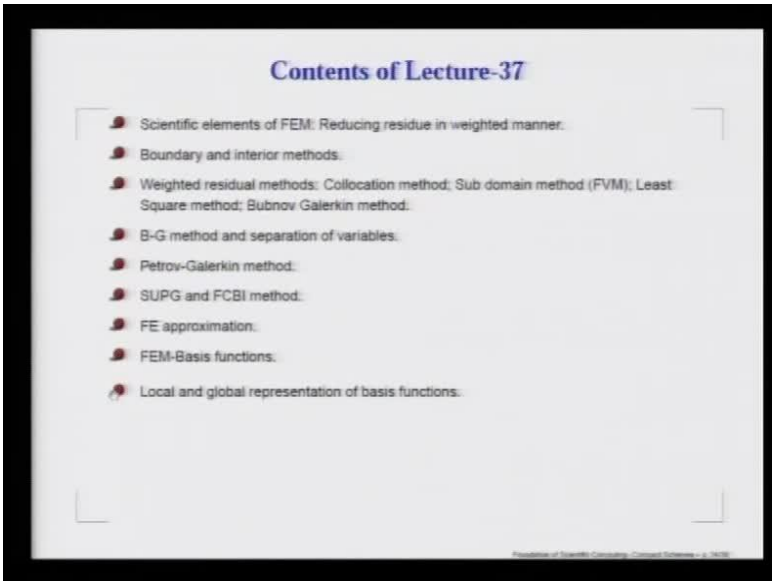

On lecture 37, we seriously start our discussion on the scientific elements of a FEM. We are not interested on engineering application of FEM, there are many such specialized courses where you could do that, but we tried to find out, how the accuracy aspect of FEM is, in the course of this lecture and the following lectures.

We notice that the scientific element of FEM is in an effort to reduce the residue, not exactly, but in a weighted manner, and we talk about various variations of this method in terms of boundary and interior methods.

We also talk about various versions of weighted residual method. They could be a collocation method or we could apply it at a subdomain or one could actually (( )) imply a least square approach or one could take the classical Bubnov-Galerkin method.

The Bubnov-Galerkin method is a non-dissipative method and which is quite interesting in its own right; so, we talk about this Bubnov-Galerkin method.

In appearance this Bubnov-Galerkin method appears to be as if, we have separated the variables, but it is indeed not so, we will highlight why it is so. We also identify that the Bubnov-Galerkin method would be sort of non dissipative, so it can offer numerical instability.

And in the Bubnov-Galerkin method, we choose the basis function itself as the interpolating functions, which are low order polynomials, but there are other ways of circumventing problems of Bubnov-Galerkin method. The one of the method is the Petrov-Galerkin method, and its versions that we talk about here is the stream line upwind Petrov-Galerkin method of Hughes and the flow control, FCBI method of Bathe and his colleagues.

We basically, having introduced all this briefly, get into the finite element approximation, talk about the various basis functions and we notice that these basis functions could be local or global and with this we will conclude this lecture.

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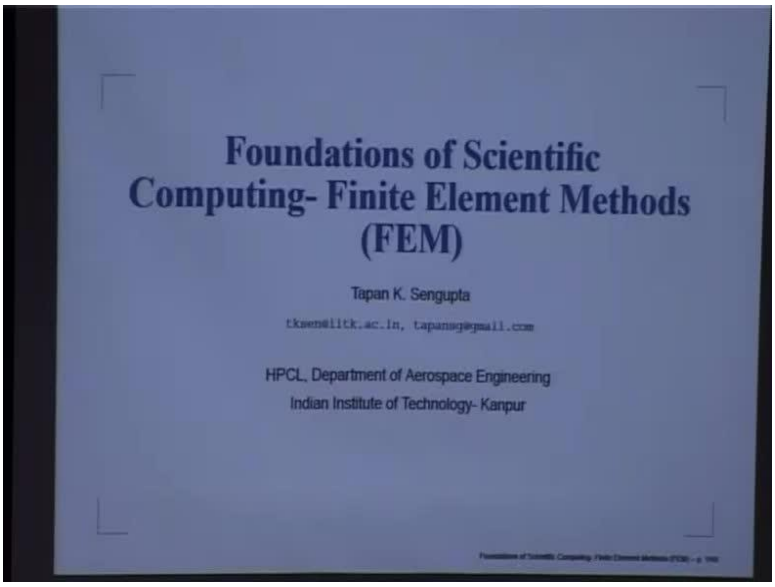

And touching up on the last topic, it is on finite element method, but I am not going to cover finite element method; we will find in many book or in many other course.

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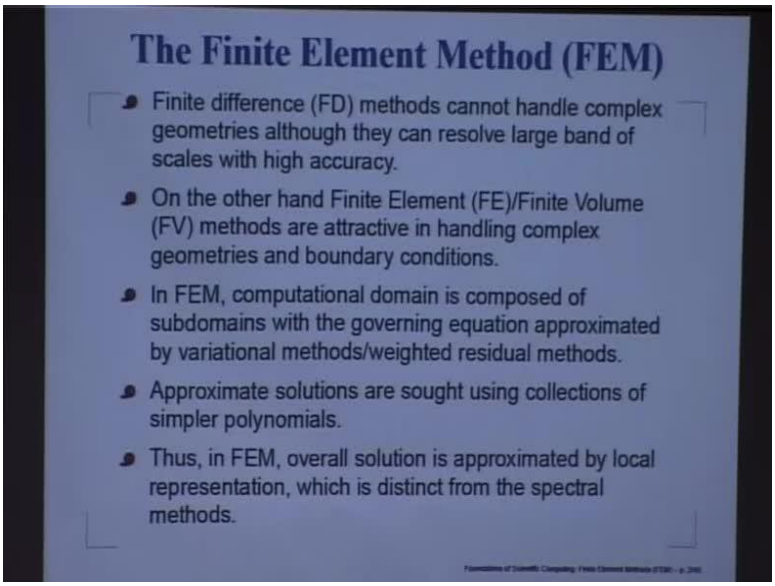

We will focus upon the scientific element of…; we are going to highlight how FEM is similar or dissimilar with other method.

For example, say we do talk about, we did talk about a lot on finite difference method throughout this semester; by and large finite difference methods are quite good in terms of resolving scales with very high accuracy.

One of the disadvantages of the finite difference method that we have seen is, it is little difficult to handle complex geometries.

On the other hand, if you look at the other methods those are available, for example finite element or finite volume method, they are considered the attractive because they are able to handle complex geometries and therefore, they can take care of complicated boundary conditions too.

Now, if that is so, we need to really find out if this method - finite element in finite volume methods - also satisfy those nice properties which FDM gives automatically, I mean, not automatically, you will have to a look for it and device better methods.

Keeping our point of view focused on FEM essential idea, the big picture is that you have a computational domain - you split that computational domain into small sub-domains and you try to satisfy these governing equations in these sub-domains in a piecemeal approach. And how do you satisfy the governing equation that is also equally

important. You just do not simply just plug it into the governing equation and say, well it is satisfied, no. There are certain developed methodologies from calculus of variation or what we are going to talk about is little bit on weighted residual method.

So, what we are saying, that in FEM we will assume a form of the solution which will call of the trial solution or approximating functions and then we will plug into the differential equation, and then we will see that the differential equation will not be exactly satisfied. Having done those error analysis, we are quite familiar that even with our best intention, we have left with some errors to handle.

At the equation level, that is what we call as the residue, and in FEM we try to reduce this residue; not exactly it is impossible, so you do it in some kind of a mathematical sense which is given by this weighted residual method; so, we will talk about that.

One of the things that distinguish FEM and this class of FEM, FVM and FDM versus, let us say, spectral method, are the solution method itself.

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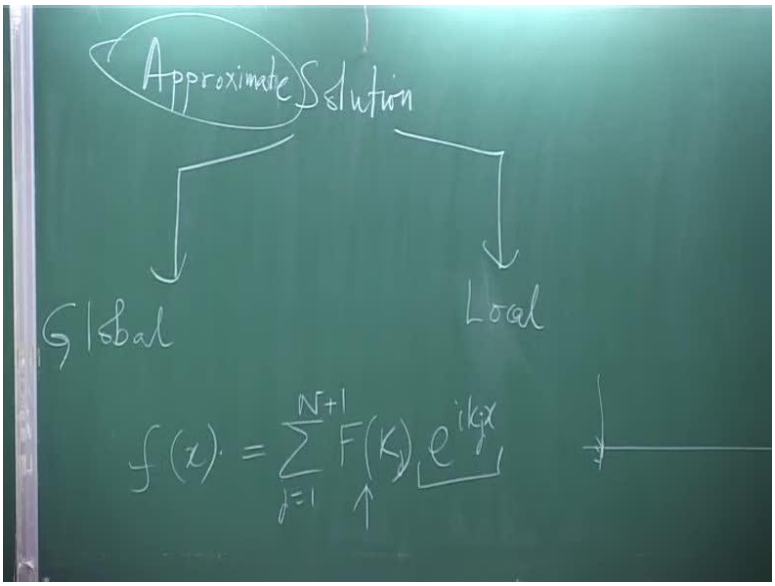

What we saw, that if we are looking for solution, then we actually start off with some kind of approximation, that is our goal. So, approximate solutions can be actually broadly classified into two distinct categories - one is the global solution, another is a local solution.

What do you mean by global solution and what do you mean by local solution? Suppose, I have a function f of x and I write it like this, then what you notice that we are writing the solution in terms of a function whose space dependence is given by this.

This is the kind of an approximation, we are talking about approximation; so, this is one set of approximation where space dependence is given, let us say, by the Fourier series.

Now, what you then try to do is, you try to find out, what this amplitudes of this Fourier series are by plugging this expression into a differential equation, and you get equations for f of k and that is a method.

Now, if I get a solution, which j equal to sum 1 to n and then I decide to add one more term, then what happens is, what about this function? If my domain is like this, let us say, my domain is like this - it starts off from here and goes on till here.

So, if I keep changing this function here, little bit, instead of k j, I change it to k j plus d k, a small change, then what will happen is that effect will be felt all over the domain; so, that is what is a global method is. So, a change in the approximate solution as a global effect across the domain, that is what we mean by global. So, one of the simplest example of this is the spectral method - Fourier spectral method, that what we wrote here.

