Foundation of Scientific Computing

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Module No. # 01

Lecture No. # 28

Spectral Analysis of Explicit and Implicit Schemes

This is lecture 28; we are going to talk about spectral analysis of explicit and implicit schemes.

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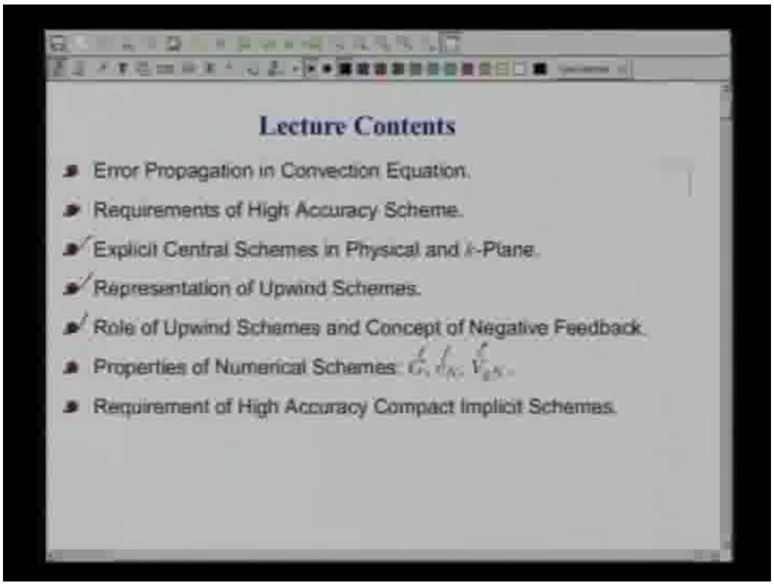

The specific content of this lecture is as follows. We will be first talking about how error propagates in convection dominated physical systems. That should set us up for requirement of high accuracy schemes and what exactly we want to do with that. Once that is discussed, we would move over and look at certain explicit central schemes. We will look at it from the physical plane point of view, as well as in the spectral plane. The spectral plane will be indicated by the wave number k. Having done that, we would try to talk about upwind schemes; we will realize that many times central schemes by themselves would not be adequate. We would require some bit of numerical dissipation coming through upwind schemes that is what we need to represent; we will find out what it is.

Once, we have done that representation, we will look at role of upwind schemes; what exactly it does? We would notice that the role that it plays has an analog with the negative feedback that we have in electrical circuits. So, that is why we will show you the connectivity of this concept with negative feedback stabilization of systems.

Having done, we have decided upon the schemes, whether we take central schemes with explicit numerical dissipation added or the upwind schemes which was built in implicit dissipation. Next job in our view would be to really obtain the properties of numerical schemes. Talking about the properties, we will be talking about the same three quantities that we talked in the earlier lecture; that is, the first thing is the numerical amplification factor, its magnitude, we need to now ascertain about whether the system is stable or not. Along with that we need to find out what is the associated phase shift that comes about with each time step of integration. That will bring about numerical phase speed, which we have called here as c N.

Finally, we would look at the group velocity. The group velocity is one of the most important parameter, we need to characterize the speed at which the energy in the system propagates and that is indicated by V g N.

Basically, having talked about the explicit schemes in the following, we would finally just setup some requirements. Should we decide to adopt similar high accuracy schemes in an implicit framework?

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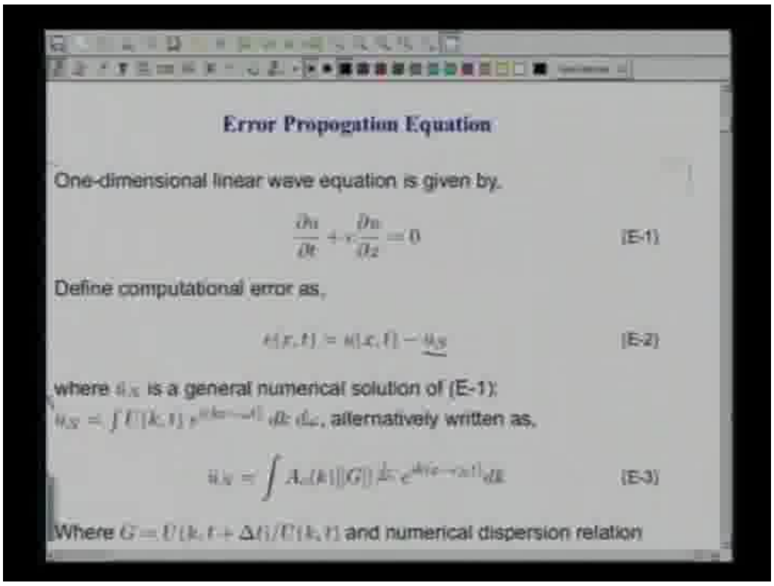

Let us begin by looking at the error propagation equation. To understand it, we once again resort to study the one-dimensional convection equation, which shows how the variable u propagates in space and time in one-dimension; c is the phase speed. As we have noted, in our discussion on waves, we have seen that this is a non-dispersive system, so the energy also propagates with the same speed c that is also the phase speed in this case.

Having decided to adopt this equation as our model equation, we can obtain the numerical solution that we have identified here by u subscript N. We take a difference of it from the exact solution and we call the quantity on the left hand side as the computational error.

The numerical solution that we obtain, we could decide to express it completely in the spectral plane in terms of the wave number k and the circular frequency omega; that is one way of writing it; that is what we have done here. Alternatively, we could write it in terms of these three parameters; that we just now talked about. Namely, the amplitude of the amplification factor, what do we mean by the amplification factor? That will be apparent when we look at the last equation.

Amplification factor basically tells you, from the Fourier-Laplace representation that we have shown here, if we look at the Fourier-Laplace amplitude at the advance time step that is U of k plus t plus delta t divided by U of k at t that would be our numerical amplification factor.

The first part of the solution is the initial spectrum that is given by the initial condition of the solution that should be kept in advanced in time, which is given by the second factor, which is nothing but mod of G raise to the power t by delta t. t by delta t actually gives you the number of time steps that we have essentially taken.

At each time step, it amplifies by factor of mod G, either it amplifies or attenuates; that would be determined by the property of mod G. In addition, this complex G that we have written here, it would also give raise to a phase shift at each time step. After N such time step, we have arrived at t equal to t that would give us a kind of a numerical phase speed that we have written here as c of N. Then, basically in this equation E-3, we are representing the solution in a sort of a hybrid fashion, because the space x has been represented in terms of the wave number k, but the time is retained as it is.

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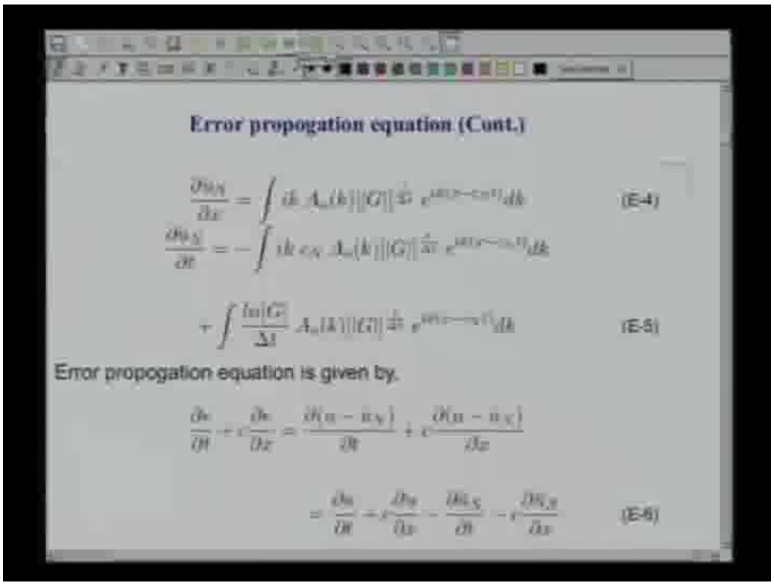

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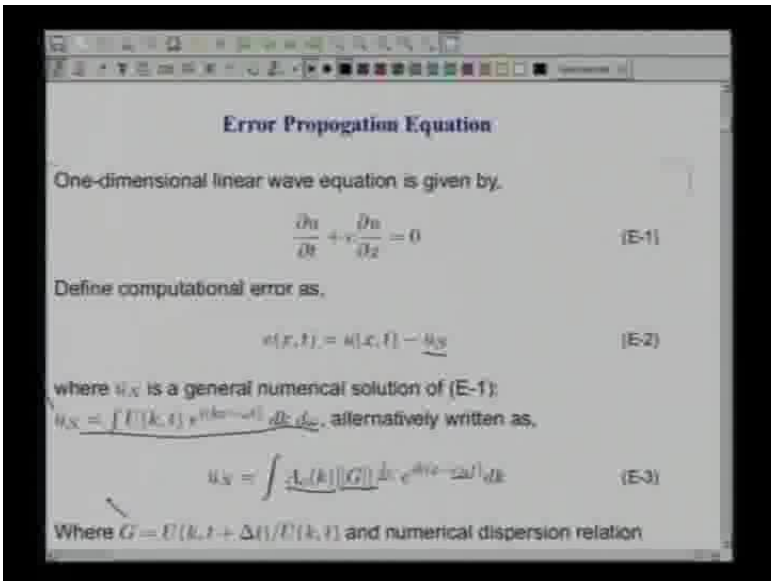

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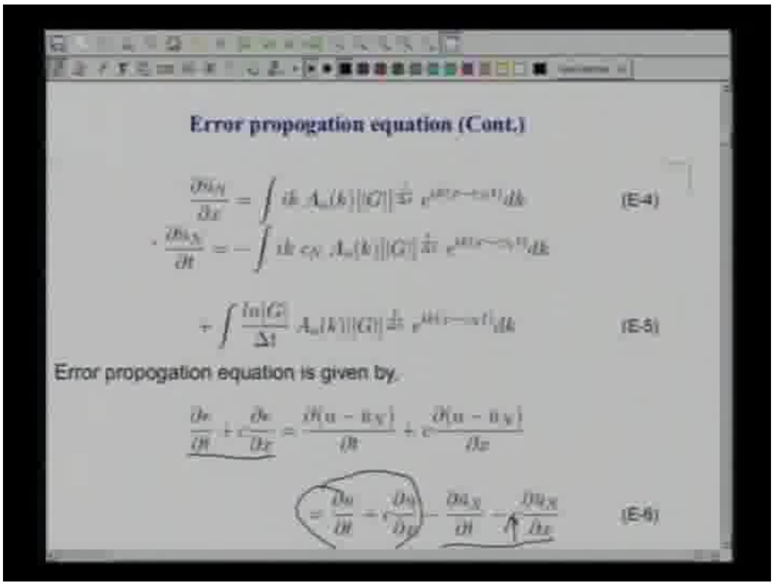

So, if we look at the expression that is given in the previous slide, we can immediately obtain the space derivative and the time derivative one at a time. The expressions are given as there. Please note the fact that in the previous slide the time dependence actually comes through the G term as well as e to the power minus i k c N t term. It is for the same reason that when you take the time derivative you end up with two sets of terms in e phi.

