Foundation of Scientific Computing

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Lecture No. # 34

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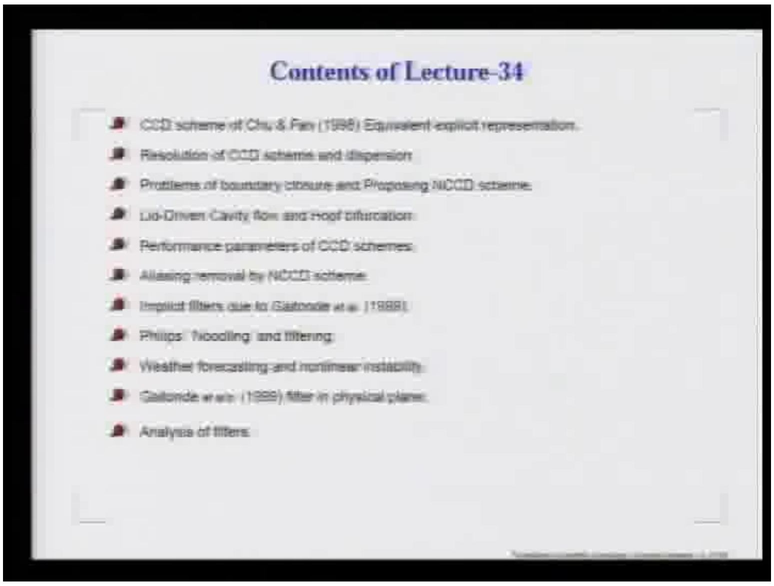

In the last class we talked about aliasing and we did talk about the role of upwinding in controlling, that we particularly pointed out, that, that we must have a uniform resolution across all wave number and in this context, we talk about a method called combined compact differencing scheme or CCD scheme, which was originally proposed by Chu and Fan. In this what we do is, we evaluate the 1st and 2nd derivative simultaneously by compact scheme and hope to expect better accuracy using our spectral analysis tool. We can represent this CCD scheme in an equivalent representation and figure out the resolution, that is obtained by CCD scheme and the dispersion, that it brings in expressing the 2nd derivative term and instability that it brings in for the 1st derivative term.

And once again, we note, that the CCD scheme also has boundary closure problem because the boundary closure used our implicit method, that leads us into suggesting a new CCD method, which we have called as NCCD scheme, which has been pioneered here and we noticed, that all we need to do is replace the implicit boundary closure by explicit boundary closure. And we solve a very interesting problem, benchmark problem of lid-driven cavity flow, which shows flow in stability at a particular Reynolds number originating from hop bifurcation and then we just show, compare the various parameters of the CCD schemes, how it is degraded in the original CCD scheme due to aliasing and we show, how this aliasing can be removed by this newly adapted NCCD scheme.

So, this more or less concludes our discussion on compact schemes of various forms all the way up-to-date and then, we switch over to our discussion on filters. We have noticed that many of the problems originate at high wave number and in this context we talk about the implicit filters those are introduced in mid-nineties by Gaitonde and colleagues.

But we must also note historically, that Philips in 1950s, while performing weather prediction used to filter the solution by physically removing high wave number component and getting a solution, which was very typical, unusual of what actually happens in weather prediction. The vertical structures used to elongate and that Philip solved and the original problem that he faced without filtering, he identified it as a non-linear instability.

Now, having established the historic context and the implicit filter of Gaitonde, we just note that this Gaitonde's filter is actually used in physical plane, but once again with the spectral analysis tool, we can analyze this filter and that is what we do in this lecture.

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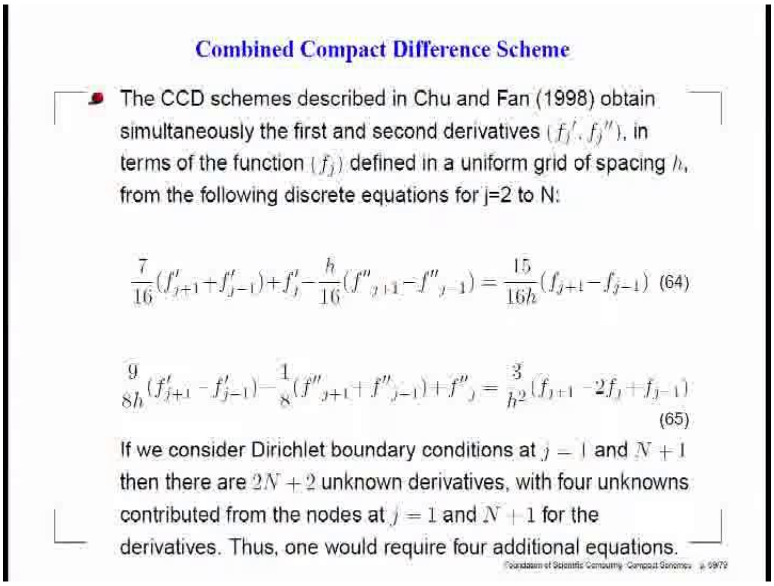

Let me first go to this general stencil that is used for this Combined Compact Different Scheme. These are the 2 equations, that you apply for j equal to 2 to N and the node points are defined from j equal to 1 to N plus 1. So, you have 2 N plus 2 unknowns. So, basically, this 2 equation, 64 and 65 gives you 2 times N minus 1 equation. So, you require actually another 4 equations to close the system.

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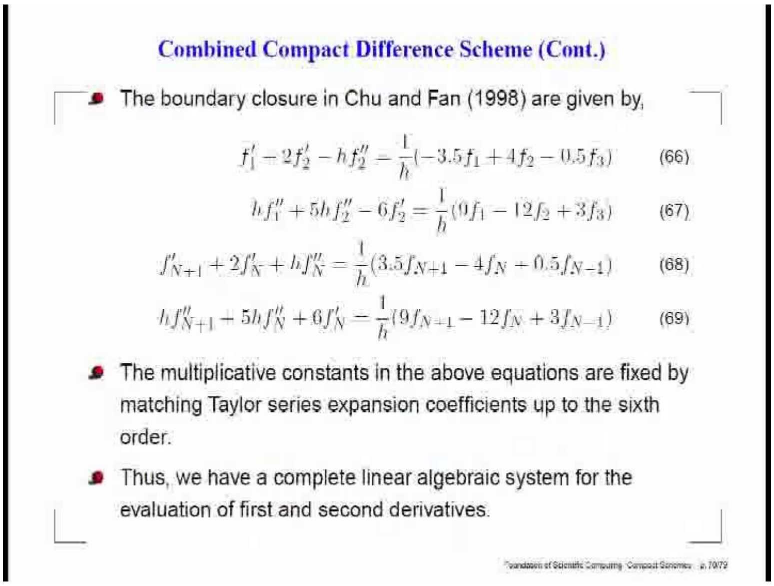

And this was what was suggested, this 4 equations, that you are seeing here or over there, you can see, they are the ones those are used by Chu and Fan to close the system.

So, basically you, you would have a complete linear algebraic system and you can solve for 1st and 2nd derivative.

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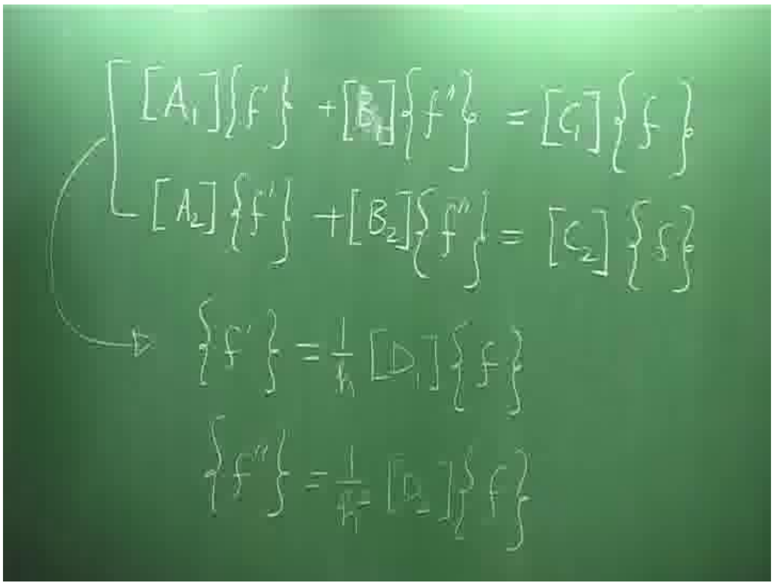

That you would be basically solving these 2 sets of equations written in the tridiagonal form, that we wrote down. So, block tridiagonal form that you can use as far as for the analysis purpose, we have written it in this form and this could be reduced to this form. I had given you the expression for D 1 and D 2 in the last class, I will tell, so we could work, or did I not give you, if it was there. So, if we, if we have seen how we could actually operate, I mean, obtain equivalent explicit equation for 1st and 2nd derivative simultaneously. One thing you might like to note, that this coefficients are fixed by Taylor series expansion and you match coefficients up to 6th derivative, and that is what will give you this equation 66 to 69.