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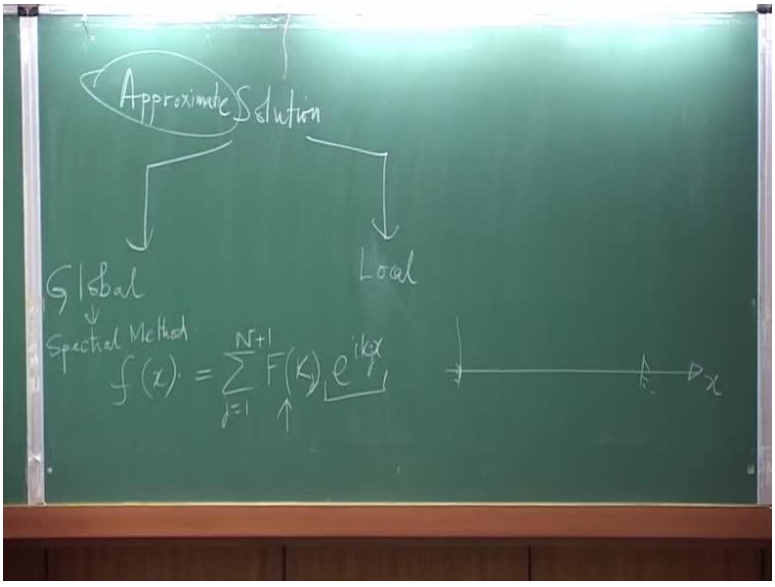

Now, there are plus points and there are minus points. The plus point is, when you apply a global method, then you could take a fewer terms, fewer terms and get very accurate solutions.

So, for example, if I am trying to solve an equation, differential equation of f and then I may take sixteen terms or twenty terms and I find that is going to give me a pretty good solution.

So, global method - one of the strongest point is that you can get by with taking far fewer number of points, that is good; but what happens is if you make a change in this global solution components, anyone of them, its effect is also global. So, that is one of the issue.

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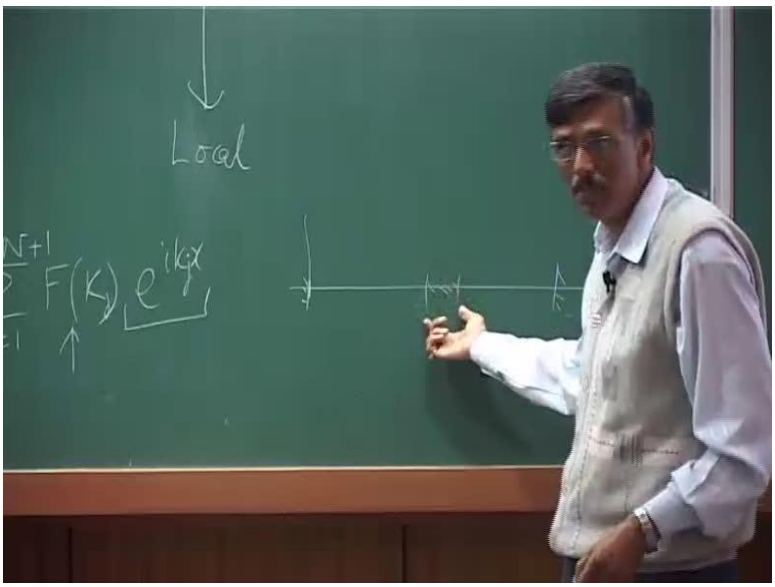

In contrast, if what you could do is you could keep solving problem in a local sense; I could identify a sub-domain and develop a method, apply it there, and then I try to solve it and there if I make some change here, the effect is kind of a local effect; it is not going to be percolated there.

In fact, having done this finite difference method in such great detail over this semester, now you have realized that when we did those explicit spatial discretizations, they are like your local method, whereas a compact schemes which we did, they are like global method because their each and every node were connected by those auxiliary equations that we wrote for this derivatives. So, you can see that even talking about in these

generalities within each method itself, we could distinguish between global and local methods.

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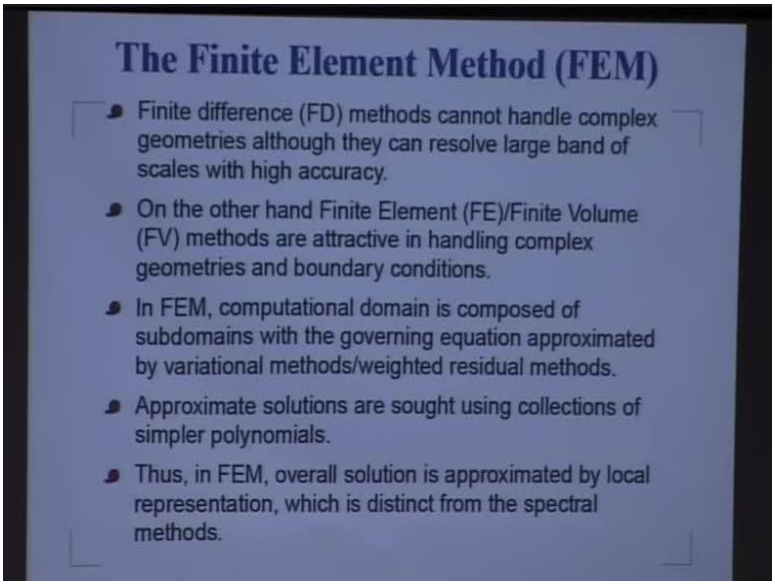

So, that is what we are saying here that in FEM we will decompose the problem into smaller sub-domains and try to satisfy governing equations, in a weighted, by some weighted residual method. And one of this essential element of FEM is this approximate solutions that we are writing, they are going to be simpler polynomials.

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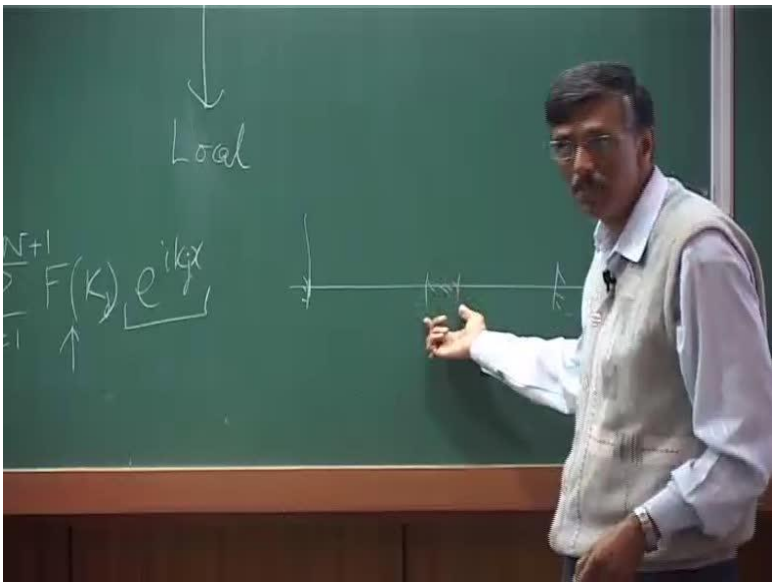

Here, it was like a cosine or sine function which is a global function, but in FEM I would probably approximate the solution in this and I could probably say, look if this is one node and that is another node, my local solution be either would be like this at decomposition of two nodes, this plus that, we will see. So, this is something.

If I take the linear basis functions for this FEM, I would probably do that; so that that is what we are saying here that we will take it as simpler polynomial. So, I am just showing, the lowest order polynomial that is possible is the linear variation. But, so you see this is the sum and substance, then FEM is a local representation which is distinct from global methods, like spectral method.

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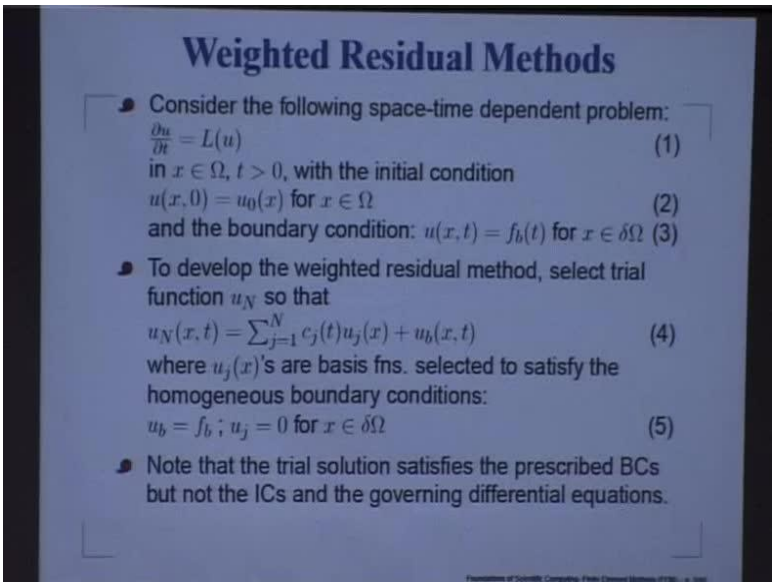

Now, what are this weighted residual methods that we are talking about? Take a look at equation one, it is your generic problem where you probably do some kind of spatial discretizations and you end up with some kind of evolution equation like the one that is given in equation 1. You define it in the domain x in omega and you also define the initial conditions and the boundary conditions as given in 2 and 3.

Now, how do we develop this weighted residual method? You have to select a trial function which we will call as u of N; u of N will have two parts. The first part relates to a space-time dependence of the problem where this u j of x is the specific qualitative dependence that we are going to prescribe, whereas I told you, like if I look at a domain

and if I say, my u j of x is like this, so these two together I could say that look, look this is my u j of x; so, I prescribe that space dependence there.

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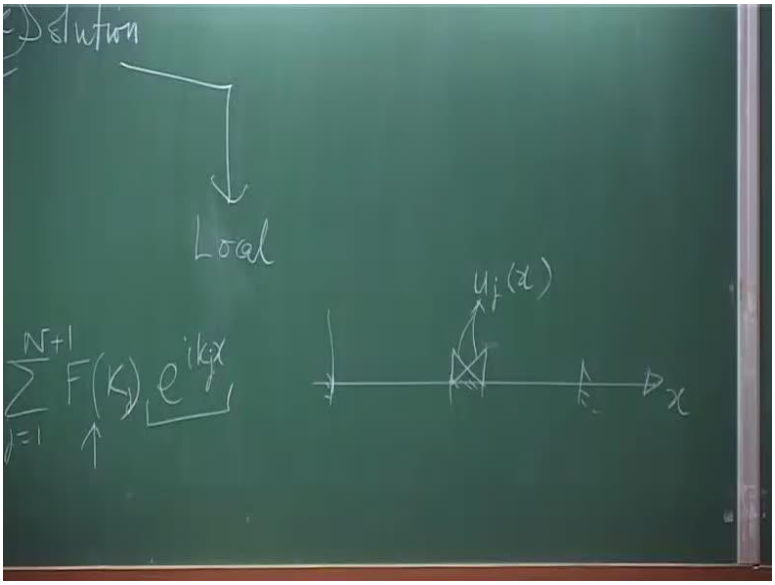

In addition, this in equation 4, this last part - u b of x 2 is put in there that actually helps you in satisfying the boundary conditions.