What you could do is, basically use the definition of the computational error that we have shown and try to set up error propagation equation. The idea is simple, if the governing equation is given by this differential operator on the left hand side here, we want to see the error also does analogously the same thing or different.

Please note, this is a linear equation given by the standard work done by previous researches. It was conjectured that the error would follow the signal itself. However, what we are showing here is that if we represent e as u minus u bar N, then basically we will have two sets of terms in E-6. You notice that this set of term is analytically equal to 0.

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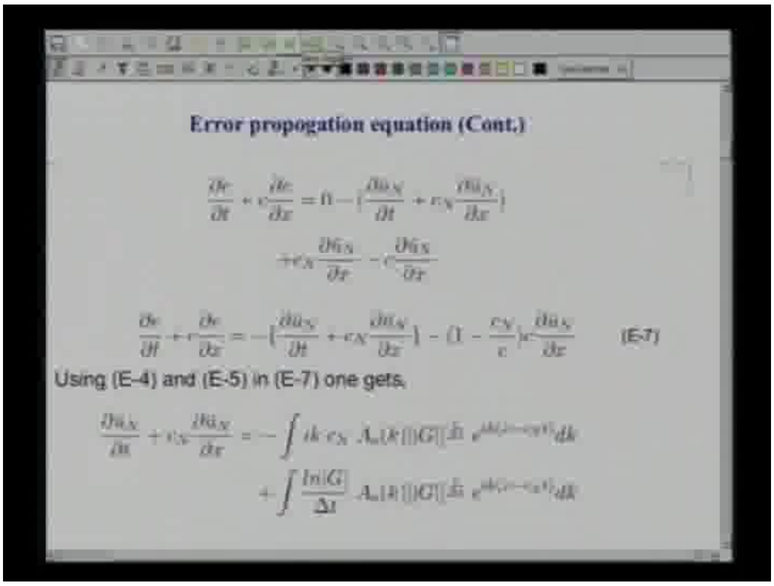

What happens is, basically the error is actually dictated by this two terms that comes about numerically. So, this is essentially the idea of setting up this error propagation equation. So, that is why we rewrite.

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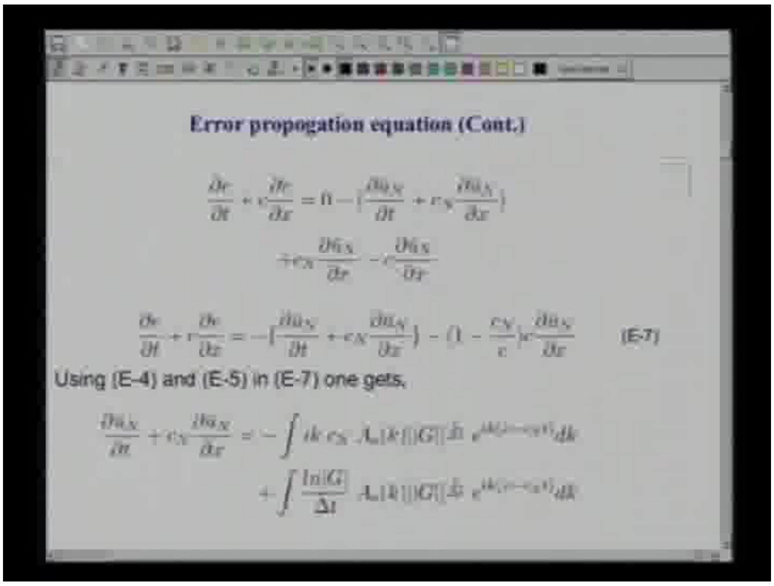

If you note in the previous slide, this right hand side had c here, whereas as we mention in computation what will happen is that c will be replaced by c of N - the numerical phase speed. So, that is why we have done this following manipulation.

We wrote down the first line here, in terms of c N. In the second line, we have subtracted it out and this is what we get. Essentially, then what we can say is the error is governed by the left hand side operator, which is forced by the following set of terms that is on the right hand side. Since, we know what the expression for u bar of N, we can substitute those quantities. If we do that in E-7, we get this following representation here.

So, one sort of term depends on the numerical phase speed, the other set of term depends on the modulus of G. Look at the natural logarithm appearing over there.

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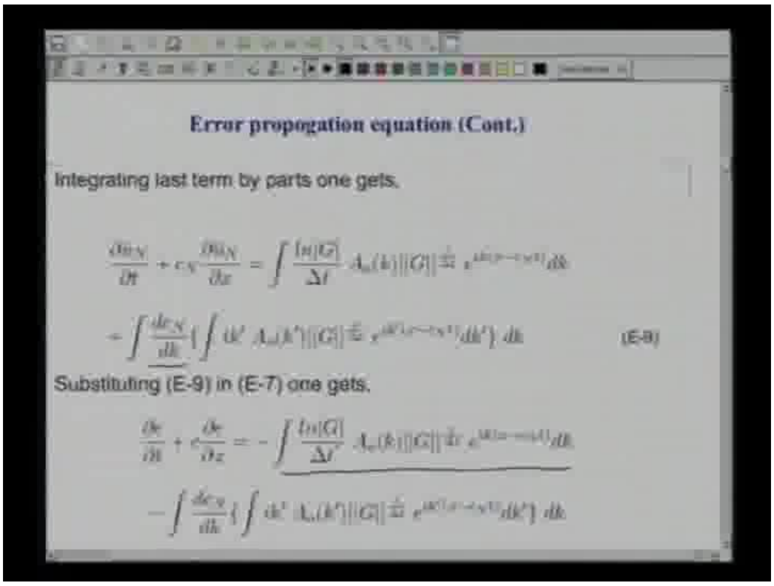

The last term that we had is the phase dependent term, we can integrate by parts. We can see that the equation further more simplifies in terms of this. You notice that naturally a term comes, which is given here, which is nothing but d dk of c of N.

So, this basically tells you that your original problem, c was a constant. What we are noticing here is that we are providing the possibility that c of N the numerical phase speed need not necessarily be a constant; it could be a function of the wave number itself. If that is so, that would be contributed by this last term given here on this equation E-9.

Basically, we are looking at this expression. We can substitute it back for this set of term in the error propagation equation; we come out with this set of terms. We will notice one thing very clearly that the first set of term that we have here depends on numerical amplification factor. If the error has to be governed by the same equation as the signal itself, then we must have modulus of G equal to 1.

What it actually tells you very clearly is that if you really want a well behaved algorithm, then you must have modulus of G equal to 1. Any numerical scheme for which modulus of G is not equal to 1, we will contribute to forcing of error through this term.

The second set of term tells you that numerical phase speed is going to be a function of the wave number k, if it is so that also will give raise to additional error. This property where c of N is a function of k is an attribute of a dispersive system.

If c N was not a function of k, we would have called it a non-dispersive system. Our original equation was a non-dispersive equation, because there c was constant, but here numerically we have converted a non-dispersive system to theoretically a dispersive system.

Now, individual algorithms will provide us an estimate for modulus of G and the expression of c N. From there we should be able to find out whether the method is the right one in terms of the error due to instability or stability, or error due to dispersion.

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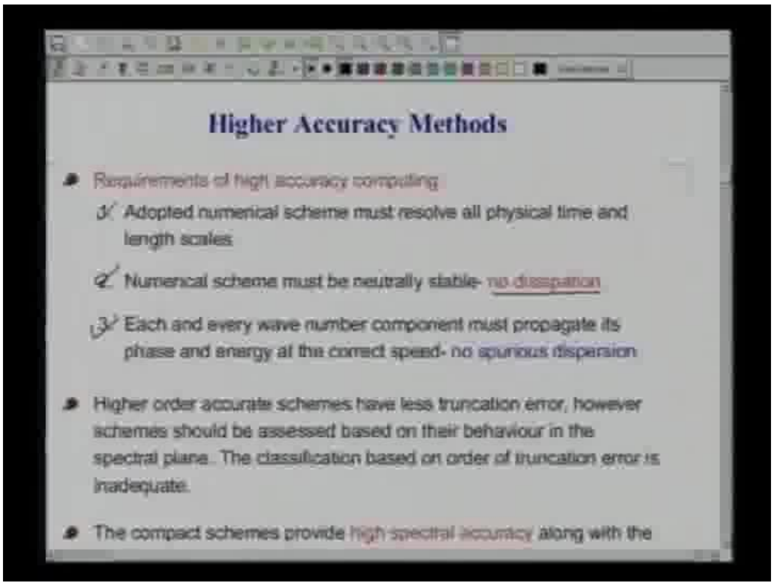

Having laid this ground work, we can now state what exactly we really want to do, if we want to perform high accuracy computing. First and foremost, we agree that what we require any space and time scale that is relevant for the physical problem, they have to be resolved. The first term is very obvious; we need to resolve all physical time and length scales.

Now, the second term is what I just now described to you that we need to have mod G equal to 1; that means, we need to have a neutrally stable system, we cannot have instability, then the solution will blow up, you will immediately know, but what is not all is realizing that you cannot also have so called stable dissipative system.

The basic idea is the following. Numeric should not impose its role on the physics of problem, so numeric should stay neutral. Physics should dictate, whether the system is stable or unstable. So, that is why numerically we do not want to add any dissipation, this is absolutely unwanted.

Now, we have also seen in the previous slide that there is another source of error that is where the numerical phase speed becomes a function of wave number. We say that is a hallmark of a dispersive system, we do not even want that. That is what we have stated in the 0.3 that each and every wave number component should propagate its phase and energy at the current speed and that I have stated to you that is, the group velocity. We will go ahead and figure out how these things are evaluated, and we will find out how to avoid spurious dispersion.

These are the three prime requirement of high accuracy computing system. Next, what we also realize that when we are replacing a differential equation by its discreet version we are essentially indulging in some amount of truncation of terms; that error is what is called as a truncation error. It has become quite common place for everyone to look at the property of the numerical scheme in the physical space. Then people try to talk about the order of the discretization. It is hoped that if you have a higher order of the discretization, lower will be the error. However, this is somewhat misleading, because the error behaves like this.