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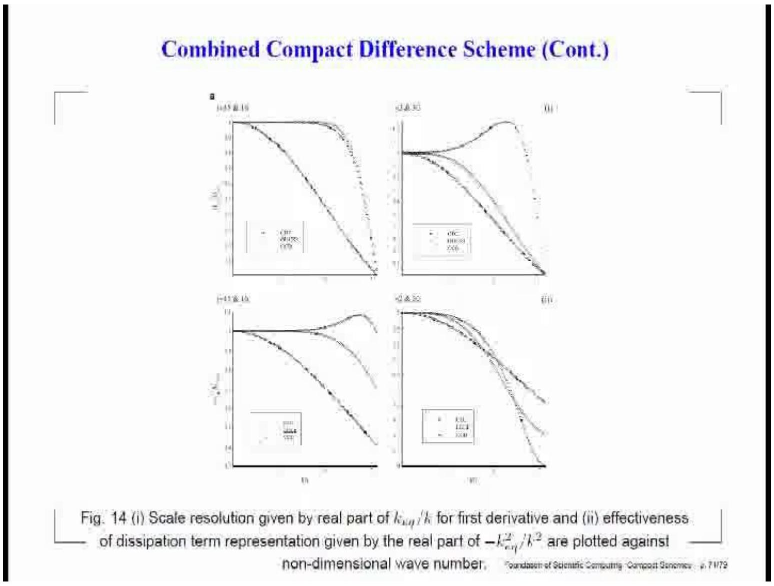

Now, so this is what was done by Chu and Fan in describing this CCD scheme and the results of such a scheme is displayed here, let us go over slowly. The left hand side you are seeing the properties, that is given for the 1st derivative on this side and this is the 2nd derivative. These are for the interior node, the, so if I have taken 30 points, 31 points, so these are the 2 middle points, 15 and 16. The top one is for the 1st derivative, the bottom one is for the 2nd derivative, suitably non-dimensionalized, that we equate k equivalent by k and this is k equivalent by k square. And of course, the x-axis is familiar, this is, that non-dimensional wave number k h, we write it from 0 to 5. Now, let us take a look at these figures slowly one by one.

For example, in this figure, this line, the lowermost line corresponds to your traditional CCD scheme, that is, we have seen time and again. Then, we have the extreme scheme, the curve, the top curve that corresponds to that optimized compact scheme that we have seen. Now, we have optimized it, borrowed the idea from Harass and Tarsal and improved up on the (( )) closure, and that gives us the best resolution as you can see remains flat all the way up to 2.3, 2.4 and then it slopes down to 0 at 5.

Whereas, this compact difference in scheme is the one that is just below it. So, it also gives clearly a decent accuracy, clearly decent resolution when it comes to representing the 1st derivative. And if I now focus my attention on the 2nd derivative, you can now see, this is the lowest most curve is the CD 2 curve. And last class we actually noted down, that this value, the value at the Nyquist limit was 4 by 5 square. So, this is about 40 percent of the resolution that you get.

And the curve that we have here in the middle, this is a scheme that we wrote down given by Lele directly obtaining the 2nd derivative, so that you can see, that you get even at the Nyquist limit more than 70 percent. Interestingly enough, that CCD scheme actually is the topmost curve and it has a very peculiar feature, that it remained 1 and then it never actually comes below 1. It actually has a bit of an overshoot at high wave number. So, we have this property here for CCD scheme, we will talk about it, we will show some results and then we will discuss it.

Whereas, if we now look at the boundary points, so this is what we are showing on the right (( )) panels are the properties for j equal to 2 and the corresponding right hand side, point j equal to 30. So, (( )). So, 2nd and the 2nd last point, we are noticing and once again you can see, CD 2 scheme is plotted here just for you to give a reference.

You can find out why we have been focusing up on compact scheme (( )), you can see here, if you can resolve without any attenuation, only this part for CD 2, the other schemes give you about almost 10 times and this is what we have emphasized before also. If we have a two-dimensional or three-dimensional problem, each direction you are going to get, by this account you are going to get a benefit of 8 to 10 times. So, you can imagine the kind of saving you can bring about.

That reason, that we are showing you, just this CD 2 result to show you a sort of a scale, that where we have moved in time over last 10 years or so. Now, what you are seeing is that the 2nd line corresponds to the OUCS3 scheme as we obtained for j equal to 2 or 30. And this is the CCD scheme again, is a kind of an over shoot even for the 1st derivative also. Whereas, if I look at the 2nd derivative quantities, that you notice, a very interesting feature, that the CD 2 begins lower compared to the other 2 curves and the other 2 curves are corresponding to that Lele 2nd derivative scheme and the CCD scheme. So, CCD scheme is the one that is here with the inverted triangle, whereas the Lele scheme is the middle line and CD 2 is here.

So, what you notice, that the high wave number range, even a CD 2 scheme seems to do better near the boundary. This was not so for the interior stencil, interior stencil it remained there. So, we got to understand, that this is what happens when we plot the real part of the k equivalent as non-dimensionalized wave.

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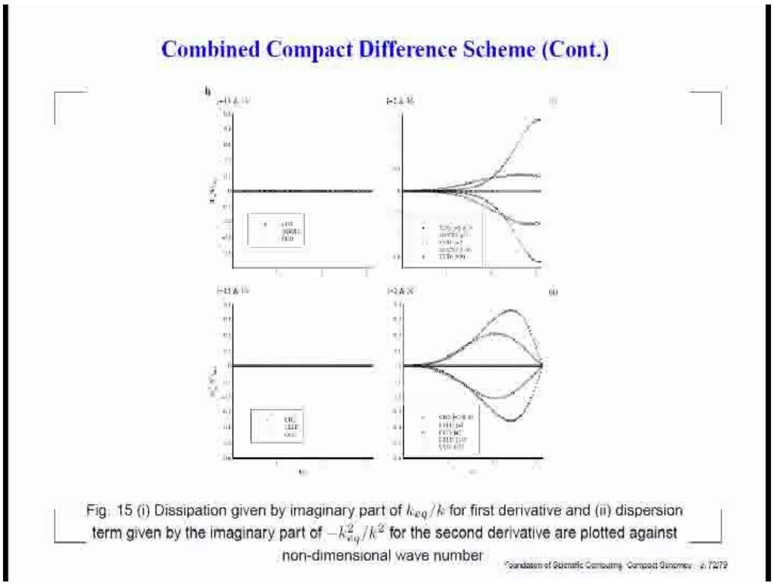

So, k equivalent also would have imaginary part and that we have plotted here. Well, if I (( )) the real part of the 1st derivative, sorry, the imaginary part of the 1st derivative. Then, you see all the 3 schemes, that we are showing here CD 2, OUCS3 and CCD, they remain that in the middle, I mean, they are non-dissipative central scheme. So, that is what you would expect.

And the same thing happens about the dispersion as sort of experience by these 3 methods for the 2nd derivative term. So, that also is nonexistent. So, that is a good attribute, that we do not have any dispersion error because of the 2nd derivative discretization.

However, if you look at the boundary points j equal to 2 and 30, when it comes to the 1st derivative, we notice, that for the various points, for the various schemes, that we have shown, some of them actually, this 2 line show instability. This will contribute to numerical instability (( )) as an imaginary path being positive contribute to numerical instability have been talking about and these are the 2 points, which are the near (( )) plane, they are overtly dissipative, that you are seeing here.

And when it comes to the imaginary part of k equivalent by k square for the 2nd derivative, you do see, that we do have a bit of dispersion on both sides, most of the time it is ok, but for some of the schemes, that you can see, the inverted triangle in CCD scheme, the topmost, which actually has a dispersion for high wave number and which is quite more than your, let us say, the Lele scheme, that you have here in the middle. So, CD 2 of course, remains flat, you have little problem there.