So, what you are trying to do is one's part of the trial solution is geared towards satisfying the differential equation, another part is geared towards satisfying the boundary condition. Well, I said that it is geared towards satisfying the differential equation that is not necessarily guaranteed because we are not specifying what this c j of t is; so, the time dependence is kept as it is.

We do not know a priory, what it is, but space dependence - we are making some kind of a local guess. This is like what we did in our explicit method, we locally fitted a polynomial, so the that is like fixing your u j of x. If I take a second order central difference scheme, then basically I am prescribing this u j of x is a kind of a quadratic polynomial; so that is the way we do.

However, here what you are doing is almost like your separation of variable - there is a space dependence path segregated from the time dependence path. However, I would not explain it to you write away, but please do make note of the following fact that the time

dependence path, if it is truly time dependent and not space dependent, then what this subscript j is doing there.

This is often not very clearly explained, we will come back to it later, but please pay attention to this, it is just not simply separation of variable the way we understand. If it was a pure separation of variable, I would not identify this c of t with individual node.

There is an implicit space dependence also built in there; so, this is something we will come back to it. In fact, I failed to see in most of books from finite element where this part is highlighted, but that is a one of the strongest point of FEM which people are not sort of really highlighted.

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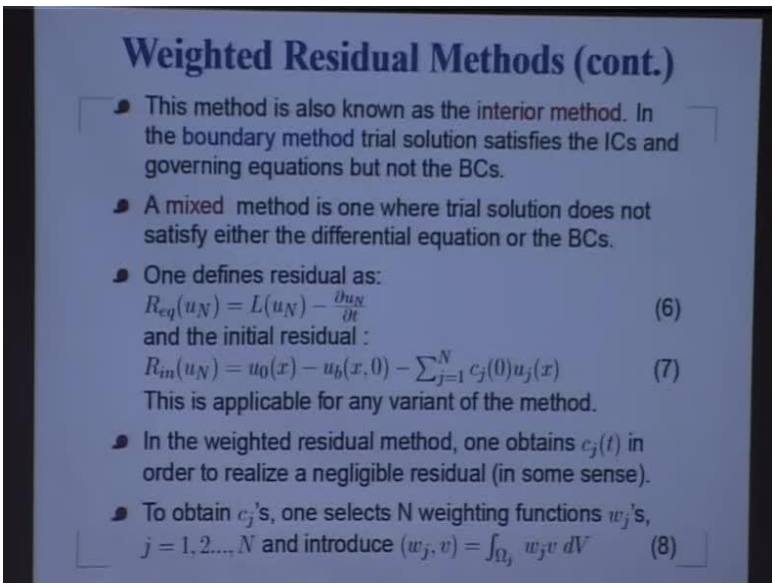

Now, what we do? We prescribe a trial function u of N given by 4, such that the second part of 4 automatically takes care of the boundary condition, and then we try to classify in a sort of a rudimentary way, all this collection of methods, which we can use, where actually we satisfy the boundary condition explicitly by the choice of this u of b; then we call those methods are the interior methods.

In contrast, you can also have boundary methods by the trial solution is chosen in such a way, that it tried to satisfy the initial conditions and governing equations as accurately as possible, that does not satisfy the boundary condition.

So, boundary methods do not satisfy boundary condition and interior methods satisfy the boundary condition; this is the essential difference.

The third category could be a mixture of the two where this trial solution need not satisfy either the differential equation or the boundary condition.

Now, if I, even if I decide to choose the trial solution in such a way that my intention is to satisfy the differential equation and plug in into the differential equation as given in the right hand side of 6 here. Now, what you would be noticing that even with your best intention, that quantity will not be equal to 0 and that is your residue or the solution error that we talked about. So, our equation is basically residue of the equation, as obtained, with respect to the trial solution that we have chosen.

Now, we can actually plug the same expression for our initial condition and you would be even surprised that, that also may not be the, the, satisfied even with the interior method.

So, that is why I said that is a kind of an artificial way of classifying methods because even with your best intention, you will find this residues are not going to be equal to 0. So, that is something we need to keep our self cognizant about it, so we should be careful about it.

Now, we did not say anything about what this c j of t is going to be? So, what we are going to do is, we are trying to find this quantity c j of t by looking at the residue, and looking at the residue is not only going to be a just looking at it, we have to basically minimize this residue in some sense. We need to minimize this residue to some negligible value in some sense and then we try to find out what this function c j of t are.

One of the way of finding this functions, c j of t is to select some kind of a weight functions. Well, the weighted residual method that fabricates itself (( )) mean tells you that we are going to talk about some weights; that is what we are going to do.

Let those weighting functions be w j; and how many c of j we have? Well, I mean, as you have seen, we have taken N number of such functions in 4, so what we need to do is we need to derive equations for N number of such equations, for evaluating N of this c j's.

So, we will choose first step is some weighting functions, what they are? We are going to talk about it shortly, N of them, and then we define sort of a norm.

Some kind of a norm of a function V with respect to this weights, which will be defining by this, separated by a comma within a bracket and that is nothing but (( )), trying to evaluate that quantity over the whole domain with the weight functions multiply; so, that is what we are going to do.

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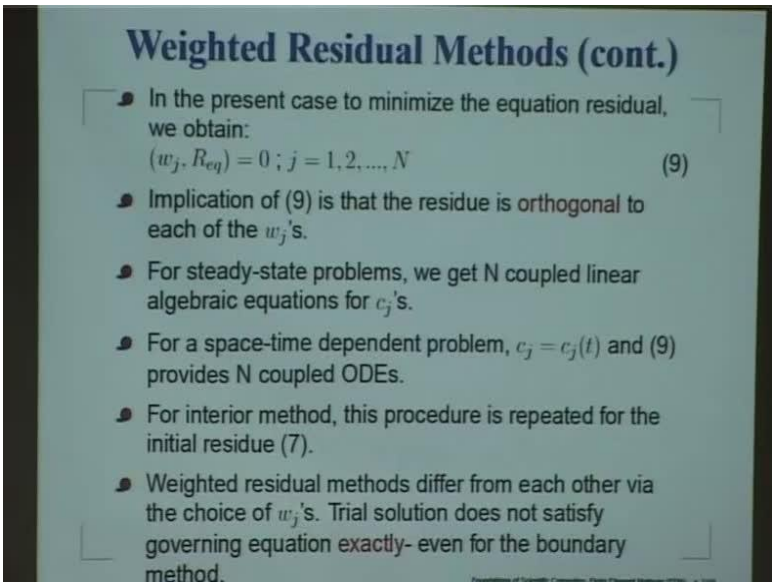

Once we define this norm, then we would like to minimize the equation residual with respect to this weights, that is what this equation 9 is; that is the cardinal principle on which we will be working on this weighted residual method.

So, we will basically reduce this equation residue with respect to each and every one of this weight functions w j.

Well, whenever you have such a norm equal to 0 that means what? That means that this weights w j's are orthogonal to this solution residue, that is the mathematical definition of orthogonality.

If I have two functions, if they are orthogonal to each other, I can take a product of it and integrate over the whole domain; if they are not correlated, integral cannot be 0. Correlation means non-orthogonality; orthogonality means non-correlation. If two functions are not correlated, the integral of the whole thing should be equal to 0.

So, there also we are trying to say that look, I will choose my trial solution in such a way, that the residue of the equation that I would get should be orthogonal to some particular choice of w j's; that is what the second point implies here.

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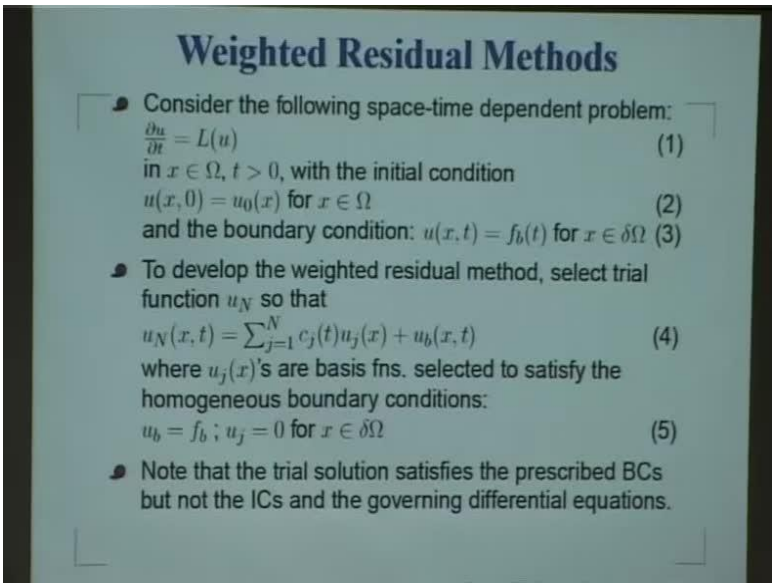

Now, if my original problem was time independent, then you could notice that we would not be taking c of j as a function of t, that will be a pure constant here; a time independent problem, we will just simply write c j of t times u j of x.

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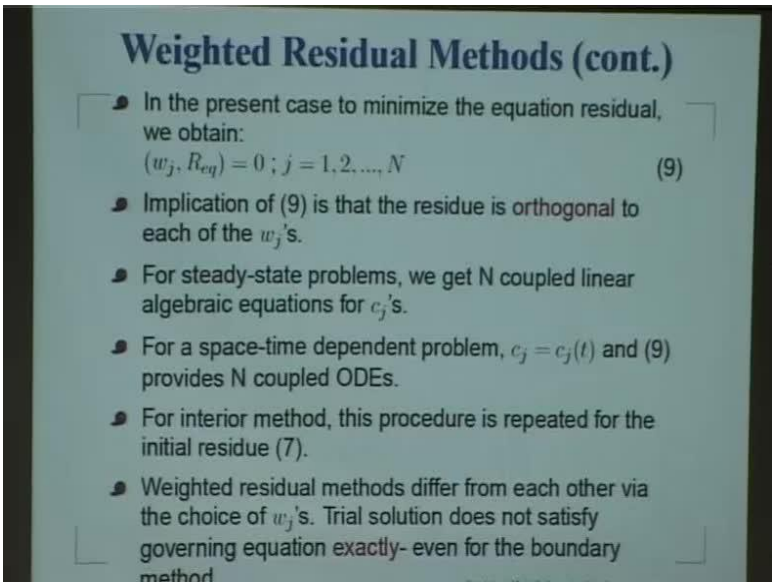

And in such a case, equation 9 would give you what? Some kind of an algebraic relation for the c j's, depending on the type of equation that we have, but that they will still be a linear algebraic equations for c j's. Whereas, if you have a complete space time dependent problem, then this c j's will be, of course, a function of time and since we have already prescribed the x dependence and put it into the differential equation and evaluated the residue and then performed this integral with respect to w j, w j's are also functions of x. They are also space dependent functions and you are integrating over the whole domain. So, space dependence parts is integrated away, what remains is only the time dependent.