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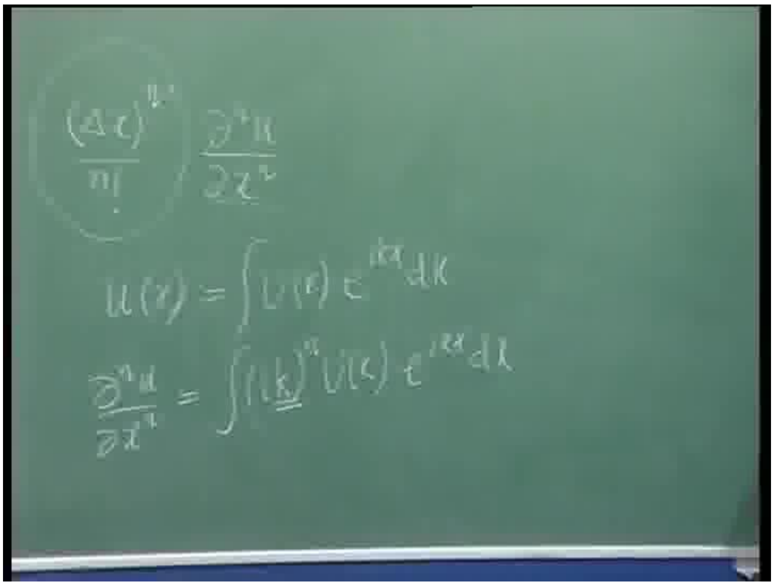

Suppose I have nth order system that would be given by a term, which will be dependent on the order of the system. There could be probably say n factorial term coming in there. Then what we could have is that we will have associated higher derivative term. What is basically done in most of the time, when you look at higher order accurate system, you are basically talking about this term.

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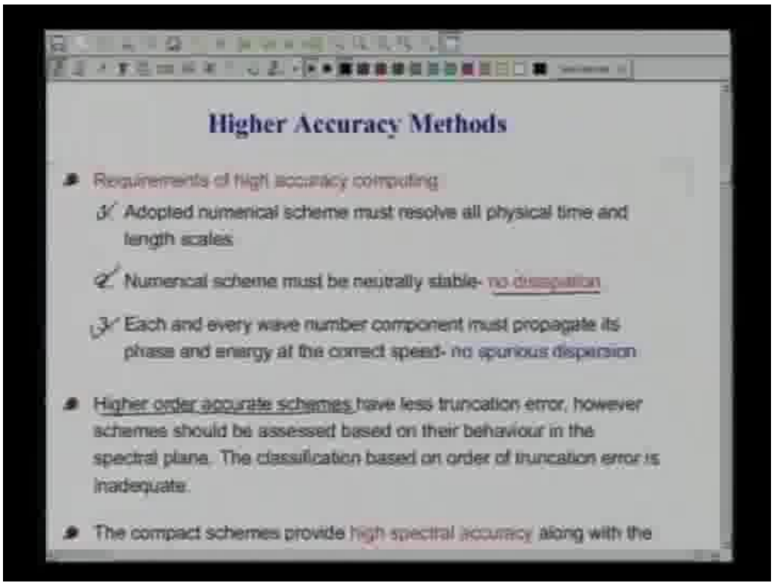

You expect that if this exponent is high, delta x being small, the contribution will be small, but the essential point remains. What we need to look at is basically the product of the two. It may so happen that I have n equal to 6, then this is ok, then I am looking at this. But, I can derive another scheme, where I would have a term here, which is instead of 5, it becomes 2. But then, the associated term there would become the third derivative. For that particular physical problem that third derivative could be much lower than this quantity.

There is no guarantee that higher the order of this term, lower is the magnitude. In fact, in most of the physical system, it happens the other way. If you look at the higher order derivatives, they also carry higher amplitude. This you can very clearly understand if you look at a term like this. If I write its Fourier representation, then I would be writing like this.

If I try to figure out its nth derivative, then what I am going to get would be nothing but i k raise to the power n and U of K e to the power i K x dk. You can see, higher the n, you are going to get higher contribution coming from higher wave numbers. So, this is quite easily understood that this root of characterizing schemes by order of the term in terms of the grid spacing is erroneous.

What instead one should look at is representation in the spectral plane. If we neglect term then see it is negligible or not, if it is negligible, you are done. We are going to take that kind of an approach. What we see is that many times you could derive schemes, which are formally lower order, but they are truly higher order from the prospective of its behavior in the spectral plane.

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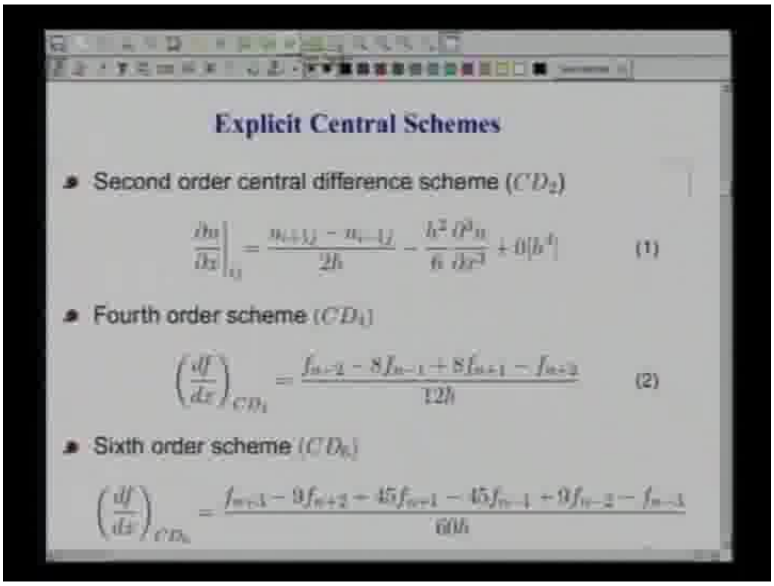

Now, let us explore this form some simple example. For example, we are interested in discretizing the first derivative term that is important in many physical systems, so let us begin with a first derivative. Suppose we perform a second order central different schemes that is given by equation 1 here, you can see that it is related to its neighboring point i plus 1 j minus i minus 1 j are divided by 2h. The highest order truncation error term is proportional to h square by 3 factorial. This is the third derivative and rest of it is order of 4 and above.

For the same way, we can also obtain a fourth order scheme that would involve more number of points. The essential point you notice is migrating from a second order scheme to a fourth order scheme, your stencil size becomes bigger.

For example, equation 1 involves 3 points, equation 2 involves 5 points. If you look at the corresponding sixth order system, it actually involves 7 points. This is the essential feature of explicit scheme; you want to go higher order, your stencil size increases. We will talk about why this may not be always good to go for higher and higher order schemes.

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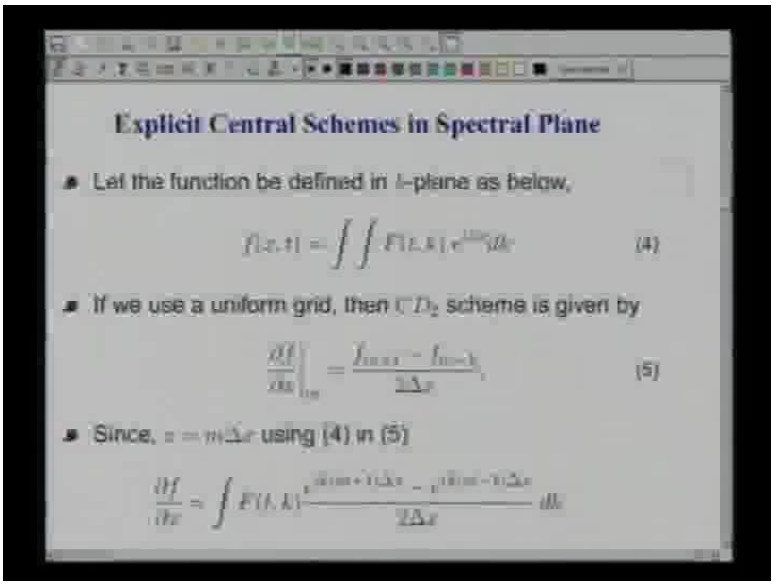

Now, if I want to view these central schemes in k-plane, then what I can do is, I can represent a function f of x and t in this hybrid manner, retain the time dependence as size it is, introduce k instead of x and then we perform this integral.

If we use a uniform grade, then what we usually do is we represent the second derivative like this; that we have just now seen. If we now substitute 4 in equation 5, noticing that x is equal to m times delta x, then I would get the Fourier amplitude retained as it is. In the phase, I will write e to the power ik m plus 1 delta x minus e to the power ik m minus 1 delta x. You notice that e to the power ik x is nothing but e to the power ik m delta x.

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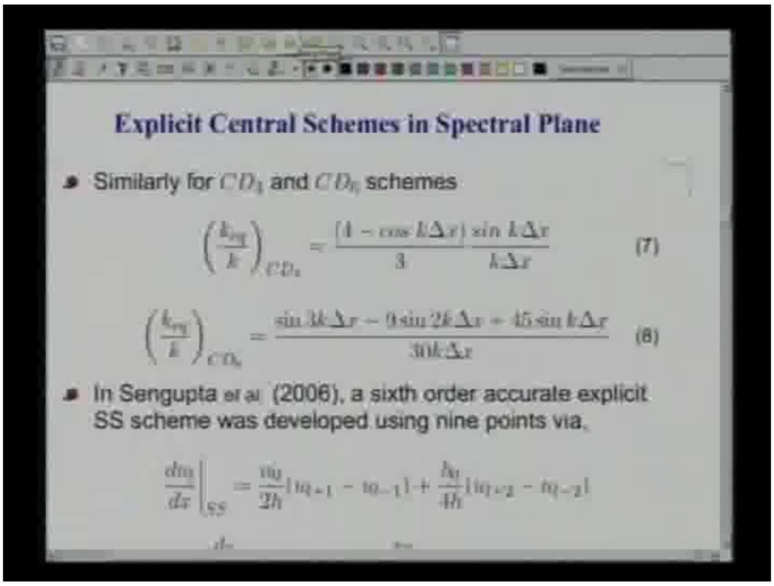

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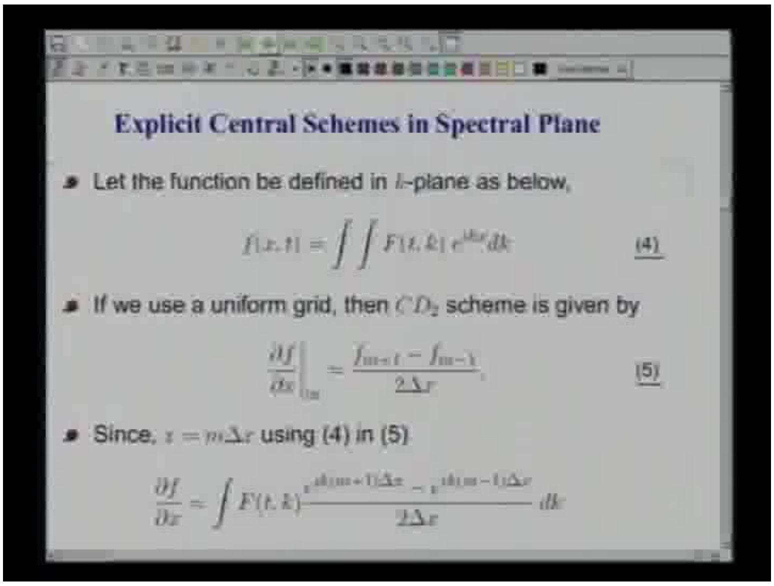