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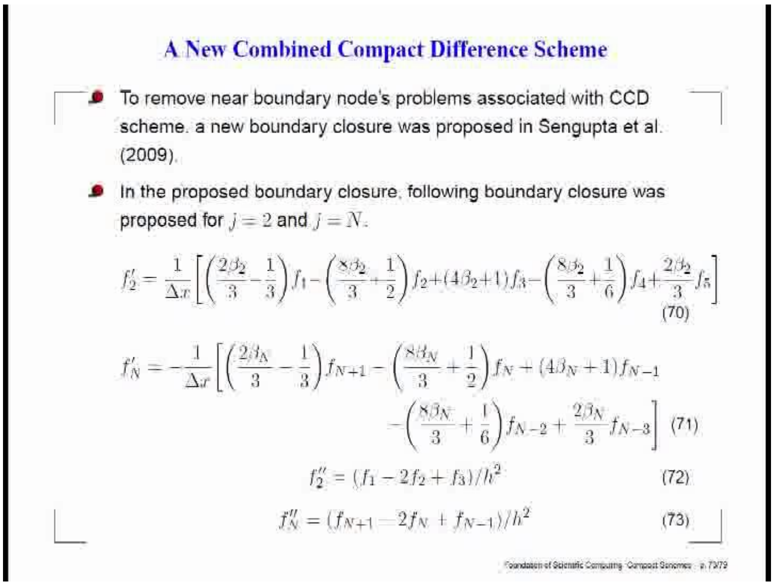

So, what does happen, that the CCD schemes appears to use except the fact, that inside the boundaries we have the old instability problem that we have seen with the compact scheme for 1st derivative alone. We figured out, that most for the scheme to the analysis we figured out, they were not (( )). And we did propose a method of recouping it by noticing, that once again this boundary closure, that we have here, these 4 lines we have written here, they are implicit in nature and one sided nature of this closure actually bringing problems for 1st derivative through numerical instability, and for the 2nd derivative we have the dispersion.

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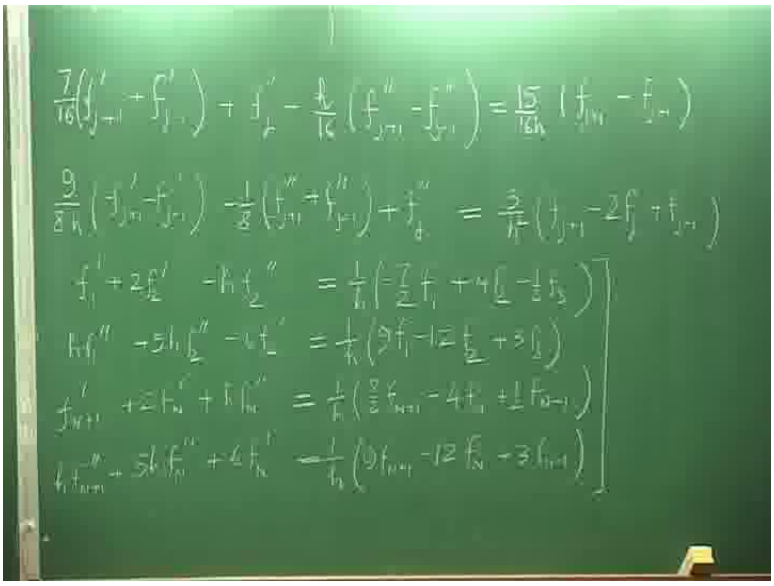

So, of course, we have the knowledge and experience how to (( )) problems. We have problems coming about due to implicit nature of the closure; we switch over to explicit schemes. So, this is what we intend doing. So, replace these 4 equations by those 4 equations and as you can see, they are nothing but simply explicit evaluation of f prime and f double prime at j equal to 2 and at j equal to well N. That should close the system and we know what this beta 2 and beta N are? These are those constants, which we can actually manually tune and see the global property of the (( )) will be ok.

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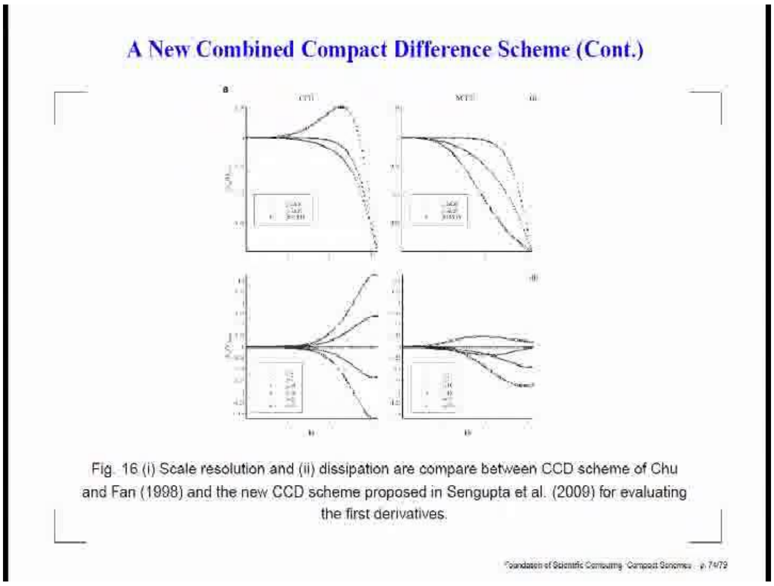

And that is what we do get. So, this is the Chu and Fan's CCD scheme and this is what we have mended, we have improved upon this. So, we just simply called it a New Compact Combined Compact Difference Scheme, NCCD scheme and you can notice, that this kind of overshoot is not there. So, at j equal to 2 we decide to accept lower resolution because we do not (( )) inherit problem on the 2nd derivative issue.

So, these are basically what we like to do and this is the real part of k equivalent by k and this is the imaginary part k equivalent by k. So, you see this point is the only culprit, but that is once again at j equal to 2.

So, I have probably not mentioned it clearly to you, but what you actually do is you evaluate this derivative. Then, what happens is at j equal to 2, you discard value that you obtain from this. Instead you just take a non-dissipative explicit scheme. So, basically you can prove this j equal to 2 scheme, that we are seeing here to cause some problem, rest of the points we have perfect stability. You have no problem to worry about, that only this can be replaced by an explicit term.

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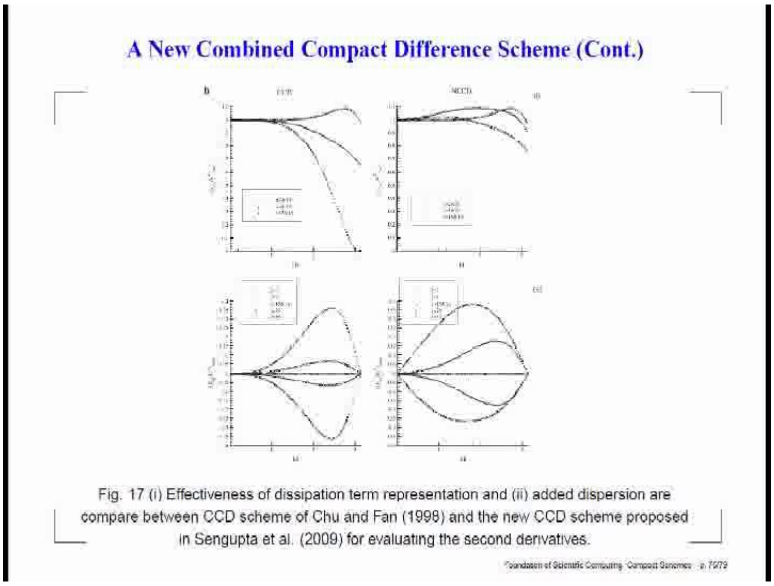

Now, if you look at the 2nd derivative, this is what you see, that Chu and Fan's scheme versus the scheme we have proposed. Now, we have noted, for different nodes we have different behavior. The central points have this overshoot, but most of the places it is equal to 1. So, this was a positive that we had in this original CCD that is also carried through. But you can also see what we have done? We have been able to (( )) the effectiveness of the 2nd derivative at near (( )) points. Here, you can see they are significantly better.

And the figure below shows the corresponding dispersion error coming through k equivalent by k square imaginary part.