So, that is what we are saying that for this space-time dependent problem, equation 9 would essentially give you some kind of an ordinary differential equation for this coefficient function c j of t.

So, that is the essential difference between a time independent and time dependent problem. In one case, you will get a linear algebraic equation; in the other case you will get a coupled ODE's.

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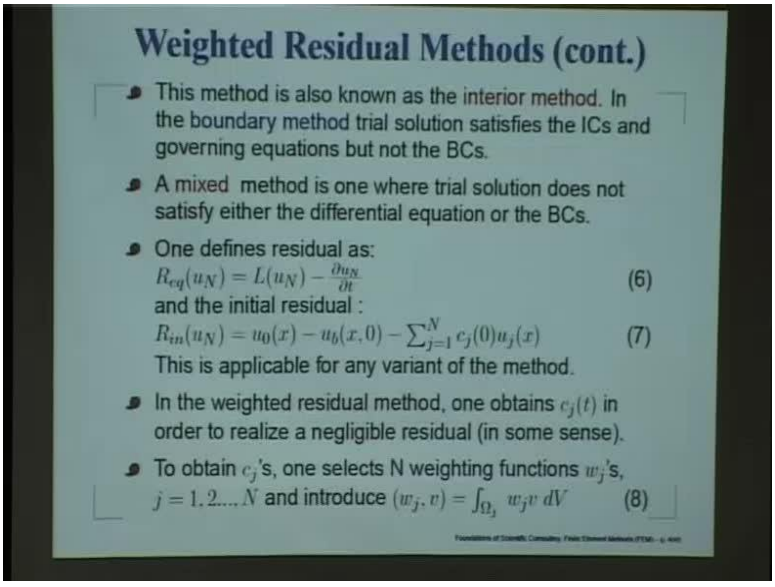

Now, if you have a space-time dependent problem, so we are going to have ODE for the c j's. So, you would require initial condition for those equations that you get it from your initial residue, that we have already defined. We have shown here in equation 7, what the

initial residue is? So, from here I could get some relationship for c j at t equal to 0; so that is the essential idea of using that initial residue as well.

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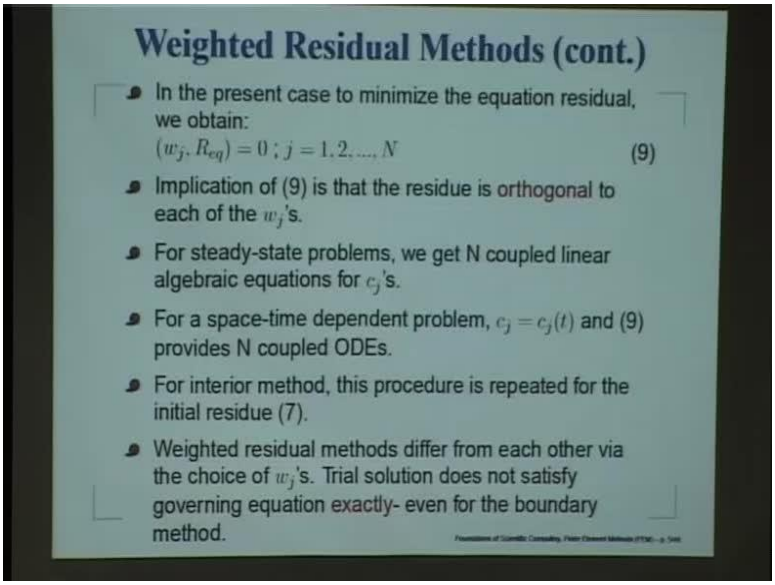

Now, as I told you that we will have a whole range of weighted residual methods, each one will defer from the other by the type of choice that we exercise in picking up in this w j's. We will talk about it shortly. However, we must note this, after all these things the trial solutions would not satisfy the governing equations exactly, even for the boundary method and that is why, we are getting this R eq, R of eq is nonzero.

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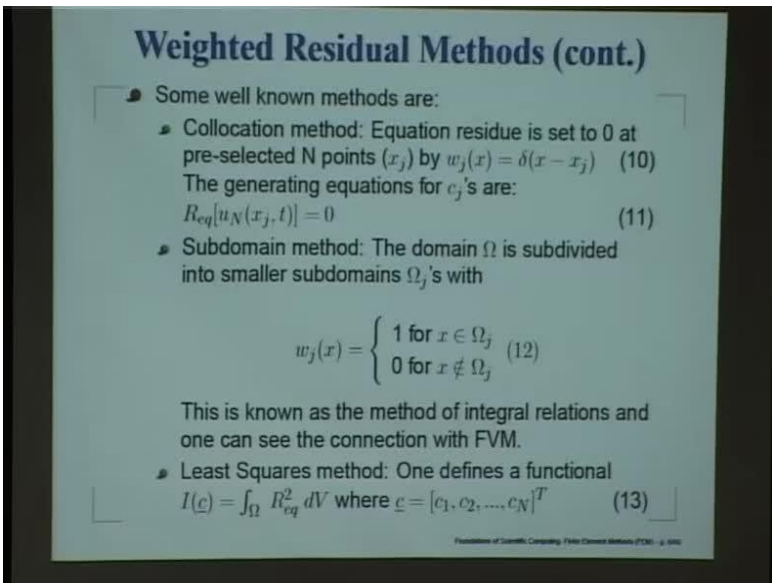

So, let us now move over and look at some of the generic classes of methods those are used. The first such method is the collocation method.

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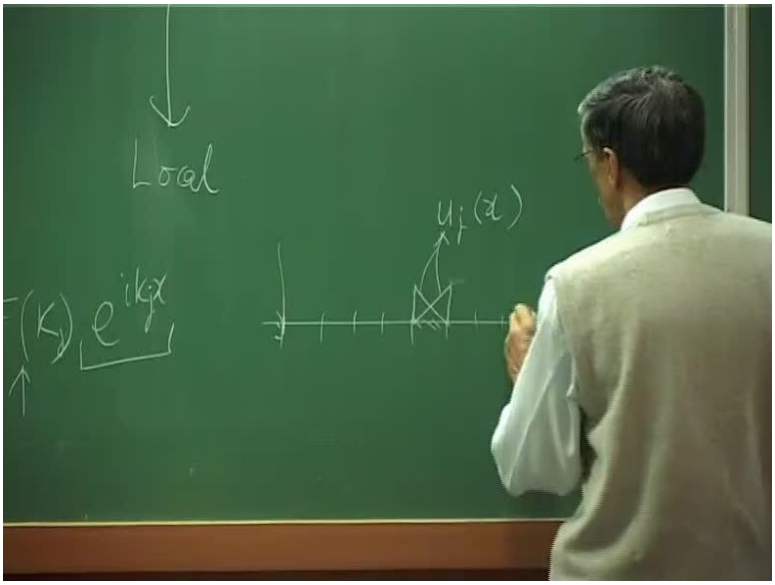

Collocation method implies that, let us say, I have the domain like this and then I have a discrete nodes like this and I decide to satisfy this differential equation at discrete points, those points are called the collocation points.

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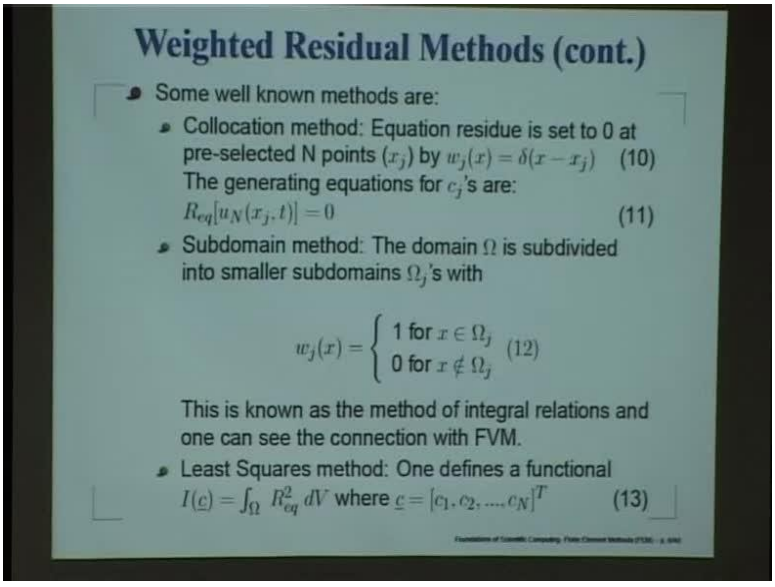

For example, I could choose a point here. I could say that this is where I want to set the residue exactly equal to 0, that means what? I am actually performing an integration of

the weight function with those solution residue, that the weight function itself is a delta function; so, that is nonzero at x j and anywhere else it is 0. So, that is the essential idea of collocation method and you can very clearly see this is what you did, we did for finite difference method.

Is not that what we did? We looked at the differential equation in a discrete form, that what we called as the difference equation and we equated them as those nodes at the finite difference nodes, so that is equivalent to your collocation method; so, that is what we did.

However, we could also do something slightly different and which we did not have time to do is, we could divide these domains into smaller sub-domains like what I indicated here.