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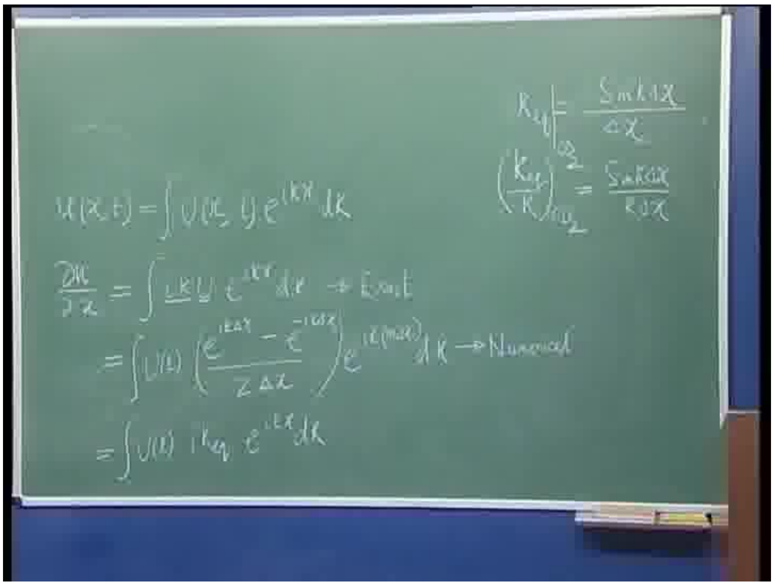

So, what happens as a consequence? We get an expression for k equivalent. Let me explain what this k equivalent is, if I represent u of x and t in terms of U of k and t e to the power ikx dk, then del u del x can be very easily written as i k U e to the power ikx dk. So, this is your exact representation.

What we just now seen, numerically what we have got? Numerically, we got U of K, written as it is. We had obtained e to the power ik delta x minus e to the power minus ik delta x by 2 h. Then, we had e to the power ikx that was m delta x that is what we say and that is our x and dk. So, what happens here, this is your numerical representation.

What you are noticing is that in getting the exact representation what you did? You had the Fourier amplitude; you just simply multiplied by ik to get the derivative. So, if I look at this, I could write this equation also in terms of U of k as it is. This quantity that we have here, let me write it as ik equivalent and then we have e to the power ikx dk.

So, what happens? You see, doing the numerical operation is equivalent to replacing ik by ik equivalent. What is this ik equivalent? The ik equivalent is this quantity. What is this 2i sine k delta x by 2 h(Refer Slide Time: 25:37). So that I will get as equal to i sine k delta x divided by now mixing up, so instead of h let me write this as delta x, so I am going to get this as equal to ik delta x by delta x.

What happens is then you can get rid of this, then this is an expression that we have derived it for CD2 scheme. So, what we are going to write here as k equivalent by k for this CD2 scheme is going to be sine k delta x by k delta x.

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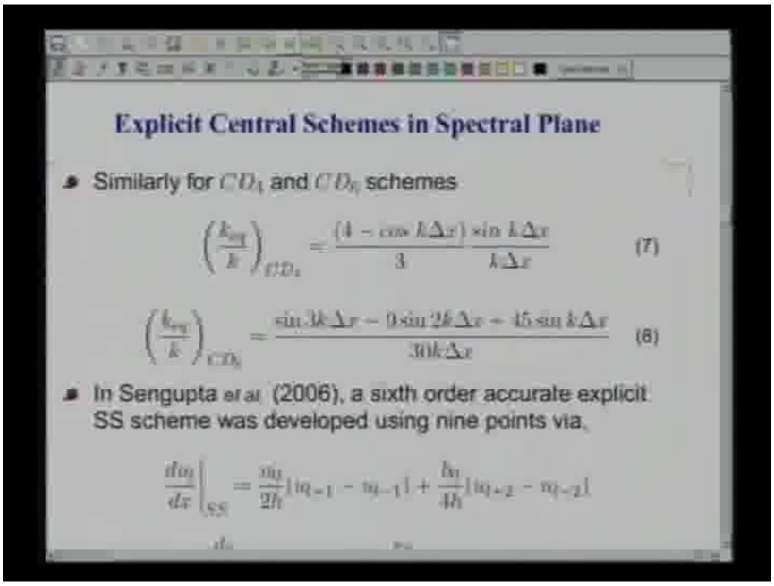

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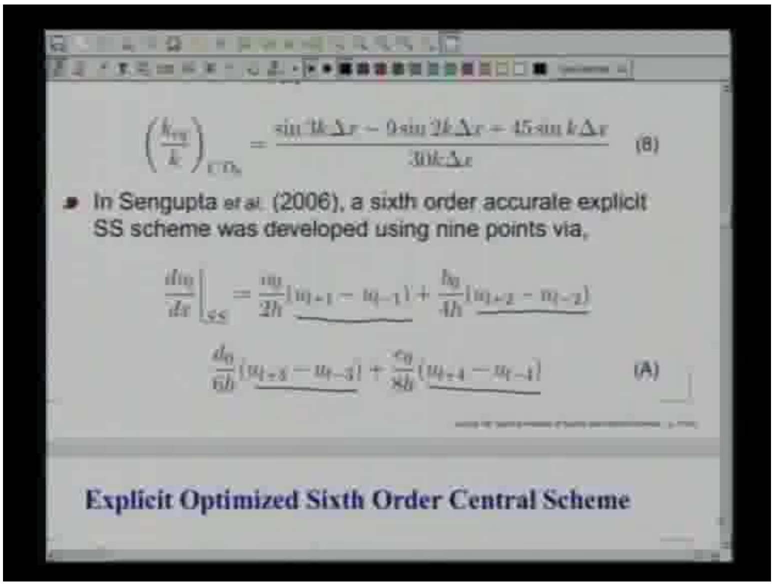

We can carry on with this exercise for CD4 and CD6 scheme. What we notice is that we can get some expression for k equivalent by k for CD6 scheme, given by sine 3 k delta etcetera, which is given by your equation 8. So, we are collecting expressions for this in the k plane.

Now, let me tell you about a method that was developed here, few years ago, where we actually wanted to develop in 9 points schemes. We want to explore a 9 points scheme, but what we wanted to do was we wanted to optimize the scheme. So that the error of this 9 point scheme is much lower than many explicit schemes.

Basically, what you do is what I have written down in this equation A. I will write it as dul dx, it would be a naught by 2 h. You can see the terms appear pair wise. For example, l plus 1 and l minus 1 are coupled together, the same way l plus 2 and l minus 2 terms coupled here. You can see without the presence of u l we have 9 points here.

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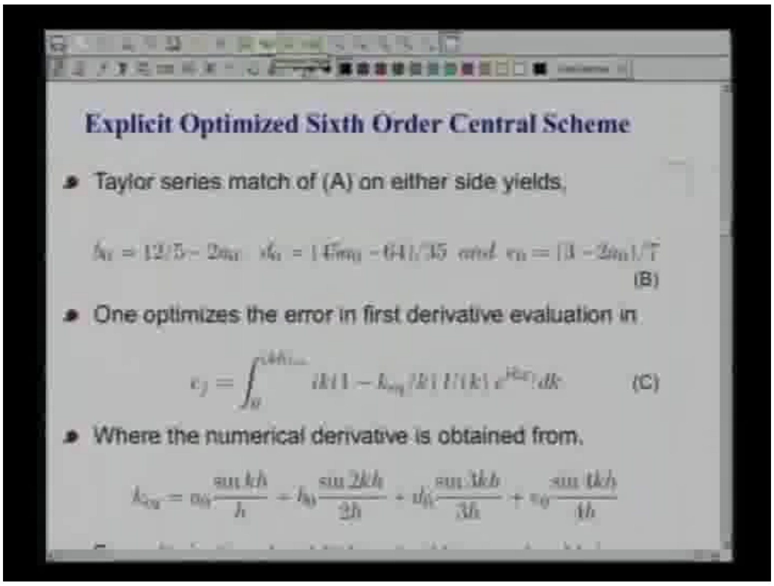

Now, how does this scheme behave or how we go ahead and do it; that is what we are going to see. We have written down those expressions in the physical plane. Now, what we could do is on the left hand side we have du dx, on the right hand side we have those 9 points. So, we can write down the Taylor series and match term by term.

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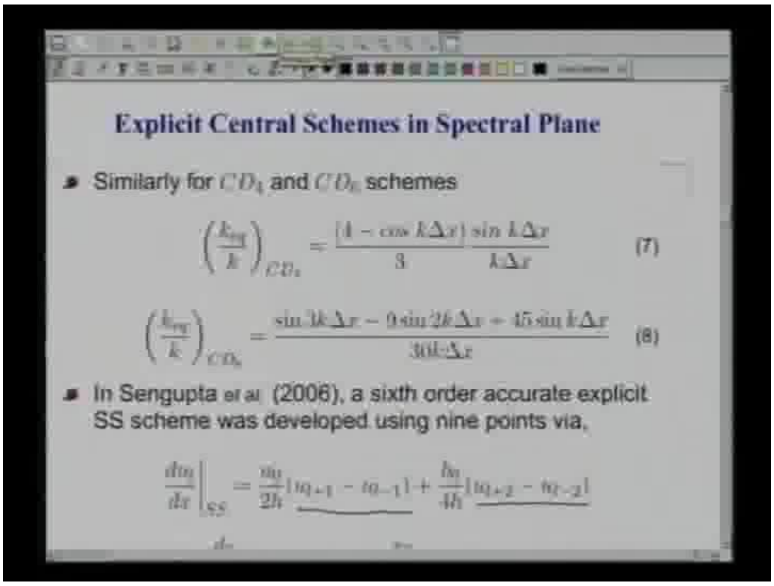

When we do that we are going to get various odd derivative terms, even derivative terms will all becomes 0, why? Because, you can see that the way this terms appear pair wise with a minus sign in between, so if you do a Taylor series this will only be written - the odd derivatives, all even derivatives terms drop out. Then, you successively equate the odd derivatives - all kinds of derivatives on the left and right hand side, write everything in terms of one of the parameter.