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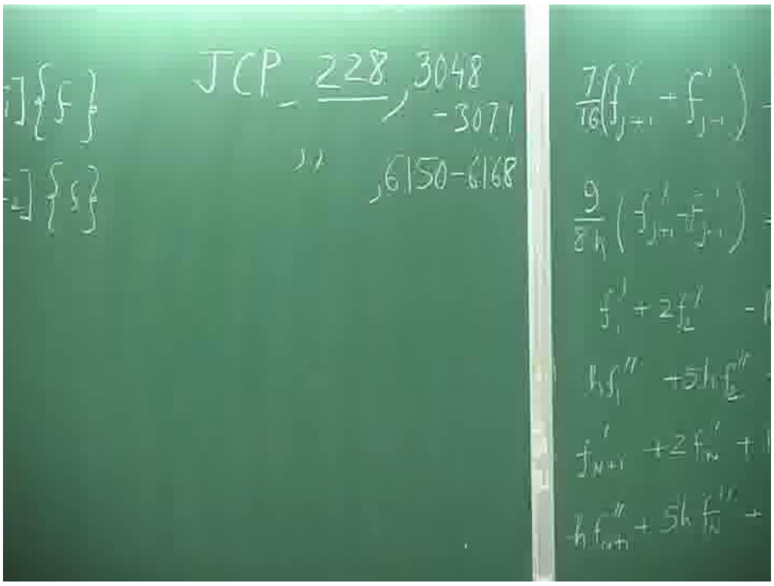

And you can see, that these are fairly decent things, these have been only very recently announced as you can see, a few months ago, we have worked it out. You are interested you can look at this journal, Journal of computational physics, volume is 228 and you can take a look at page 3048 to 3071. This you can download it from (( )) directly.

We have not only written, that we have a followed paper also in this, this year only, that is a short of paper, you can look at the same volume, but at a later month. These 2 papers actually tell you what is the current status that you have in this particular method, this I will probably come out on the couple of months ago.

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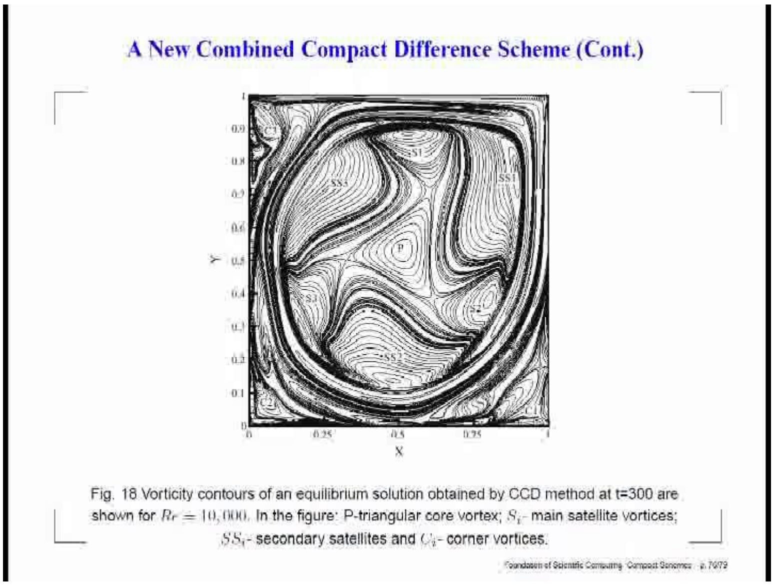

So, you see, we have done this and what we get as a consequence? As a consequence, I will show you a very interesting result, which was anticipated for a long time, but people could never see. This is what is called a driven cavity problem in fluid mechanics. This is often used to calibrate numerical method because this particular flow is very well-defined.

See, the flow is occurring inside this cavity, a square cavity and the top lid is continuously moving. So, through that you are actually imparting energy to the system and that movement, the movement from left to right actually causes vortices to form. And what you notice is, this is a solution at a, sort of a developed time of t equal to 3 non-dimensional time.

The Reynolds number is defined in terms of the speed of this lid; this is also called the lid-driven cavity problem. So, we have a cavity that is driven by the lid. What happens is, the vortices do get formed due to no slip condition on these boundaries and then, they actually keep rotating about in the cavity. But what is interesting is you notice that in the center actually, obtain a triangular vortex. People have seen it in experiments in a sort of a fleeting glimpse, glance for few seconds, but this is the definitive first theoretical result, where we actually could capture this triangular vortex by using this CCD scheme.

And you have the central triangular vortex and you have 2 sets of satellites, which we have shown here: S 1, S 2, S 3, these are the primary satellite they keep (( )) about. So, the, this primary vortex, this triangular vortex, this also goes slowly at a particular speed, but this S 1, S 2, S 3, they go at a different speed. And superposing all this you have that secondary satellite that you are seeing: SS 1, SS 2 and SS 3. And the walls are stationary, except the top one. So, that also causes this corner vortices (( )), which we have shown here C 1, C 2 and C 3.

Why I actually wanted to show you this figure, was not only to talk about the topology of this flow, this is of course, was a very satisfying exercise, but I wanted to bring to your attention, something you can see on the top right corner, if you strain your eye and look at the top right corner, you would notice there are some small wiggles, wiggles where, right there, did you see that wiggles, very small wiggles. And now, if you look at that grid size, they are of the size of the grid spacing. So, these actually correspond to k h, almost close to (( )), these are grid scale variation, grid scale variation would be that. If I have a grid spacing of h, then the maximum k h is 5, that would correspond to that events occurring over 2 delta a.

So, those kinds of points give rise to what? A source of errors that we talked about. What kind of error that we talked about? The aliasing error; so, aliasing error is a very, very significant player. You have heard me tell you many a times that in nature, there are no free launches. If you are seeing some effect, there must be cause and here all this motion that you see are set up is because of some source of disturbance and that source of disturbance happens to occur in this top right corner due to aliasing.

If I can reduce this aliasing, then you may actually see the same flow topology, but not necessarily at k equal to 300, it may occur at much greater. So, what I wanted to tell you is that this flow has been especially for this false number. This result has been used by many, many people over the years. The first set of results was published in 1981, where people said, this flow is steady and now we can see, with the passage of (( )), we have come to a state where we know, that this is indeed not true. There are some additional phenomena occurs, which are called Hopf Bifurcation.

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Now, this is essentially an instability solution breaks down if you have an equilibrium flow. It breaks down and go to, goes to a new equilibrium state through a bifurcation, which is attributed to Hopf. So, this flow actually suffers from Hopf Bifurcation.

(( )) Reynolds number crosses above 8000 and unfortunately, people have been actually using those old steady results for a long, long time. So, this result actually sets up a new benchmark solution for people to really look at and try to calibrate their method with respect to this result and see, if they can calculate this.

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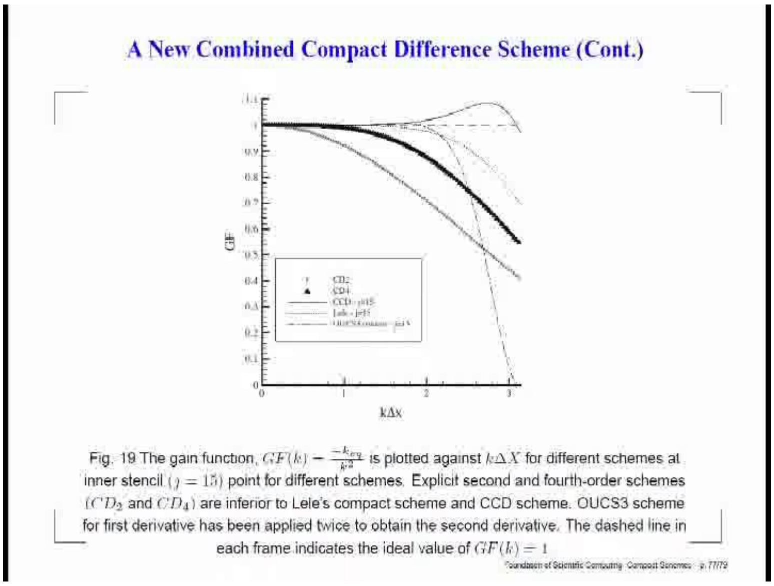

Now, what happens is, I told you the source of problem is some aliasing occurring near k h equal to 5. Now, I really like to analyze in about what is really happening in this case? Then noticed, that the dissipation that we introduced by difference method, as shown here, we have the CD 2 method, CD 4, the CCD scheme, the Lele scheme and even the optimized compact schemes applied twice and this is what we get, that k equivalent by k square, which I have just simply called it as a kind of a game function.