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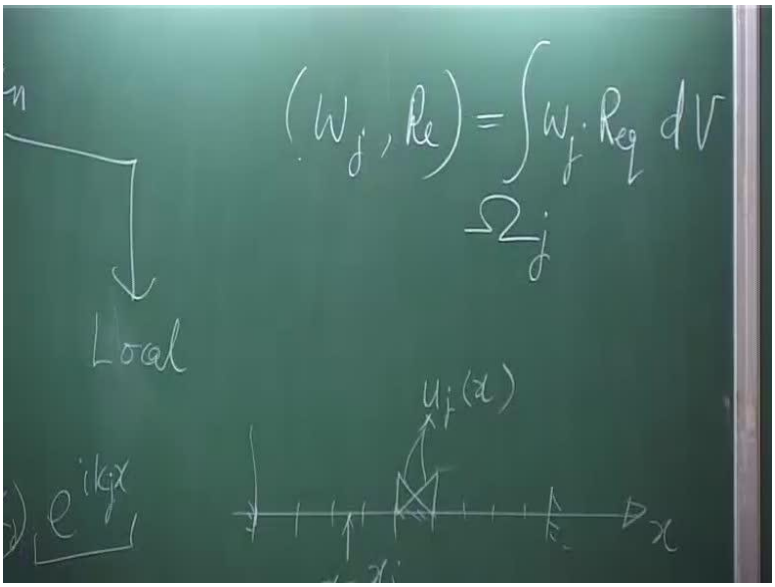

If I divide there and I decide to set W j equal to 1 in that particular sub-domain omega j and everywhere else it is 0, that means what? I am integrating the residue in that sub-domain instead of doing it over the whole integral, recall that is what we have defined. So, if I talk about this, so this would imply that I am doing W j times R eq and this I am doing it over the whole domain.

Now, in the sub-domain method, I will just do it only over that particular subdomain there, then what does it mean, that in a sense putting W j equal to 0 everywhere else

except that particular sub-domain and that is what we have given here in equation 1. Now, this method is what is called as the finite volume method.

So, you can see the connection of the FEM with different methods, like as I told you here, FDM would be more like your collocation method, FVM in the finite volume method would be more like your sub-domain method.

Now, this particular method is also called method of integral relations and one of the interesting aspects of it is the following, that you see - whenever we develop a new subject, could not start always from scratch, we start with what exist before. For example, when we try to solve a computationally problem, what we do in FDM? We actually start off with a differential equation.

What does it actually tell you that it gives you a conservation principle has applied to a single small infinitesimal element, that is your differential equation with the element size vanishing. However, when you go to compute, it is, it is not that you are doing in an infinitesimal element, you are doing with a finite size element or domain or whatever you call it.

Then what happens? Why should we then take this circuitries route? First derive a differential equation and then again approximate it over a domain and then we satisfy that equation in an integrated sense over a finite domain.

So, in finite volume method what is done is this first part is eliminated. What you do is you try to satisfy those conservation principles in those finite domains itself or finite volumes; that is why it is called finite volume method. So, what you do is you identify a small element or a small volume and then apply your conservation principle directly to that finite volume or finite element.

That is not the same as starting off from a point and then integrating over a domain. I do not know if you appreciate the fact, but this is of prime importance that we appreciate what we are doing. It is not just a simple, I would say, splitting the terms just for the sake of it, but this relates to quite a significant difference in the different methods that we get

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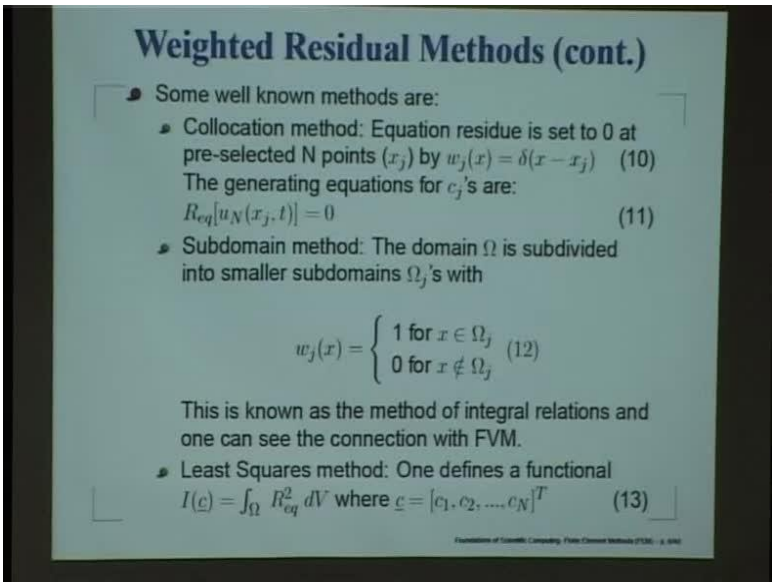

So, now, we can see under the umbrella of weighted residual method, we can classify FDM or FVM also as a sub-case.

Now, in each of these two methods that we just now talked about, you still trying to get the residue equal to 0 either in a particular point or over a sub-domain; that is what the name suggests. Whether you are doing collocation or you are doing integral method of integral relation, you are still trying to put the residue equal to 0 exactly at some of those points.

However, you just do not want to do that because you are aware that it is very difficult to make the residue exactly equal to 0. What you could do is you could accept that there would be some residue, but let us try to minimize that residue. One of the way is minimize the residue in a least square sense. So, what I am going to do is I will define a functional, which I am calling it I of c. This is a some kind of a vector, which is defined in terms of those coefficients in your approximating solutions, then what you are saying that I will choose this c 1 to c n in such a way this I of c is minimum.

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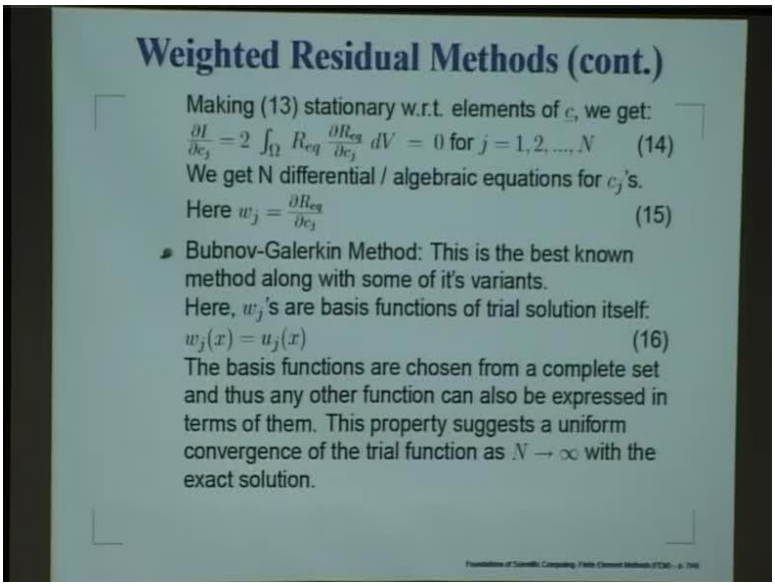

So, if I do that what I would need to do is I will just simply differentiate the objective function of the functional I, with respect to each of these c j's and that would actually give me this because your function was R square. So, when I differentiate it with respect to c j, I will get 2 or equivalent times the partial of that R equivalent, an equation with respect to c j integrating over the full domain and that is equated to 0.

So, of course you can very clearly see from equation 14, if you look at this, this quantity, this partial is itself is nothing but your weight functions.

So, this actually gives you some kind of a basis for choosing the weight function. You choose it in such a way in this method, so that your residue is minimized in a least square sense. Now, this is one way of doing it.

One of the oldest and the classical method is due to Bubnov and Galerkin. This is one of the best known methods and it has a few variants also, we will talk about them.

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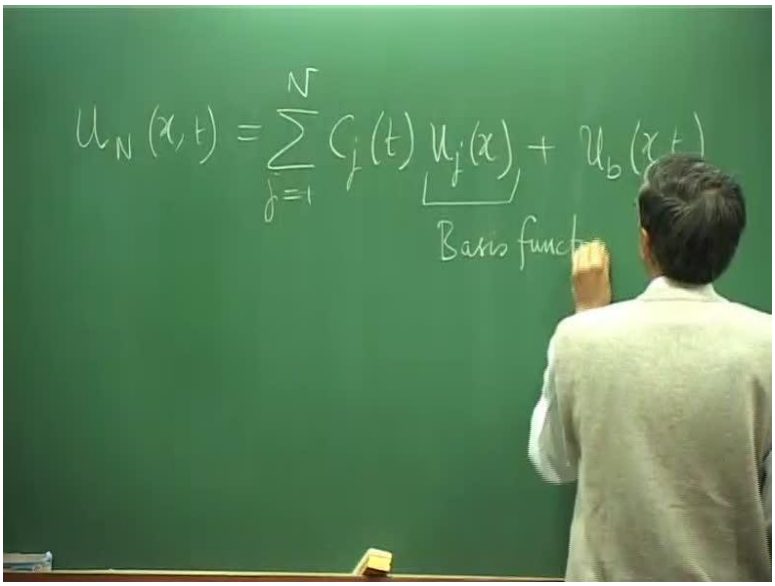

In Bubnov-Galerkin method, you choose this weight functions as the basis functions that themselves; so, if you recall what we did, we wrote the trial function as this. So, what we are saying now that these are what are called as the basis functions.

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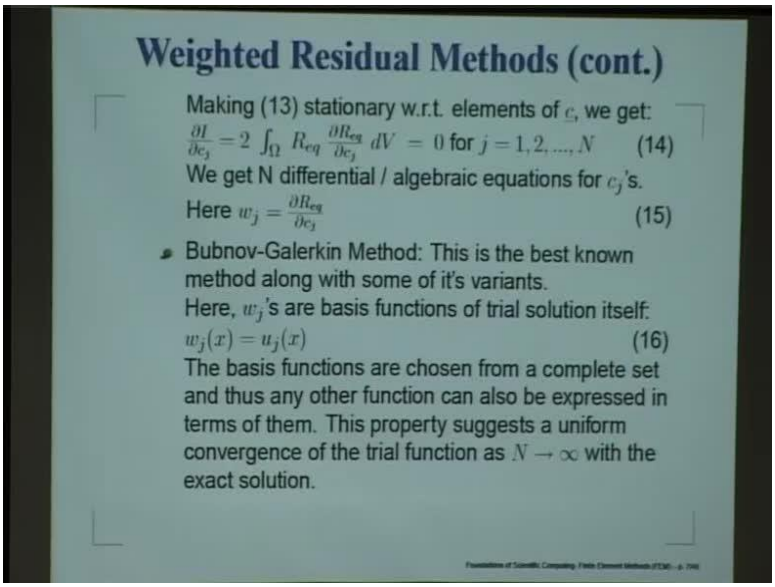

So, in the Bubnov-Galerkin method, we choose the weights themselves as the basis functions; so, w j's are nothing but u j's. Now, if you try to relate this Galerkin method, you can see the necessity for choosing a complete set for this basis function, why? Because you are trying to obtain a trial solution which could be very arbitrary and if I am

trying to show it as a linear combination of a set of functions, it is quite expected that I try take it as a complete basis functions, like what you do in your Fourier series or Lagrange polynomial or Basel functions; you would, you are familiar with some of those complete sets.