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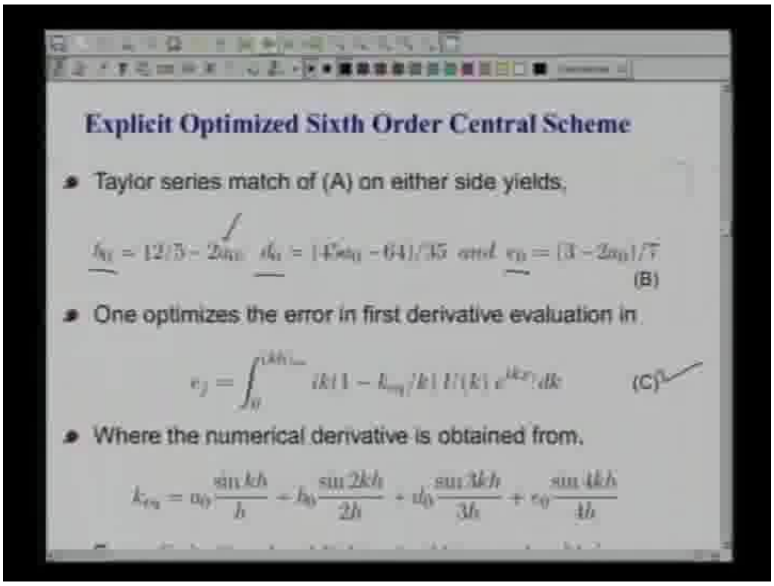

Here, we have used a naught as the parameter. Matching the Taylor series, we get expression for b naught, d naught and e naught in terms of the parameter a naught.

Now, what we want to do is as we have seen here, the first derivative is given in terms of ik U for the exact quantity. For the numerical quantity, it is ik equivalent U (Refer Slide Time: 29:58). So, the error would be difference of the two. That is what we have written in this equation C. If you look at equation C, the error is the difference between exact and numerical estimate.

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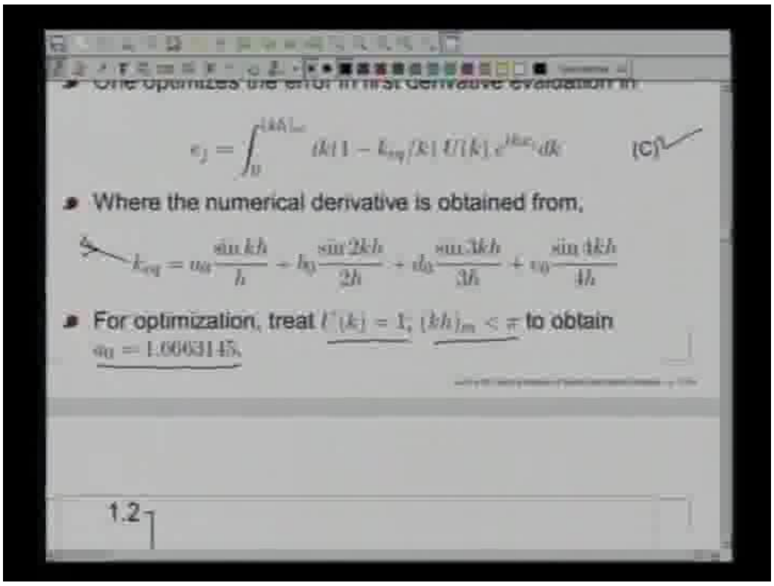

Now, what you could is we have the expression here; we can use that in the expression for the derivative that we have written there. In the previous slide, what we can see here is given in this equation. So what we could do is, basically we could substitute the Fourier-Laplace series and we will get this expression that we have indicated here.

Now, the problem is set, you have the expression in c that is going to be a function of what? Only a naught. What we did was optimize this error as a function of a naught. We have done two things; we have taken U of k equal to 1 that is true for delta function excitation that is the most conservative estimate one can get. The other thing is we did not go the full limit from 0 to pi; instead we have taken a limit, which is slightly lower than pi. In performing this optimization in equation C, we end up getting a value of a naught.

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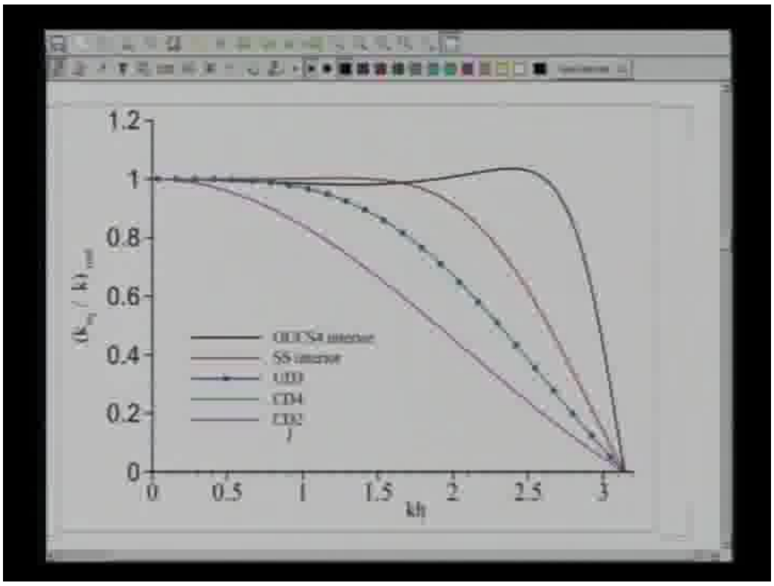

We have now introduced some schemes here, they are essentially plotted here. What you are seeing is a host of method. The purple color is the CD2 method and that gives you that k equivalent by k. It is the performance parameter; ideally you wanted it to be equal to 1. What we are noticing here that CD2 scheme falls off from the ideal limit for a very small value of k h itself.

What is kh? kh is nothing but non dimensional k. k is a wave number, its dimension is 1 over length. h is delta x, which has a dimension of length. So, kh is non-dimensional length that we have shown it in a limit of 0 to pi.

Now, once you notice that we have shown here the CD2 scheme, CD4 scheme is shown by the green line, which is shown here. UD3 is a third order upwind scheme, which we are going to subsequently define, but please take a note at this SS scheme that we just now obtained as a optimization exercise. You can see its performance; it is a 9 point formula, but what happened, in getting those conditions for Taylor series, we actually have matched up to 6 order.

Basically, it is a 9 point, but it is sixth order scheme, because that is how we match the Taylor series, because we kept a naught floating. However, this k equivalent by k we have plotted here against kh. This performance parameter is better if you would have taken a CD8 scheme, because that is what we want to do. Because, you have taken a 9 point and what you really want is a benefit that is more than that you could get with a CD8 scheme. That is what we mention that you should not focus your attention in the physical plane; instead you should look at in the spectral plane. That is what we are seeing, in the k plane this nominal sixth order scheme performs better than an eighth order scheme and this is basically an explicit scheme.

I have purposely drawn another line here, which is shown here by this black line. This is an implicit scheme that should motivate us to look for them, as it appears here that this implicit scheme is far more accurate than this explicit scheme. We are going to talk about it as we go along.

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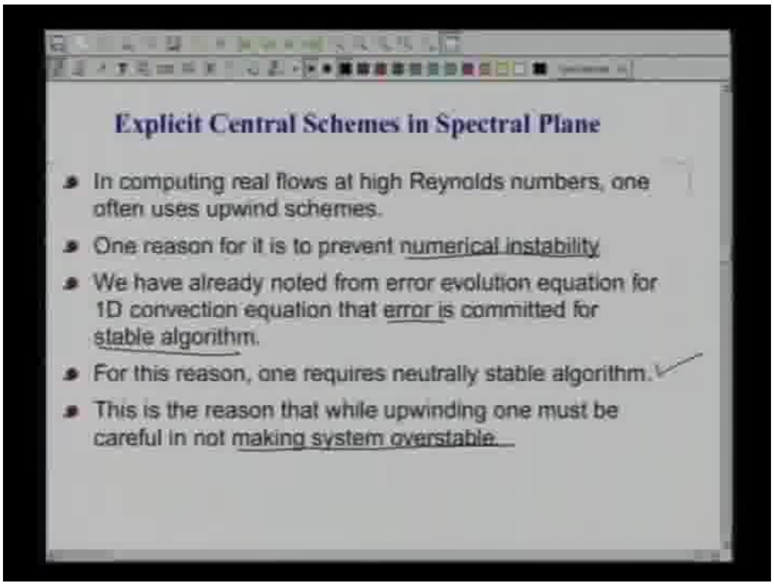

Now, if I try to see its usage for all this derivative operation, we want to compute a real flow at high Reynolds number. If we use the central scheme, it may not work, because what happens is the property - combined Properties of the space time discretization will show that such central schemes are susceptible to very high wave number error. You need to stabilize your numerical method against those high wave number errors.

One of the ways for utilizing upwind scheme is to introduce numerical dissipation that prevents numerical instability. This is the motivation for abandoning explicit scheme; instead going for upwind scheme. Explicit central scheme will be abandoned in favor of explicit upwind scheme.

We have to be careful, because we have already seen in the derivation of the error revolution equation that in our zeal. If we want to make the method stable, we can also introduce error. So, that is what we have to be careful. Even stable algorithm creates error, so we will have to worry about it. That is why we have said very clearly that we need numerical stable algorithm.

So, taking upwind scheme is an absolute must, we have to be rather careful in adding just the right amount of dissipation, which will take away those high wave number errors, while it should not tamper with the physical nature of the problem. That is what we want to make a comment here. We do not want to make the system over stable that is something we must be careful.

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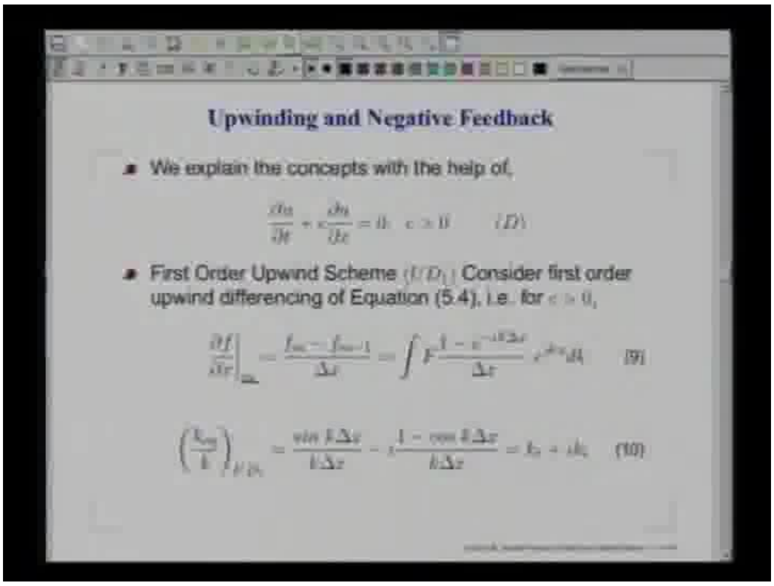

What does upwinding does? Again, let us try to explore it with the help of all model equation that convection equation that we have shown here, del u del t plus c into del u del x.