If you do not want to have this value, anything other (( )), that we can see for different methods, we get different effectiveness of discretization of the 2nd derivative term. And we have talked about CD 2 method, we know it; CD 4 you can work it out, that would of course, (( )) better than CD 2, as you can see by these dark rectangles, filled triangles and then, on top you have this Lele's scheme also we have seen. And what you notice is that OUCS3 scheme applied twice, if you look at that, that really performs very well compared to these other 3 methods that we just now talked to. About a value of close to 2 after that, of course, it falls up to 0, that is fine, but look at compact scheme, CCD scheme, this has an overshoot. So, what does it do? Not only represents the physical dissipation what you wanted to be, but it actually puts in a type of overestimate. That is what it means, it goes over 1.

So, what happens? Is this kind of an overestimation, is it good or bad? Well, we have seen, that if we are trying to solve a problem and if the grid is not able to resolve all the phenomenon correctly, then we get aliasing and this aliasing mostly occurs at very high wave number. So, we should have a means of avoiding (( )), aliasing. One of the ways we have talked about in the class is by upwinding, but suppose, we take some of this central scheme, then this CCD scheme has this unique ability to provide that additional dissipation to actually control aliasing.

So, this is something that we should look at as a (( )) for the CCD scheme. This overshoot should not be considered as a liability, but it should be considered as an aid, if we imaginatively can view it to control aliasing. So, this is something we must keep in mind.

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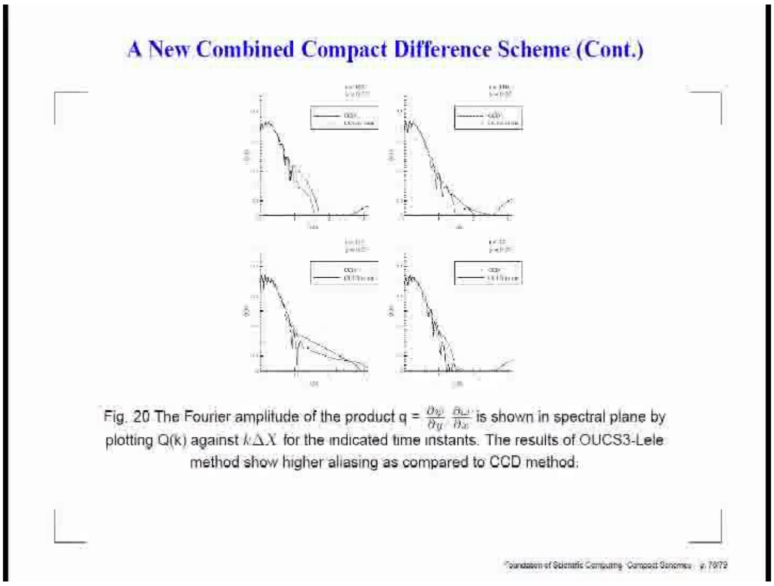

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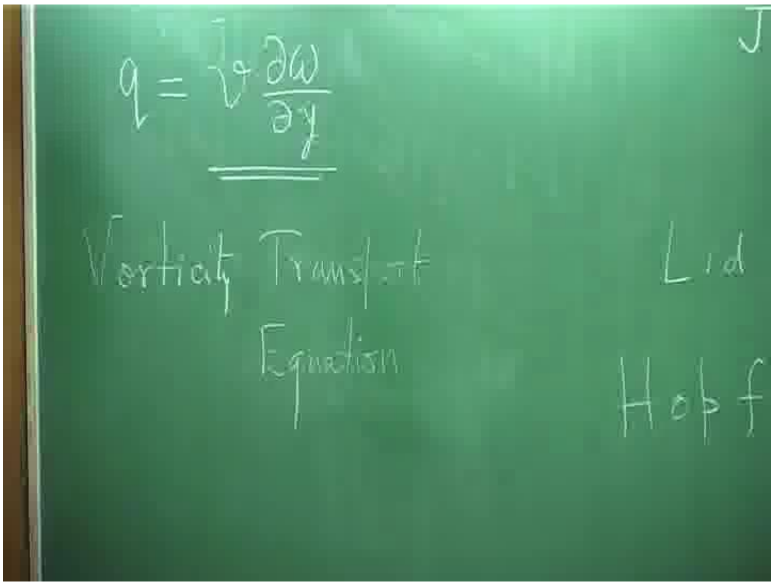

Now, to show you what really happens, we have actually shown you a product of 2 quantities that appears in this equation of the solution, that we have seen there. If we look at (( )), the physical quantity, the vorticity, how it, how it transports? There would be a term, that would be like V times d omega d y kind of a term. The basic idea is, that you can see, that this is a product term, that is what we are showing here, that is, q is del psi del y is basically the V velocity times d omega, sorry, d omega d x. Well, no, it is, this is correct, it is this will be d omega d y.

So, basically, then what you can do is, from your numerical solution you can obtain this value of q and then you can perform Fourier transform to look at the distribution of this q across different k. So, what you can do is, take the Fourier transform and plot q of k against k delta x. And we are seeing here 2 sets of results by 2 different methods discretizing the 2nd derivative. The one is the CCD with this dotted line and with the solid line we have, what we are using is the Lele scheme for the 2nd derivative and the OUCS3 scheme for the 1st derivative for solving this vorticity transport equation. So, this is what we are solving.

So, in solving this vorticity transform equation we get a time sequence, so we have shown the results at different time. I suppose you can see, this is starting at 300. We have gone and shown the results at different times and what you notice is that, that Lele scheme along with that optimal (( )) scheme does exhibit, what you see as a key near the I k the value.

And what this could be? This is a sure sign of aliasing error; aliasing error, as I told you will always appear in high k h range first and that too it shows up a kind of a pileup. You see, at this time, it is unphysically increasing; in no physical system you will see a spectrum of this type. Whenever you are computing something and you do a transform and look at this kind of behavior, you can be very sure, that this is an error and this is due to aliasing.

And you can see that it keeps on increasing here; it keeps increasing here. Of course, at different times, you can see some time it will come down, sometime it is up. But you can see, the Lele scheme, despite being much better than those other explicit schemes, cannot really control this aliasing problem, whereas the CCD scheme can.

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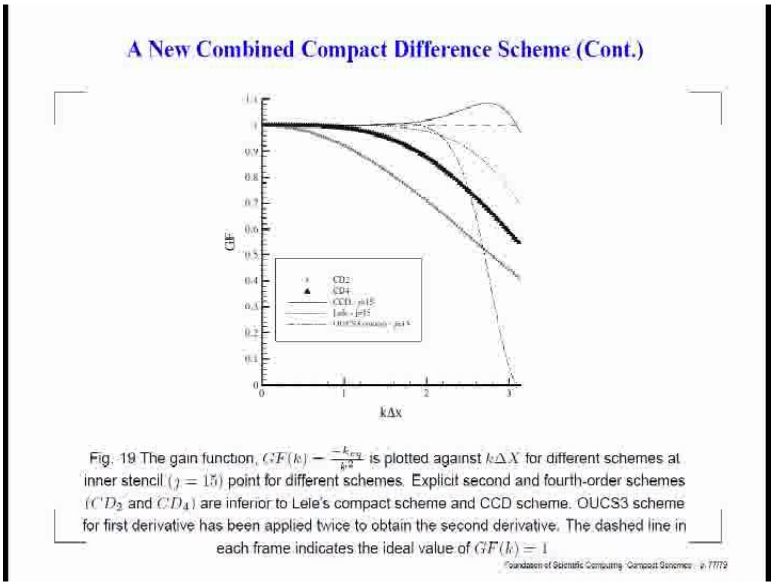

And this happened because of that, you know we talked about that, that we have this overshoot. This overshoot actually helps you in controlling those high k h error, that you are clearly seeing from the numeric solution. So, I think that should basically tell us about this (( )), this newer schemes, that we have.

This is a subsequent time event and you can see that with time the spectrum keeps changing and consistently, you see CCD outperforming the other Lele and OUCS3 scheme and this is matter of comfort, that we can actually do it.

Now, having said all this, what is the issue with the CCD scheme? I have not told you the complete story, that these are only used for uniform grid. If, if we have non-uniform grid, we will have to work little more (( )), develop CCD scheme for non-uniform grid in the transform plane, but that would require lot more additional work. There are people who have tried looking at it, none too successfully. So, basic lesson that we are trying to communicate to you here, that there is still some way to go. We still can develop some better methods, but CCD method has shown from this set of result, that you can actually use it to your advantage provided you can use uniform grid. We have talked about some other issues in both these papers.