And then what you do is, you automatically satisfy what is called as a uniform convergence because we know that in the limit n going to infinity, I can define any function; any function that we can talk about and that is the whole idea of this.

However, you can see that some of those complete sets of functions that we have just now talked about, they happened to be global functions, like if I take Fourier series sine cosine functions - they are global functions, you take Lagrange's polynomials or Chebyshev functions - all those are global functions. So they are good, but their global methods, we are not going to talk about this global methods.

So, we will have to probably not worry too much about this uniform convergence at this point in time because we have decided that that we will choose this basis function, which has to be in local in nature; that gives us some mathematical simplification and operational ease.

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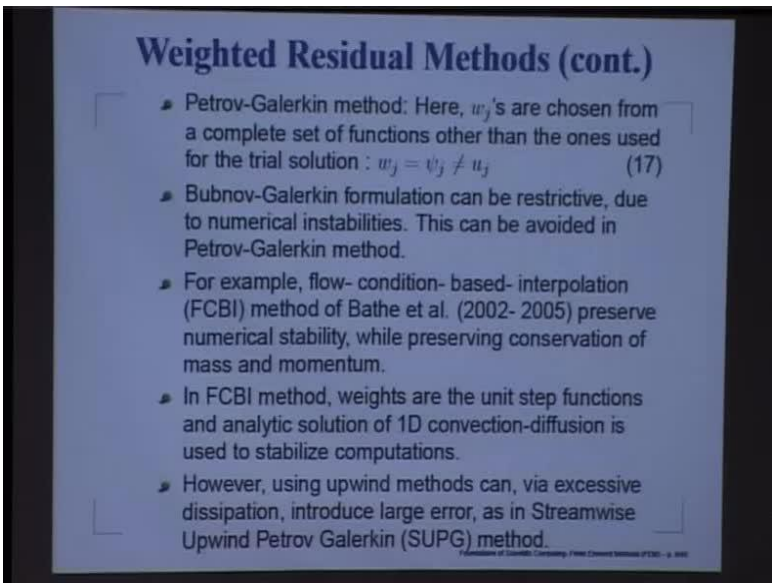

There is this other class of method as opposed to Bubnov-Galerkin, there is this Petrov-Galerkin method. You notice that these all these mathematicians are from Russia, they

have developed this subject for a long before computers came into being; they were ready with the method.

So, Petrov to Galerkin method and he said look, I mean, let us choose this weight functions as some other complete set of functions, it is not necessarily have to be those basis functions that we have chosen in approximating the solution.

So, that is what 17 says that w j is not necessarily of u j, that could be psi of j and the psi of j could be a complete set of function because your intention is that you are going to take some kind of a local approximation to the basis; whereas, we are trying to satisfy their solution residue in some sense while being able to have a solution process, which is as general as possible.

So, let us not restrict those w j's as those local polynomials; let us instead pick them up from a complete set of functions, that is what equation 17 suggests.

This was actually necessitated by the observation that many a times, one finds that this Galerkin formulation suffer from numerical instabilities. So, this has been one of the major issues that up to 60's and 70's, people are focusing their attention in developing Galerkin methods, but then the Bubnov-Galerkin variety. And people have experienced difficulties time and again, and this has to do with the numerical instabilities that we talked about and they decided that we will choose instead not the Bubnov-Galerkin method, but Petrov-Galerkin method, and we will exercise our choice of the weight function in such a way that numerical instabilities can be circumvented.

For example, this method called flow condition based interpolation method, FCBI or let us say, this stream wise upwind Petrov-Galerkin method due to Hughes in, so (( )) Bathe is in MIT and Hughes from Stanford.

So, they essentially tried to use Petrov-Galerkin method with the idea that we have seen that if we do not satisfy the physical principle, that is when we get numerical instabilities.

While discussing about the property of compact schemes, we noted, like if the information is propagating from say, left to right and my closure is such that it is going from right to left, that violates the physical principle and that leads to numerical

instability. So, this is essentially the same idea with just different acronym that has been stated here, it is called flow condition based interpolation method.

So, all like to in a fluid flow problem if the flow is going in a particular direction, I will choose my, I mean, this weight functions in such a way that is dependent upon the flow condition; so that is what it is.

It is claimed by the (( )) Bathe and his group that it does preserve conservation of mass and momentum very accurately in flow problems.

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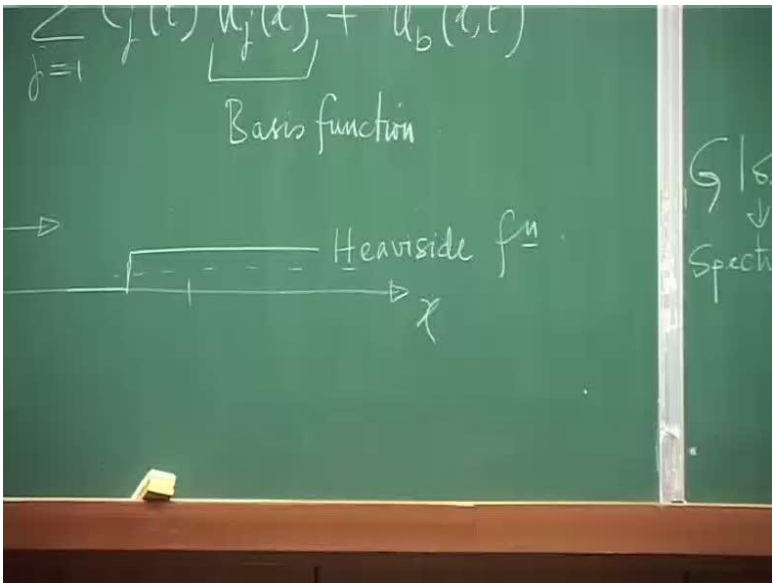

What is done here? Of course, the weights are chosen as step functions or Heaviside functions. So, if I have a domain like this and if this is a cell, I have a sub-domain like this and let us say, the flow is going like this; so, there is distinct bias of this solution. Then what is suggested in this FCBI is that you take the weights which would be like step functions, so that this is not going to affect this, but this element can affect the downstream element. This is like what we talked about our boundary layer solution.

If you recall, in the very beginning we talked about the boundary layer solution. We say, that if boundary layer develops here - not necessarily fluid dynamic kind - boundary layer of any equations, there we could see that a shear layer kind of boundary layer forms which has a directionality; that means, the solution only, any point affects the points downstream, but not upstream. So, this is what is the suggested that you get that property

through this Heaviside function or the unit step function; and the space-time dependents, and the space dependents of the solution is obtained from one dimensional convection diffusion equation; that we have ourselves investigated, yeah?

So, that is one way of doing it. However, we must be cautious about doing.., these kinds of essential ideas are nothing but similar to what we called as the appending methods.

That is what we did even in finite difference. You recall that we said that if the physics of the problem is such that this is the way it goes like this, then we do not try to do a central difference, we try to do a one sided difference because the information propagates in this direction; that is what we have seen, even that a first order accurate method in solving the convection equation turns out to be the most accurate method.

You recall that 1-D convection equation, we saw that first order upwind solution gives better solutions because it supports the physical principle. However, if you do not do it carefully enough, then we have also seen that there could be excessive dissipation coming into picture. In fact, this appending methods can introduce very, very large error and this is one of the issue with this SUPG method that is used by large number of people, they can produce engineering solutions, but as far as producing very accurate scientific solution are still not in their domain.

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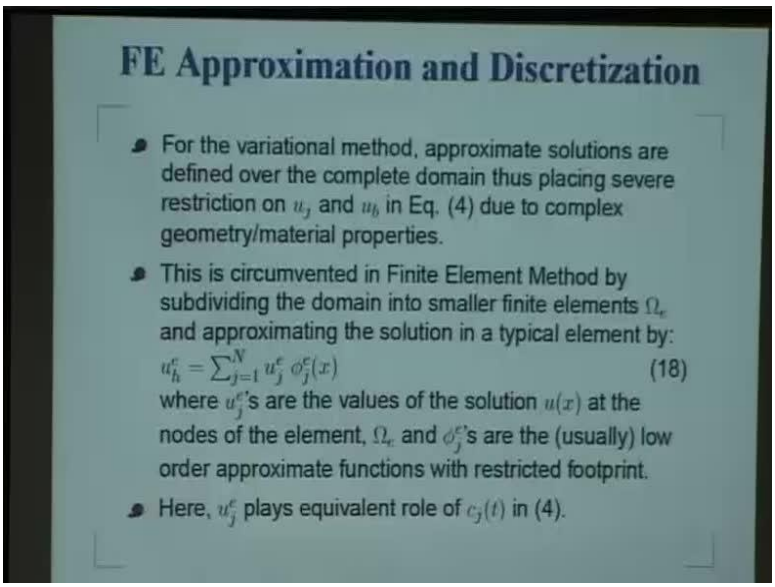

Now, when we go to finite element approximation and discretization, we basically as we said, we take a trial solution of this kind and then what we need to do is that we subdivide the domain to smaller elements; that is what we are talking about here.

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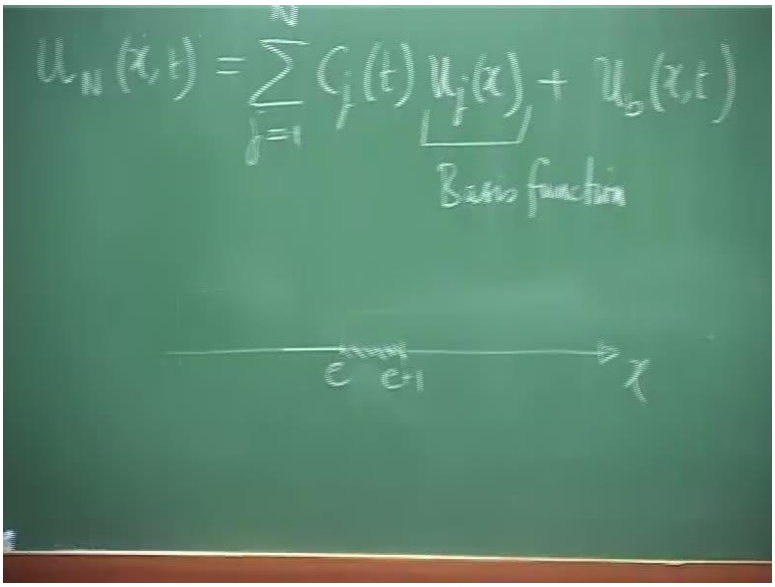

So, if I talk about a particular element here, this is the element, is defined let us say by the nodes here e, and let us say this is e plus 1; so, this is your domain element e. Let us erase this part now.