Now, let us take one of the simplest possible upwind schemes. You have noticed that all central schemes are even ordered. We have looked out the CD2, CD4, CD6 etcetera, where the stencils were symmetric and we always ended up having even ordered scheme. So, if we look at the first order scheme then what happens?

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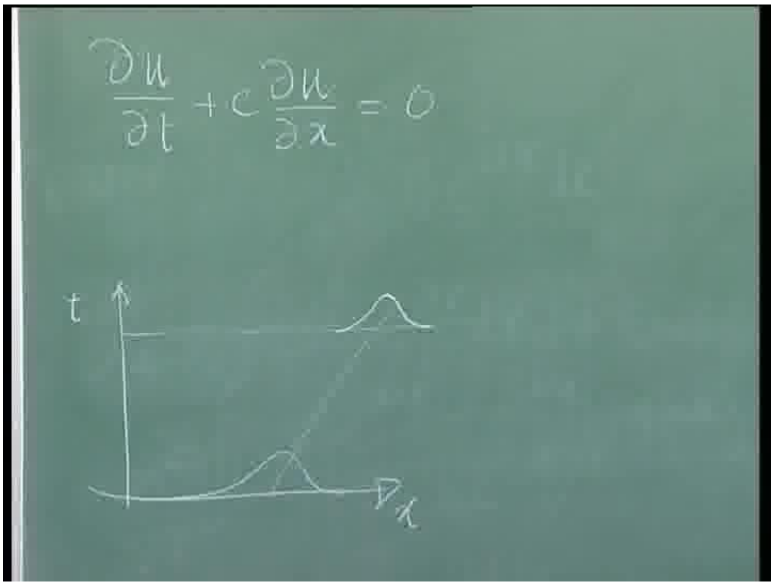

Let me explain this somewhat better, because we are looking at a problem where our governing equation tells us that the signal is propagating from left to right. Because, c is positive, if I give some kind of an initial condition in the x t plane and if I give some kind of an initial solution like this, what happens with the passage of time? This condition will move to the right.

At a later time, I might see that this solution has moved here. This is the property of this equation that it does not amplify or dissipate; it also does not disperse. So, if I have a packet like this, the packet is retained as it is.

What happens if I try to solve equation like this using first sort upwinding scheme? That is what you are noticing here. I have a purposely looked at the del f del x at the m th node, so where is the information coming from, it is going from left to right. So, what I should do is I should write an expression which should involve the m th node with the m minus one th node. This is what we are going to do; we have an expression for f that we have seen in terms of Fourier-Laplace series; that is what we have done in equation 9.

We just now seen that if we follow the physical nature while discretizing through the equation 9 we were actually following the correct trend, because information is propagating from m minus 1 th point to the n th point and that is precisely what we need to do.

Again, using the Fourier-Laplace series, what we can see that we can work out the k equivalent by k. You can work it out, what you would notice? It is a real part, which looks exactly like we had obtained for the CD2 scheme. In addition, you have an imaginary part. The imaginary part actually is the numerical dissipation that we have added to this scheme.

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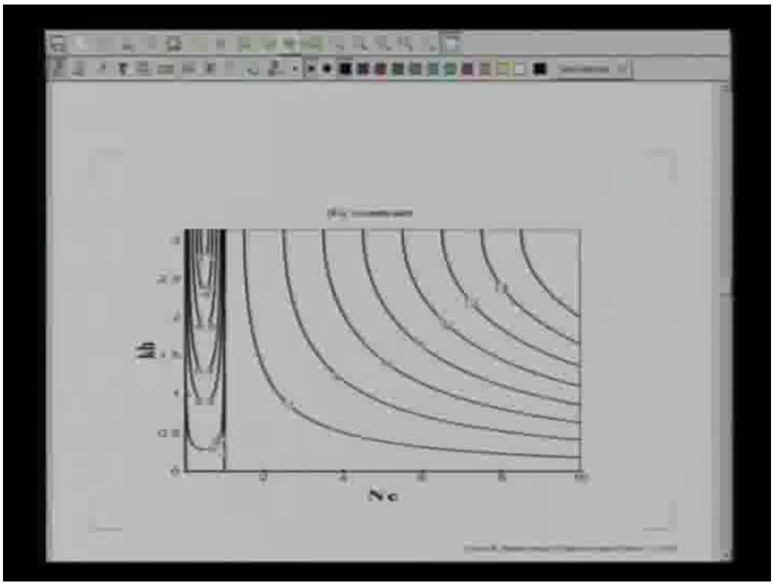

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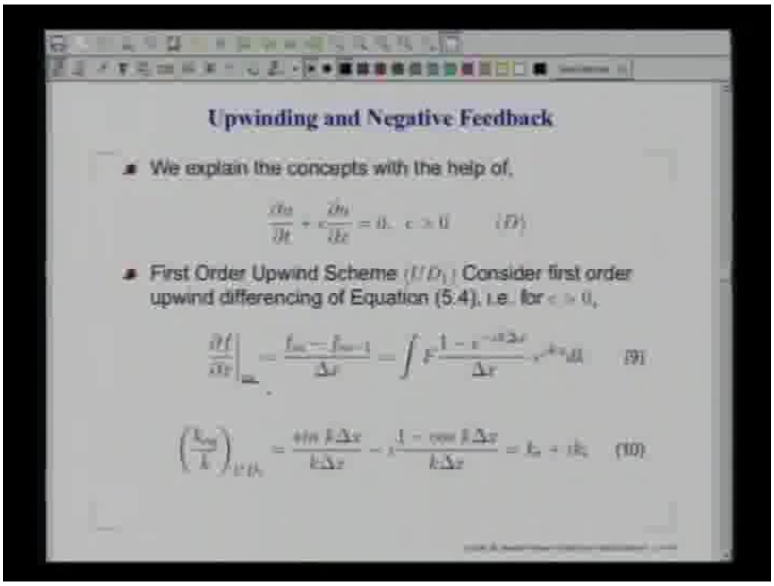

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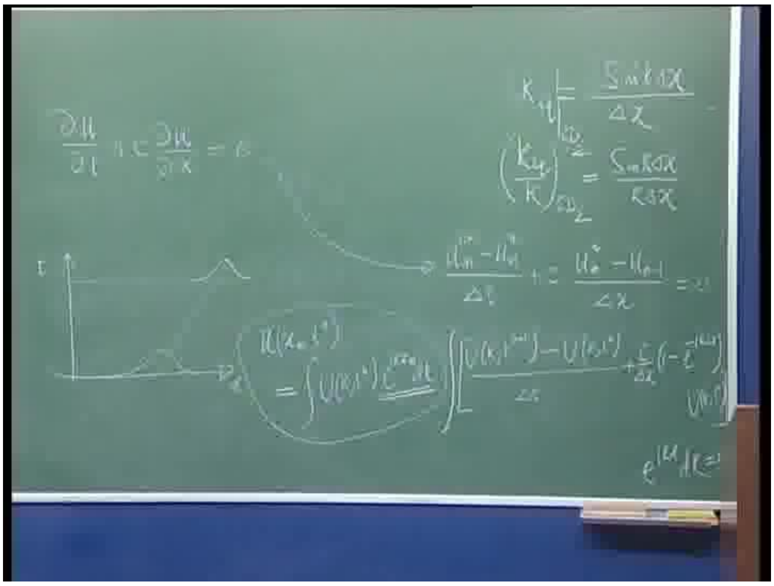

Why I am saying numerical dissipation? Which we can show it subsequently, but what we can do is, basically having obtain this scheme here del f del x. So, what we could do is basically, if I am writing this, I will be writing here u, we are looking that m th node. So, I will be writing here like this, um n by delta t, so this is my Euler time integration for the time derivative. Then, I will be adding C U m n minus, now we are doing first order up wending, so that is why I am writing it like this.

So, I can write down the Fourier-Laplace transform. Then, what we are going to do is we are going to write; if you remember we have introduce that as u of x m t n, we have written it like U of k t of n, we will write it as e to the power ik xm dk. That is our Fourier-Laplace transform expression for the variable, so I can substitute it there.

What we are going to get is, basically from here I will get U of k t n plus 1, from here I will get U of k t of n and this is divided by delta t and c. What am I going to get here? What I have just now written here?

This is the expression that is given, so what we are going to get? So, this whole thing that we are writing, it comes under the integral sign. We have performing the integral over all k. So here, I am going to get c by delta x and what do I get here? It will be1 minus e to the power minus ik delta x into U of k tn.

This whole thing I am going to multiply with respect to this (Refer Slide Time: 43:21). So that is what we have written here, it has been multiplied by e to the power ikx dk. Since, this whole thing is equal to 0 the integrant must be 0. What we have basically done say that the integrant itself must be equal to 0 and that is what we are going to write - equal to 0.

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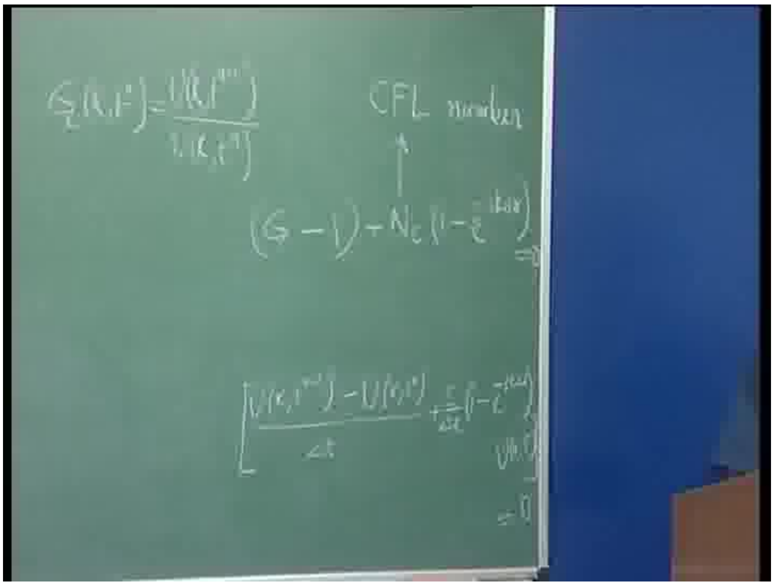

Now, if I divide this equation by U of k t n, then this term will give me G that is the definition of G. Remember, G of k t n is nothing but U of k t n plus 1 by U of k of t n. So, if I divide this equation by U of K t n, then what we are going to get is the following. We are going to get here G minus 1, if I take this what do I get? c delta t by delta x is our CFL number, so that is going to be our CFL number N c. This is well known CFL number, which we have talked about many times before, so we are not surprised. In addition, we are getting here 1 minus e to the power minus ik delta x equal to 0. So, you have an expression for G.