I do not wish to go through a special topic of fluid mechanics in this course, so we will leave it at there. So, I think we are done with the compact schemes, but to close it, we must say, the compact schemes are the schemes of great promise and we would continue to use that in times to come.

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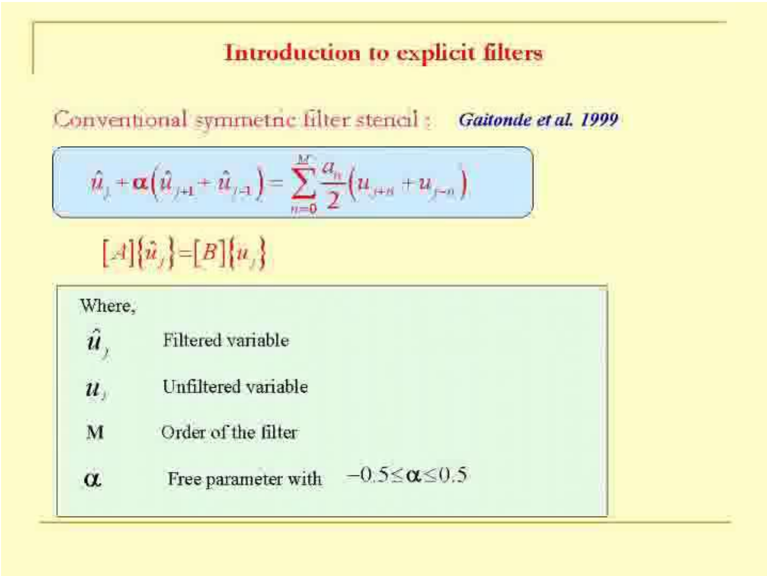

Now, what I would like to do is go over to a new topic, which is related to what we are doing here, this is about filtering; this is something that we need to do. We are going to talk about filter now.

We have seen, we have just now talked about computing and the various problems, that we encounter at high wave number that leads to instabilities and all kind of problem. So, how do we solve it? We have talked about the various sources of error, we have indicated properties of various methods, some methods are good, some methods are not so good, they all relate to the property of the numerical method as you solve the equation. There has to be some additional help coming from other directions and this is one such help. What you do is, you solve the equation first, then the solution that you have obtained at different time instant, you can rectify the solution in an offline manner, means, this has been related to the direct solving of the problem.

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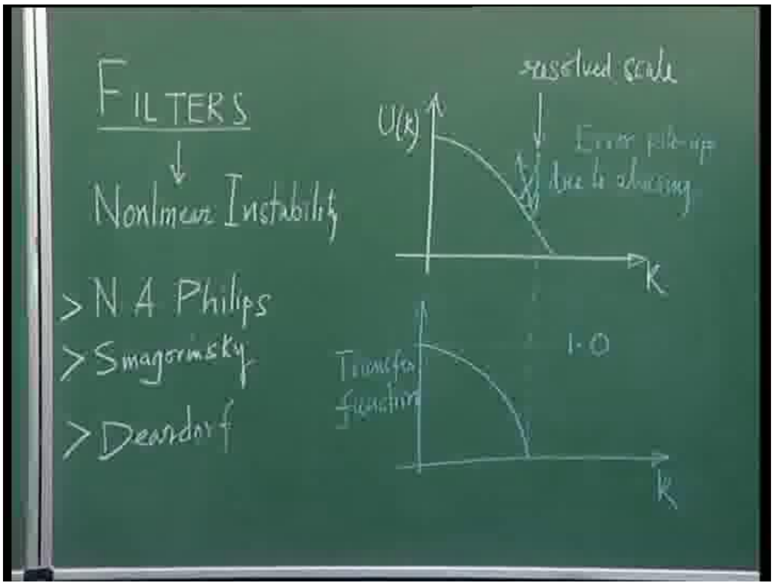

As such, this is what, is, the central theme of what we call as filters. This actually has a bit of a history, as I told you, that most of the time the computing is led by people from weather forecasting side and it all started at the Princeton Advanced Study center, led by von Neumann and his group that I mean, and so on and so forth. There whole lot of pioneers, they were altogether there, they were developing this weather prediction tools. Now, what was their major concern all the time was, that they could do some weather prediction and after a while the solution blows off. So, that they did not analyze using von Neumann stability analysis. And we talked about it, that von Neumann stability analysis was the only considered option at that time and I also mentioned, that it was considered so important, that it was classified and not allowed to be published during the 2nd world war.

So, everything that could not be solved properly and could not be analyzed by von Neumann stability analysis developed linear equation. People attributed it to non-linear instability. So, they, they said, look if I scan, the linear stability analysis does not tell me, that there is a problem and if I find the grid, then also I do not get around the problem. So, whatever that came in their way, they used to call it as a non-linear instability.

Around that time, there were a couple of gentleman by name Smagorinsky, then Deardorf and Norman Philips, they were actually part of that weather prediction team and Norman Philip actually noticed, that if you have this solution, that you are obtaining as a time series, if you filter the solution, what do you mean by filtering, that we know by now, we have seen it too often in this course. If I have a solution and the some unknown, I am trying to plot it, say in the k plane and let us say, the solution is something like this, but your computational resource does not allow you to compute the whole range because we do not have addition adequate computational resource, then what do you do? There are ways of doing it, that as you compute, if you cannot do this where will it go? We have said that it goes and folds inside and this is where it keeps on building up with time, it keeps on increasing and then solution blows up, that was what was seen. So, this was the problem.

Now, especially, when, what Philip noticed was, that if you take the solution like this and what you do is multiply this U of k with quantity, let me call, that as T of k. The property of T of k is this, that well, let us first show the property of T of k, let us say it is like this, that it (( )) with 1 and then as you come to the resolution limit, it falls off to 0. This is like our k equivalent by K, which we have been plotting; we have seen it too often.

Then, if I do this, if I multiply by e of k by this T of k, what will happen? Then, we will actually force it to come to 0 here like this. So what has actually happened, that we have been able to attenuate this growing path, this T of k. The transfer function that we are talking about here, I will define it shortly. This transfer function actually filters the solution and this is what we say as a low pass filter. What does it do? A low pass filter, it allows the passage of low wave number unattenuated; the high wave numbers are attenuated to your advantage. So, Philip was doing this and what he noticed, that earlier people used to solve for say, 5 hours, 6 hours and then, this code will blow off and he could now compute for weeks.

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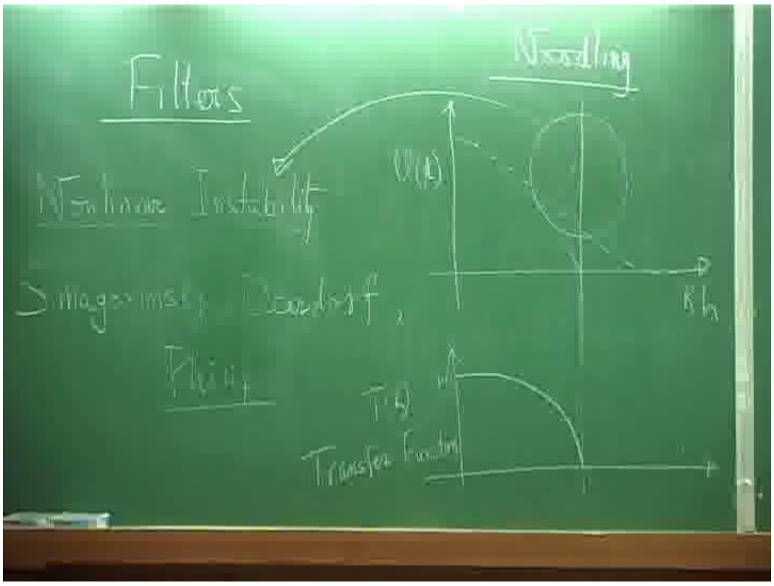

And the solutions were though, also of course, looked unphysical, they will see this vertical structure in the atmosphere, they will look very much elongated, so much so, that they used to joke about the output of Philip, they used to call it, it was not numeric simulation, they used to call it noodling, means it looked like noodles. So vertices would be stretched so, but in doing this Philip actually opened the flood gate. He pointed out, that you could take the solution and use a post processing tool like a filter with this kind of property of transfer function and you can control these.

These were attributed to non-linear instability, but now we know, over the last 1 week we talked about, now we know, that that is due to aliasing and we have also pointed out very clearly, aliasing can occur even with a linear operator. So, to say, that this is a non-linear instability problem, itself was an erroneous diagnosis by those people. So, let us not worry about whether it is a linear instability or a non-linear instability, it is instability and we also know, even the linear analysis itself was flawed.