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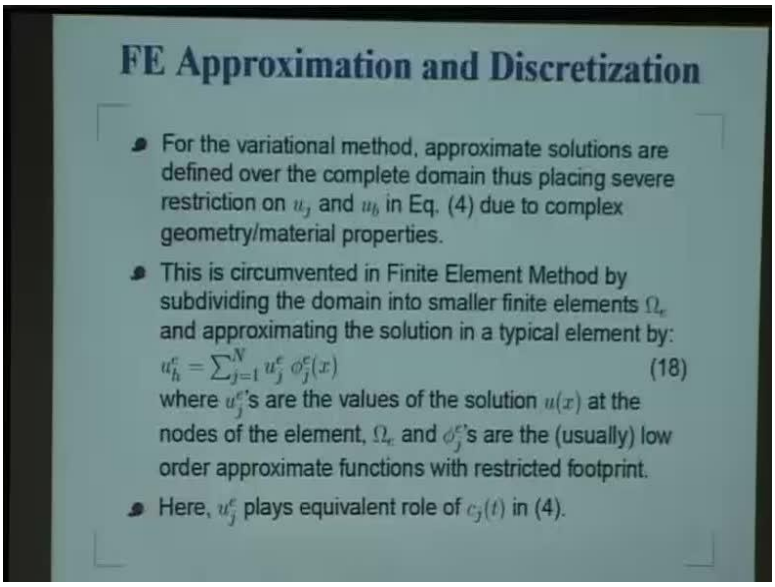

Now, what we are going to do is basically, in this element level itself we will show, the approximate solution would be dependent on where the element is and what is this element size is? We have mentioned that FEM and FVM are favored because you have complete flexibility in choosing that h size.

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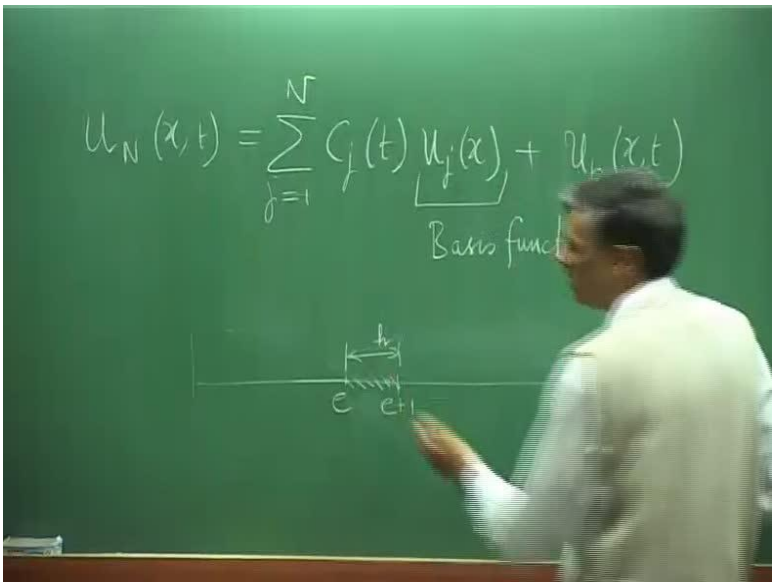

So, that is why that spacing between the two successive nodes or the width of the element is what is we are calling as h. So, what we find that we can write that element level solutions in terms of the node values.

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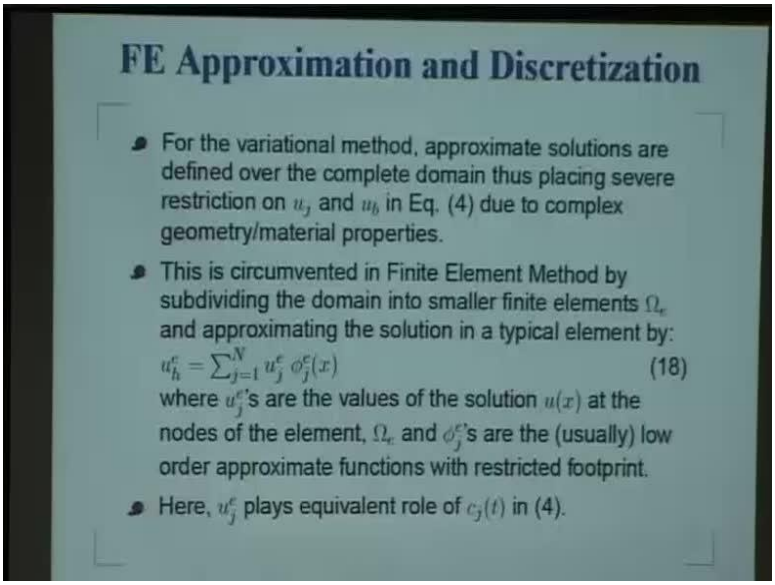

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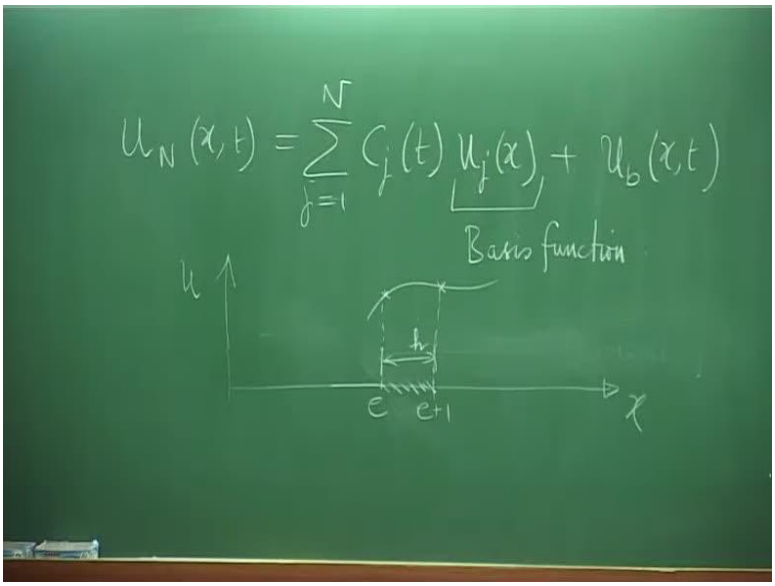

So what has happened, let us say if I am plotting u, so I have the solution like this, like this. So, what happens is at this node I have these values and those values are given here, those values are given here, u j superscript e. What is this function? This is what we are going to prescribe as it is dependent, most of the time they are taken as a low order polynomials, either it could be a linear basis functions or quadratic or in spectral element method, we can take a little higher order dependents; but this path is what you prescribed and this is the solution that you are trying to pick up, that is what you are trying to compute. So, they are your, what we called there as u j of e, that this will be u j plus (( )) so on, so forth.

So, this is what is being said that u j is at the values of the solution at the nodes and these are the usually low order approximating functions and which I have a restricted footprint.

What do you mean by restricted footprint? See, if I define a local expansion of pi j of e like this, so it is means what? It is 0 outside this element everywhere on either side and it is nonzero only inside, that is what we mean by local approximation. So, that is what we are saying also that restricted footprint of the basis function that phi j of e. Now, this you can now really see that u j e plays this role here. u j e is nothing but c j of t, that is what we have written there.

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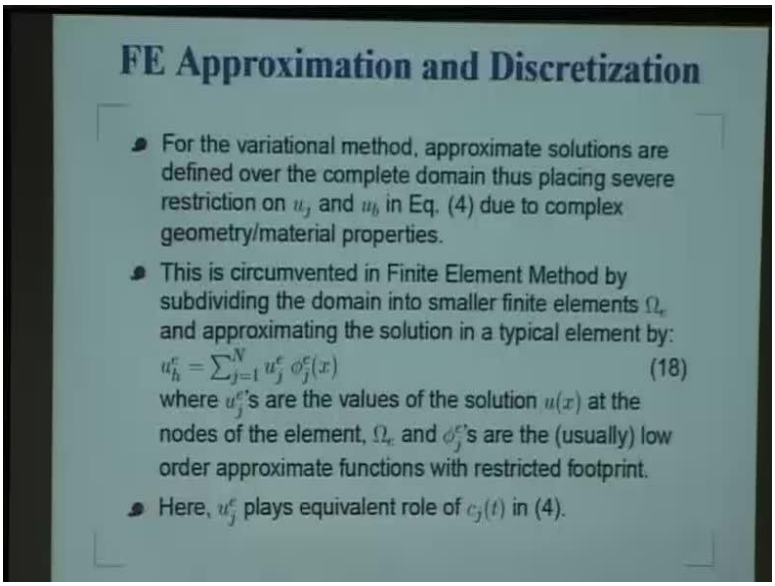

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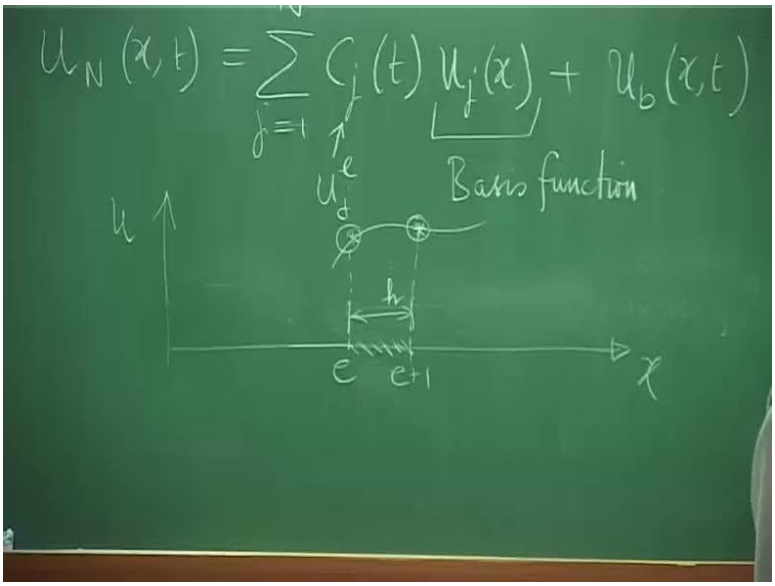

So, that is how we go about and continuity of the solution u of x across element is ensured. The moment I say that u j of e are nothing but the solutions there itself, so for this element this value is there, for the previous (( )) also I take that value; there is no discontinuity at that node point.

That is what is talked about in here that we have the continuity of the solution and that is ensured by picking up the nodal values as u j e. And then we need to put some additional constraint on this basis functions, that we will see case by case.