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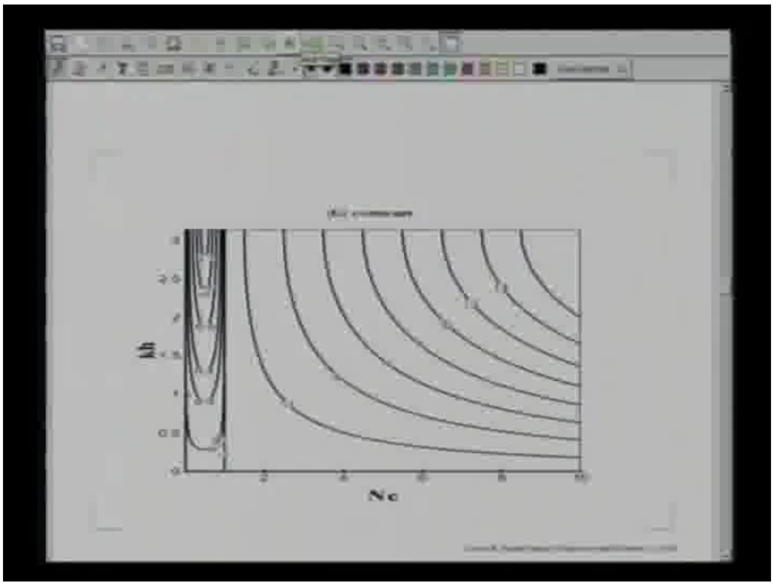

What we do is having obtained the expression for G, we can plot it. This plot is a very fascinating plot, because it says very clearly that you have a vertical line corresponding to N c equal to 1. To the right of that line the method is unstable; everywhere G is greater than 1. So, there is absolutely no mystery here, if you are adopting this method never try the value of N c greater than 1. If you do take N c less than 1, then you can see that for different wave number they are going to be attenuated by different amount. Higher the value of kh, higher is the attenuation.

Whereas, if you take N c exactly equal to 1 what you getting is G is equal to 1 everywhere and that is your exact solution. So, what we have learned that upwinding by itself is not bad, provided you do it physically. Here, doing it physically is equivalent to taking N c equal to 1.

You understand that there is nothing wrong with upwinding per say, but if you overdo it say instead of taking N c; N c it means what it is a measure of c times delta t, so delta t it basically tells you a time step that you are taking.

So, if you take a time step, which is too small, then you do not get generally more accuracy. Here, it shows very clearly to get the maximum accuracy, you will have to take delta t is equal to delta x by c. That means what? Every time step your solution is traveling by 1 node and that is the definition of phase speed c.

You can very clearly appreciate what has been achieved here; you do get exact solution for N c equal to 1. If you take more than 1, you are unstable, if you are less than 1, it is stable, but it will be erroneous.

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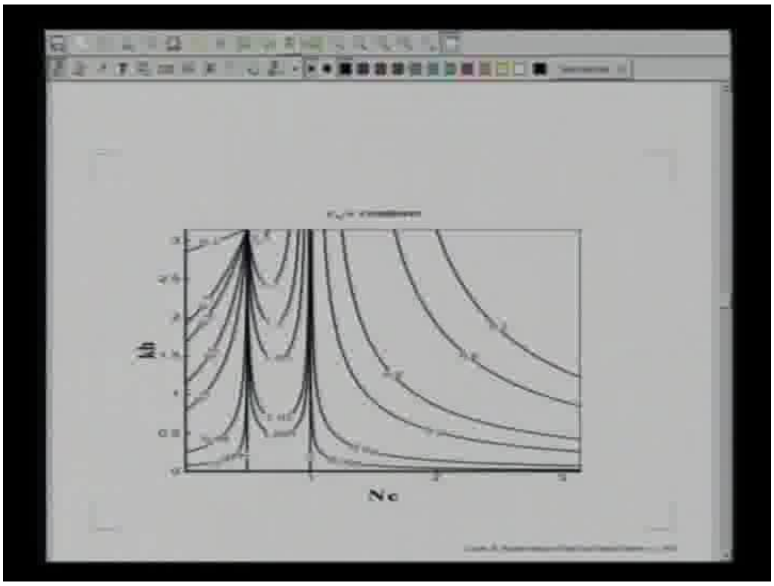

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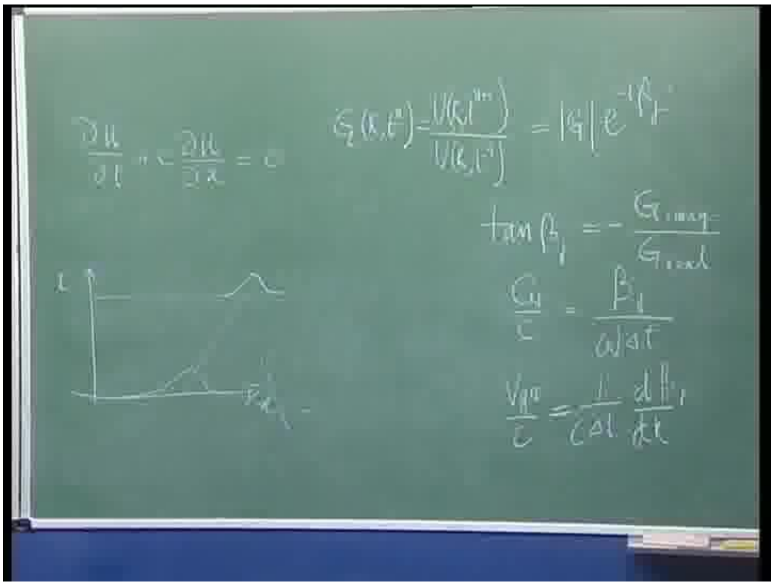

Then, we can plot this c n by c. See basically what we have written down here, G as given here. So, this could be written down also in terms of like a modulus of G; that is what we plotted in the previous slide. Then, we can also get it phase, which I write as beta j.

What is beta j? beta j is given by tan beta j, would be nothing but G is complex. We get G imaginary by G imaginary by G real, so I got this. What does it mean, whenever I am numerically integrating, I am amplifying or attenuating by mod G and the solution is shifting its phase that is what it is doing; it is moving to the right. So, every time step, I need its phase to shift and that numerical phase shift is given in terms of what you have gotten the value of G as.

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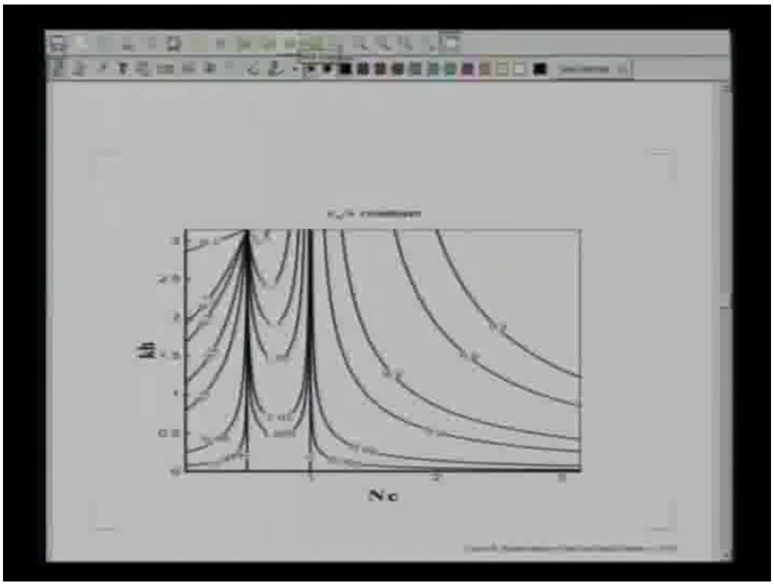

Now, once you have gotten the value of G there, inconsequence you have gotten the value of beta j, we have shown it already and that beta of j is nothing but equal to - can be related to C N by C. So, C N is the numerical phase speed that we can show. C N by C will be beta j by omega times delta t.

So, once you have obtained the beta j you can calculate numerical phase speed; they are all related. So, the basic task is to choose your numerical algorithm, obtain the value of G, collect its real and imaginary path and obtain C N by C. The same way, you can also get your V g N by C that should be equal to 1 by C delta t d beta j dk. This is what we have discussed when we are talking about hyperbolic equation.

What you notice here, in this C N by C equal to plot also there is a beautiful thing. That is, for N c equal to 1 you have C N by C equal to 1. That is exactly what you want and that should be equal to 1. Numerical phase speed should be equal to the physical phase speed.

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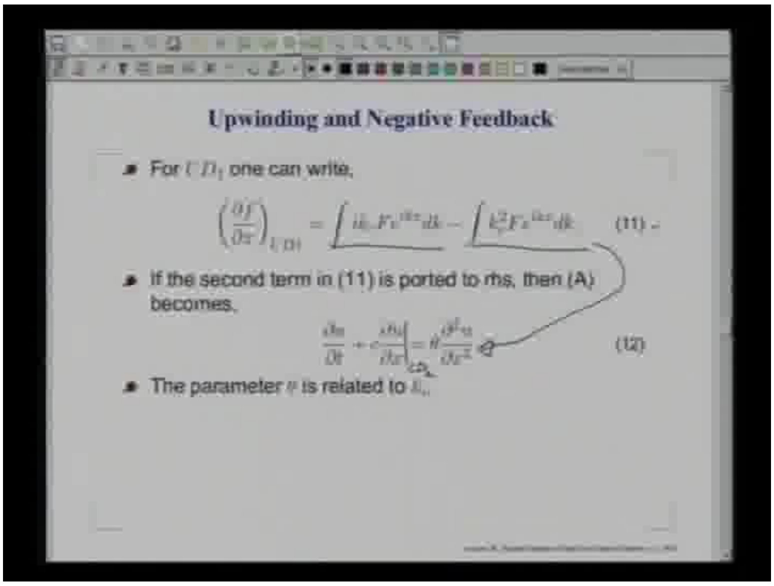

Now, let us see rather interesting thing that even for a value of N c equal to half you get the same quantity, but we have seen in the slide that there you will have G property that is not desirable. We notice that the same thing, you can look at in terms of the group velocity also. The V g N is exactly equal to C. What it should be for N c equal to 1? So, this is essentially the idea by which we can really work out.