So, there is nothing to really grow about linear stability theory. What they were using, that itself was wrong and we have also shown, that aliasing can come about from linear operator. We have shown, that if I take the, even the 1 d wave equation and I try to solve it in a transform plane, I do actually convert del u del x equal to some del u del xi into del xi del x. So, it becomes a product term, so that can contribute to aliasing.

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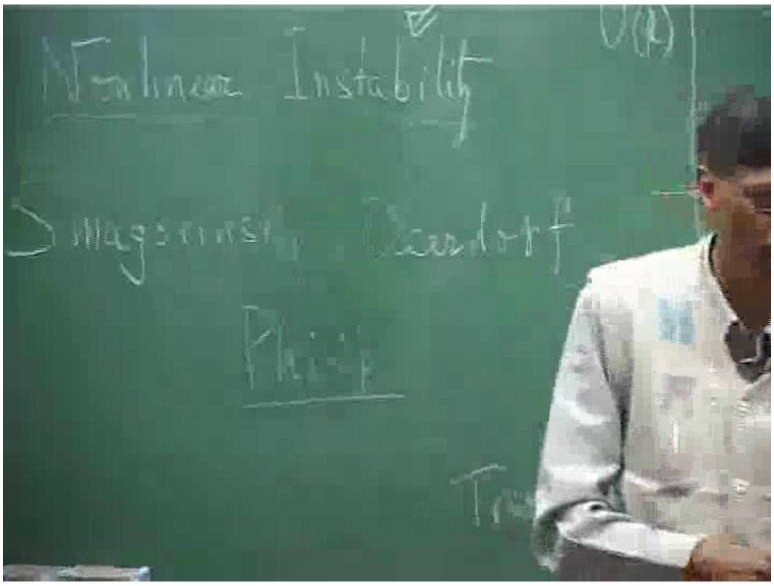

So, aliasing should not be as such called a non-linear stability problem, but they do occur as we have just now seen in our even the driven cavity problem, how aliasing was coming about by choosing some very, very sophisticated methods. These compact schemes are light years ahead of those methods these people were using. They are, they are, they are not so good, but even then, we do have this problem.

Now, how do we construct a transfer function? Well, for now, it should be, for us it should be not pretty much trivial because we have been plotting this k equivalent by k. So, we have those expressions, we can just simply use it. But that you are looking at in the k plane, in the spectral plane, but you are actually doing computing in the physical plane.

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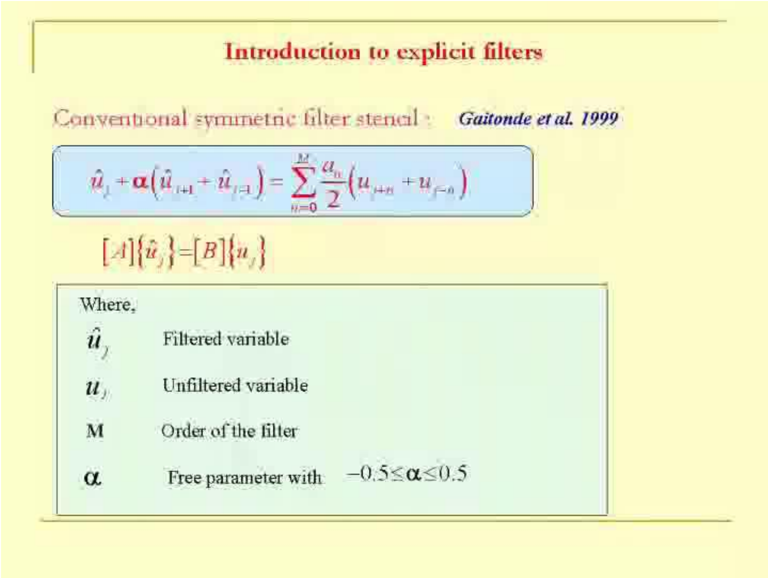

So, how do we introduce in the physical plane and that is where this gentleman came from (( )) in Ohio, Datta Gaitonde and his colleagues. They suggested, that one could use, take the solutions, which we are calling here as U j and filter the solution by using a method of this kind. Now, so the quantities with the carat or the heart will be called as the filtered variable and of course, without the 1s are the unfiltered variables. And what you are noticing on the left hand side, we have not been stupid, we have kept it as a tridiagonal matrix because we know it is easier to solve. So, we use a tridiagonal operator to give us this A matrix and then, if I know this solution, so I will basically construct this B u and then I will solve for the filtered quantity.

Now, what we need to do is fix this filtering parameters, one of which you are seeing on the left hand side is the alpha and on the right hand side, what we keep doing, we take points in a pair-wise manner with a plus sign in between. What does it do? Well, we have already seen what does it do, if I take points pair-wise, symmetrically located about the jth node, what does it do, it smoothens will be, basically it adds even derivative. So, that is what you do and you can choose the number of such terms pairs, that you would like to, that is what we call as the order of the filter M. So, we can choose this.

Now, what about the choice of this parameter alpha? Well, we have kept it between minus 0.5 and plus 0.5. What do you get? Why, why it has to be between minus 0.5 and plus 0.5? That also comes from the diagonal dominance of the A matrix. You can see, the diagonal is 1 half, diagonal is alpha, so 2 alpha plus 1 should be positive. So, alpha should be in the magnitude sense, should lie between minus half and plus half.

Now, what we can do is, take a look at that equation and make matter simple. We have started off with a case, where we will put M equal to 1. So, if I put M equal to 1, how many unknowns would we have? On the right hand side we will have a 0 and a 1. So, we will have 2 parameters on the right hand side, alpha on the left hand side, so we have 3 parameters.

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Central Second Order Filter

$$\alpha \hat{U}_{j+1} + \hat{U}_j + \alpha \hat{U}_{j-1} = a_0 U_j + \frac{a_1}{2}\left[U_{j+1} + U_{j-1}\right]$$

- Taylor series expansion provides
- LHS = $\hat{U}_j(1+2\alpha) + 2\alpha(\hat{U}''_j \frac{h^2}{2!} + \hat{U}^{iv}_j \frac{h^4}{4!} + \ldots\ldots)$
- RHS = $a_0 U_j + \frac{a_1}{2}\left(2U_j + 2U''_j \frac{h^2}{2!} + \ldots\ldots\right)$
- Consistency demands coefficients of $\hat{U}_j$ *and* U_j should be matched.

$$1 + 2\alpha = a_0 + a_1 \qquad (A)$$

And that is what we have written here and this we can write down in the Taylor series. So, left hand side gives us the quantities like this; we have the function value itself plus as you can see, the 2nd derivative, the 4th derivative and so on and so forth.

The same way, the right hand side also can be written down in this form. Now, basically, what we are trying to do? We are trying to filter U to get the filtered quantity U hat. So, of course, that would require, that we must satisfy some kind of a consistency condition, so that the coefficients of U hat and the coefficients of U on the right hand side should match and that seems to give you this last equation, that we have written below, that 1 plus 2 alpha should be equal to a naught plus (( )). So, this is something that we must satisfy.

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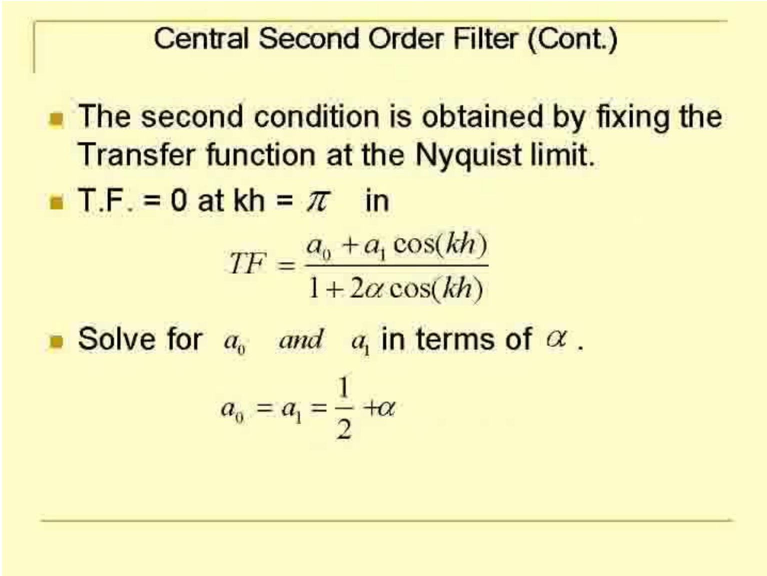

Now, what we can do is, we have talked about alpha as the free parameter. So, we are trying to design a family of filter with some degree of freedom to our self and that is the choice of alpha itself. So, what we really want to do is solve for a 0 and a 1 in terms of alpha. So, basically the consistency condition gave us one condition, one set and we have to generate another condition.