Now, this nodal values themselves could be time dependent; that is what we said u j e of e could be like c j of t. So, then when I plug this kind of expansion into the differential equation and then do those weighted residual method, I end up getting an ODE for u j of e as a function of time and we need to solve that; so that is what it is.

And since we choose these elements like this and when I calculate the residue with the help of this kind of basis function, so then what will happen is we are going to see that this nodal functions, that we are looking at here because of this low order of phi of e, are going to be like what we have seen already in say, FDM; it will involve only fewer neighbours on either side. That is the whole idea that once we do that we are going to get a sort of a discrete equation and that discrete equation would involve only fewer points, fewer points.

And then, we are going to do that process for all the elements, then we will have a coupled sort of equations. We solve those equations as an ODE and then we get the amplitude functions. And since this is already prescribed, you are now obtaining this, so you get the your solution; that is essential is the whole idea.

So, elements are basically related in terms of their neighbourliness, you would not see one element is coupled to another element which is far distant; so they are continuous, they are in neighbours of each other.

So, what do you do about this u j of x, that is what we are talking about u j of x, or what we talked about is phi j of e of x. We do take them as an algebraic polynomial and based on their order, we can actually identify additional node points in defining this phi j of e. Well, this may seem little abstract now, but when we go and pick up specific examples we will be able to understand.

So, whenever actually we satisfy the equation residue in an approximate manner, this is what is called as a weak form of the solution whereas, if you look at finite difference method, what we do there? We satisfy the differential equation, so called exactly at some distinct nodes, whereas in finite value node, finite element method, we try to satisfy the differential equation in an integral sense over a finite volume or a finite element.

So, whenever you do that, that is called a weak form of the solution, that is what we are talking about. So, whenever we adopt this weak form, then we are satisfying the

differential equation in an approximate form and if there are any derivative boundary conditions, those are satisfied; that is the way we choose those phi j of e h.

Whereas, if you have some derisley type of boundary condition at the end of the elements, they have to be specifically satisfied by the approximate solution; this do not come out by the nature of the choice of approximate solution. And we need to establish that these approximate solutions show solution convergence and they are able to represent any kinds of solutions that you may expect to get.

So, one of the thing is to show the convergence of the solution, this approximate solution must be continuous and differentiable in the weak form. Also, you understand that the essential difference in strong form versus this weak form is - in the strong form, whatever may be the order of derivative in the differential equation, you expect that out of your approximate solution.

You would satisfy the differential equation at discrete nodes and you expect all those derivatives exists, whereas in finite element or finite volume method when you actually integrate it, you are actually reducing the order of the system. So, weak forms are that is why called weak forms; they are weaker in ensuring the convergence of the solution.

So, suppose I have an equation with say, second derivative term present. So, in a differential equation form as a strong form, I expect this approximate solution to be, at least have second derivatives continuous in the whole domain. Whereas, in the weak form if I integrate over the thing, I would actually reduce this requirement by at least one because I am integrating over the domain. So, if there are second derivatives and I am integrating it over a sub-domain, I am actually removing one order of the derivative; I come down to a lower order.

So, this is what we are saying that we, for convergence approximate solution still must be continuous and differentiable in the weak form, but this requirement is somewhat lesser requirement, less respective as compared to the strong form.

The algebraic polynomials that we choose to approximate the function must be complete. What do you mean by must be complete? Suppose, here I showed that in this element I take the basis functions which is linear, does it mean that it will not allow you to satisfy a solution which is constant?

That would be dangerous, that would be dangerous even though I am talking about it is an a linear variation in x, I still should be able to compute a solution where at the element level the solution may be continuous. That is what it means that it must be complete.

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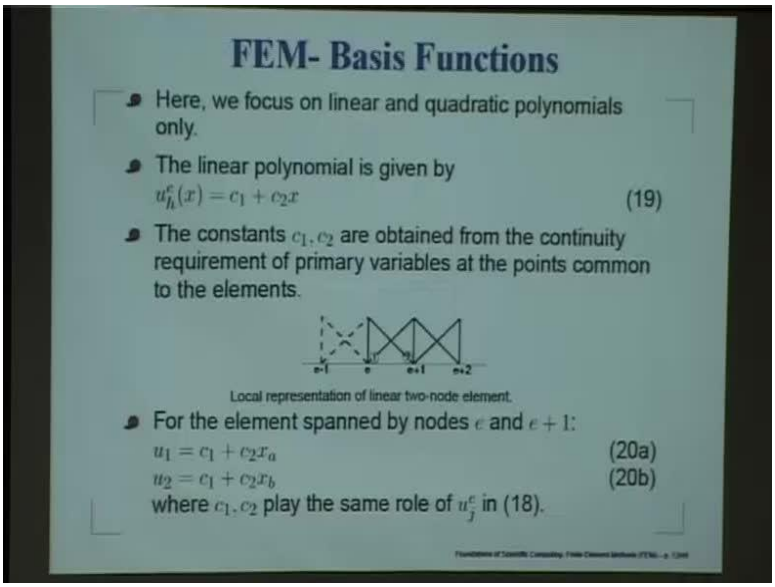

So, whatever may be the order I may decide to take it, say, cubic polynomial here, that would still give us the flexibility to satisfy any functional variation which could be, may be, a constant, which could be linear or a quadratic, but it would surely be able to do that; if it does not, then we are in all kinds of trouble.

So, this is what we must ensure that our approximate solutions must be complete, it must include all lower order terms up to the highest order representation that we have used.

So, let us now come to a very very simple example. It is simple, but we will focus not only on this linear, we will also take a look at quadratic polynomials where we define this space dependence of this function at the element level by a linear polynomial.

So, what you need to do in this equation nineteen then? Basically, we will be looking for this constants c 1 and c 2; if we can obtain these constants, then we know what it is going to be.

So, the local representation of the solution is as given here, let us say I identify the e element is find by this point e and e plus 1 to the left hand point i, let me call it as 1, the right hand side point are let be called that 2. So, basically then what we are talking about

that we try to find this constants c 1 and c 2 by fixing those values at those point 1 and 2; let us say, u 1 and u 2 are given.

And you know the coordinates of this nodes e and e plus 1 as x a and x b, then it remains to be found, how we can get this value of c 1 and c 2 and this is exactly like what we talked about as those nodal values. So, this c 1 and c 2 would be that.

Please note that we choose this function u e of h in such a way that at this node and at that node they are always 1.

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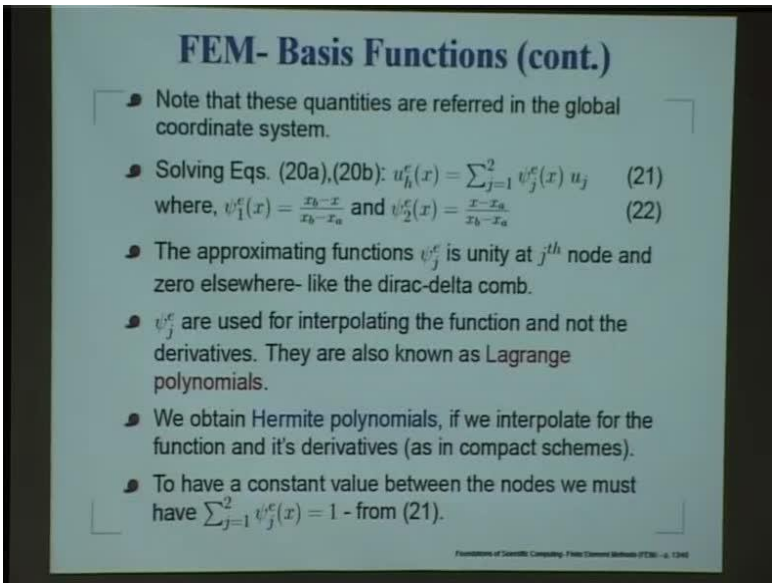

The scaling factor comes from this as we will see in the next slide itself; the scaling would come from here. So, what we are going to do is we are going to show these at the element level, the solution would be written like the space dependent path times the functional value of these nodes.

So, this psi j of e is, they are basically true in number because that is what we saw c 1 and c 2 are there. So, we are going to see the u e of h would be a combination of two linear functions, which is given here as psi 1 of e and psi 2 of e.

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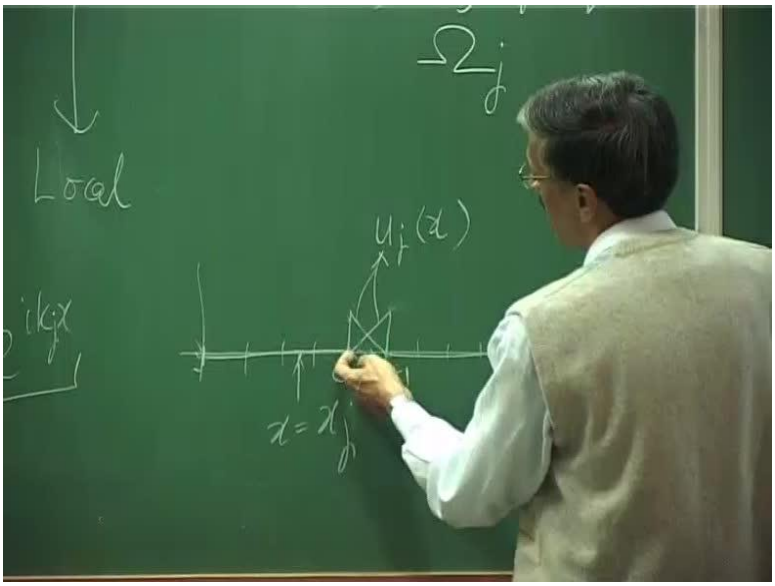

So, one would start from this points, so this is if I call it as e and this as e plus 1, so what I am doing? I am fixing origin, let us say x is equal to 0 here, so I just see that this psi 1 of e and psi 2 of e are such that at x equal to x a psi 1 of e is equal to 1 and at x equal to x b psi 2 of e is equal to 1 and that is what I have shown here.

It starts off from 1 goes to the 0 in the other end and in the other way its starts off from 0 and goes to 1; that is what these two functions are.

So, basically if, when you take a look at this, these approximate functions that we have chosen, they really work like a Dirac delta function because at the node where you are talking about, that is where they take the value 1, everywhere else they are 0.

So, this is the property that we are doing node wise, but at the same time in between the nodes we are saying it is actually not like Dirac function, but it is a linear function.

So, we will, we will stop here and we will wrap it up and there are little bit of discussions needed about interpolation, we will do that.