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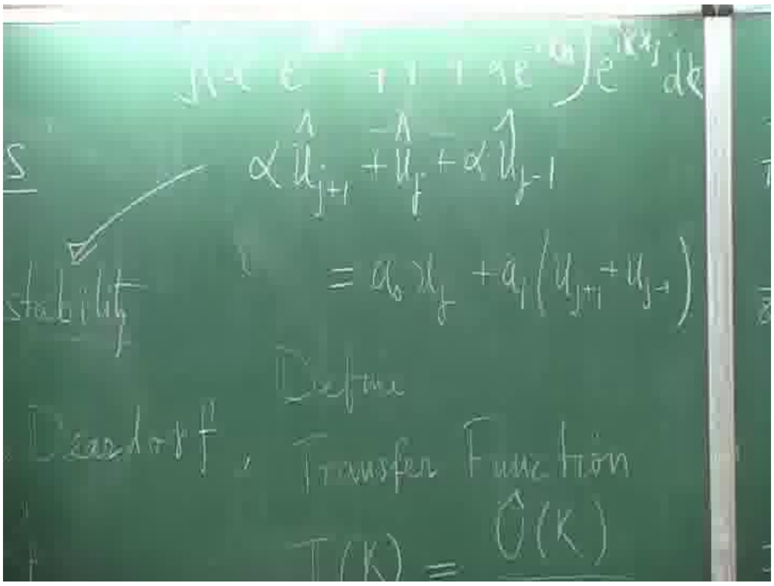

Another condition is is the following.

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The other condition is, we have written, that U j plus 1 plus U j and plus alpha. This is this and then we have, a 0 U j and plus a 1 U j plus 1 plus U j minus 1. So, this is our 2nd order filter.

So, what we basically would like to do is we define the transfer function; we define the transfer function in this form, in the k plane, in the k plane, that is what we would be doing. So, if I write it, this with the lower case in the physical plane, in the k plane, I would write it like this. Well, it could be the jth node, so I will write it like this. Well, we do not need to complicate it, let us just keep it like this, that hat by U of k.

So, is the basically quotient between the filtered with the unfiltered quantity and what you can do is, you can use the Fourier Laplace transform and then you can write this down. So, what this will give you, if you follow what we have already done before, this will give us i k h and there would be 1 here and there would be alpha e to the power minus i k h, that would be. So, this 3 term will give me that and that will be multiplied by i k x j and whatever we have, we can integrate it for all possible k. So, that is what the left hand side would look like; that is what we have written.

So, same way I can write it from here. From this I will again get e to the power i k h and plus e to the power minus i k h. So, what happens is, this plus this will give you 2 alpha cos k h and that is what you are seeing here in the denominator, 1 plus 2 alpha cos k h. So, that is the transform of this part, so this is what you are doing. You are taking the function, you are using an operator, that operator is given in the downstairs here and on the right hand side, whatever you do, that operator is given here. And since I think there is a factor of 2 involved here, so that is what we get, a 0 plus a 1 cos k h.

Now, this transfer function I have just erased it, but we did plot it some time ago. We need a particular property of this transfer function that it should be very well behaved for small k and for the large k it should progressively decay and become equal to 0 at the Nyquist limit.

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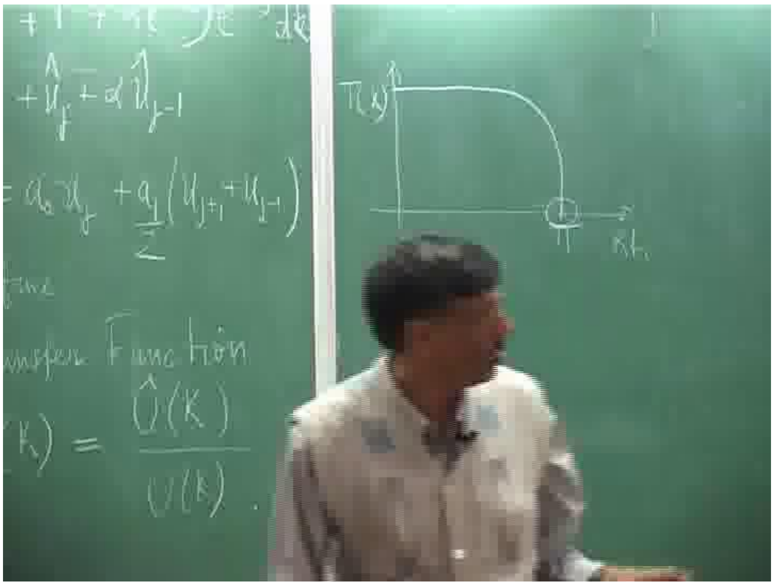

So, that is what (( )) T of k should be equal to if I plot it versus k h. So, I would like it to happen, like it to emphasize this property, that at small k it should remain 1. Then, as we go along, it should come to 0 so that enforcing the condition here, that it should be 0 here provides us the 2nd equation.

The transfer function, that we have written here, this should be 0 at k h equal to 5. So, of course, that would imply, that a 0 should be equal to minus a 1; that means, a 0 and a 1 should be same. And then, if you look at the previous equation a, if I have this, so a 0 and a 1 is obtained. So, this is very simple, that here a 0 and a 1 should be half plus alpha. So, this is the way the 2nd order filter works.

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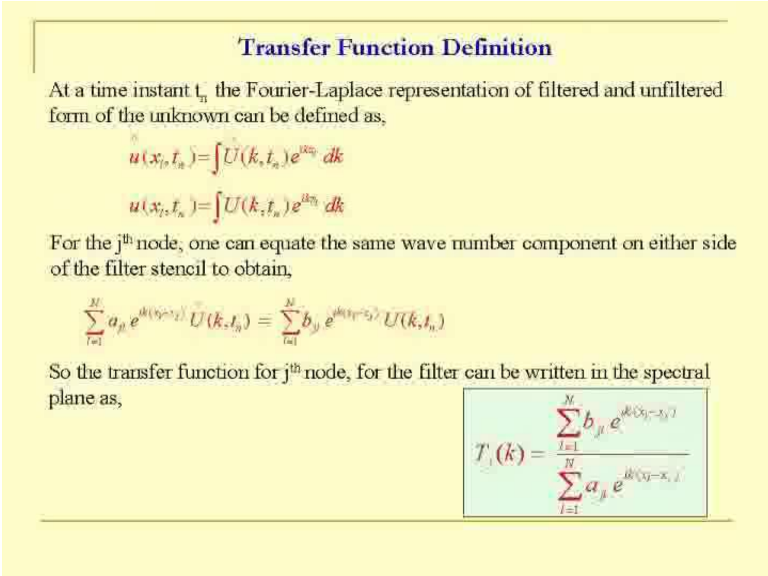

Now, there are certain properties of the most general type of filters and as you can see, these filters, that we have introduced here, these are all central filters. I am not talking about explicit filters, but explicit filter you can immediately construct for yourself by taking alpha equal to 0. So, this set also has the explicit filter as a subset by just simply looking at alpha equal to 0, but in general what we would write? The filter equation, not necessarily restricting our self to a 2nd order filter of this kind.

Why did I say it is a 2nd order filter? That we have seen, that we have just simply fitted the, equated the coefficient of u and u hat and the things that we neglected were of higher order. So, that is why, we called it a 2nd order filter. But suppose, if I would have taken additional pair of term a 2 by 2 U j plus 2 plus U j minus 2, that would increase the order of the filter by 2. Every time I add 1 extra term, my order of the filter increases by 2. I just simply showed you the 2nd order filter because it is easiest to construct.

That does not mean, that you cannot form a general filters and this is what a general filter may look like. So, I could write it in terms of this generic equation. So, I may have some various nodes involved, all the nodes involved in the domain and for the filtered quantity and the same way, I could also involve all the nodes for the function on the other side.

So, basically, then this is the generic definition of the transfer function, that would come from here and as you can see, we have projected all the points at the jth node. So, that is why this projection operator e comes, appears in numerator and denominator.

I think we will continue with this on the next class.