Now, let us look at the consequence of this UD1 scheme. I told you that k equivalent has a real part and imaginary part, so that is what we have written down here in equation 11. We have shown it, the real part gives raise to this, which is like your CD2 scheme that we have noticed before. This imaginary part actually gives you an added dissipation. Why because, on the left hand side I have this term. Now, if I have some quantity, the positive quantity with a negative sign I put it on this side, then I am going to get this term that is what a positive quantity k i square times f, this actually plays this role.

If you have the Fourier-Laplace transform presentation of a function, you do take the two derivatives twice and then you end up by getting an expression of that kind; minus k square u of k. So, this is exactly what has been achieved.

It is all over interesting thing to get a solution which is numerically correct. We added up some numerical dissipation, which was not there, but we have chosen in such a way that delta t and delta x, we have kept N c equal to 1 and we have achieved a perfect solution.

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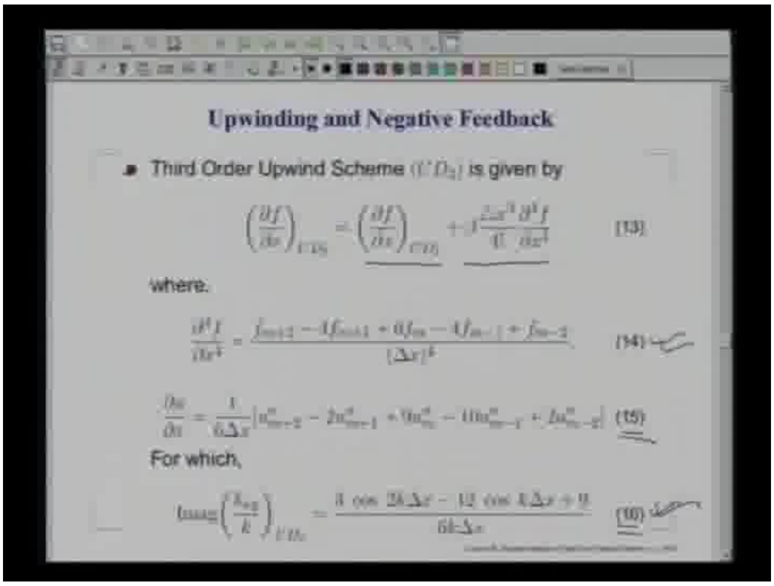

Basically, once we have made this case for first order upwinding case, what we have shown? If I look at the first order upwinding case, the first order upwinding case is equivalent to doing this here. This is your CD2 scheme on the left hand side and to that you have added a second derivative term that is your first order unwinding, so we can generalize it.

For example, we generalize it to a third order upwind scheme. Third order upwind scheme will be, I will take the derivative here, a fourth central difference scheme and to that I will add a fourth derivative that is your third order upwind scheme. You can successively generate fifth order, seventh order scheme, so on and so forth.

Where we have already seen, what del of del x CD4 expression is. I have shown you what this fourth derivative expression is in terms of that. If you substitute all of that together, you have del u del x by third order upwind scheme, it is given by equation 15. So, this has been done for a case where the signal is propagating from left to right.

Having done, we have the Fourier-Laplace series and we can substitute it. We will get the real part of k equivalent by k for UD3; it will be exactly same as CD4. In addition, what we will get? An imaginary part and that is what is shown in equation 16.

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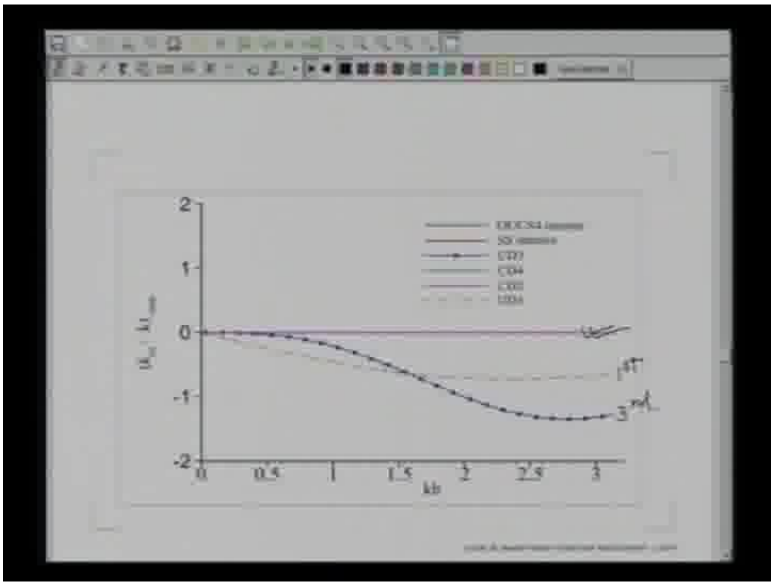

So, one can work it out, one can see that third order upwinding is equivalent to adding this term. This is what you get to see, if you plot all the k imaginary by k plot against kh. If you look at all central schemes, they all fall along 0 lines; these are what you get for all central schemes. Whereas, this brown line is for first upwind scheme and this is your third upwind scheme.

What you notice interestingly enough that there is a qualitative difference. Third order upwind scheme starts to become active at relatively higher rate number, whereas first order upwind scheme become active right from the beginning.

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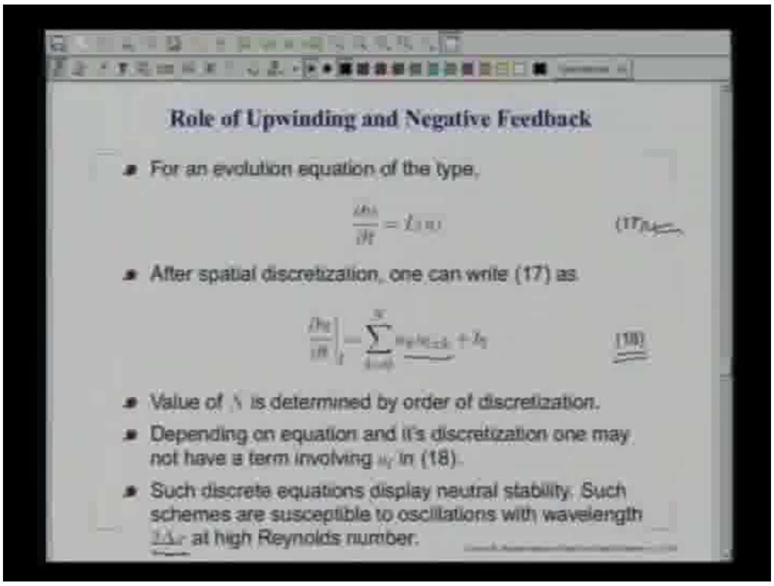

What we could do is, we can try to rationalize what upwinding is doing? Upwinding; let us say we look at some kind of an evaluation equation like this. Then we can think of L of u includes all the spatial derivatives and we can represent it. Then what we could do is we can write down the spatially discretize scheme like equation 18.

You see there are all kinds of terms involved here, k goes from 0 to N; N determine by the order of the scheme, we have seen. If we take a second order scheme, we need 3 points; if we take fourth order scheme, we need 5 points; if we need sixth order scheme, we need 7 points. So, N is determined by the order. What we have done? We have done a similar analysis for space term dependent equation and we can work it out in that particular fashion.

Basically, the discrete equation what we really need should be stable, but what we find that central schemes are neutrally stable, because just now we have seen that k equivalent by k imaginary 0. So, it does not add any dissipation, it is neutrally stable, but if there are numerically instabilities at high wave number due to some high k source then we need to suppress that.

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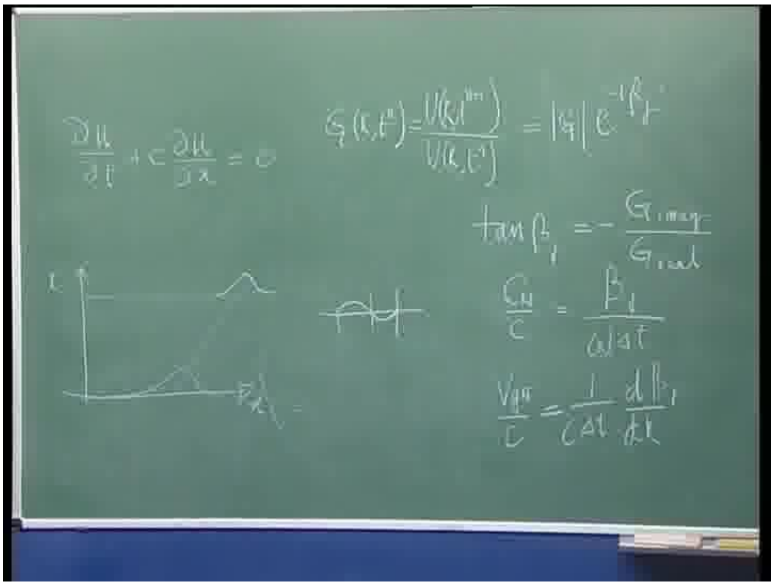

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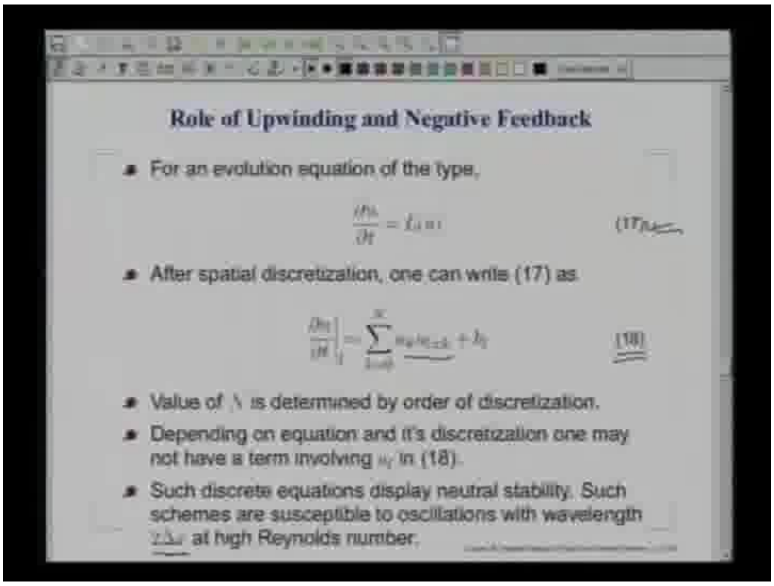

What happens in case of a central scheme? You always see oscillations with a wave length with 2 delta x. This is the highest wave number that you can resolve in that grid. If I have a grid points like this, this is the smallest wave length wave that I can represent (Refer Slide Time: 56:52). So, that is what we are seeing, so the oscillation is related to 2 delta x.

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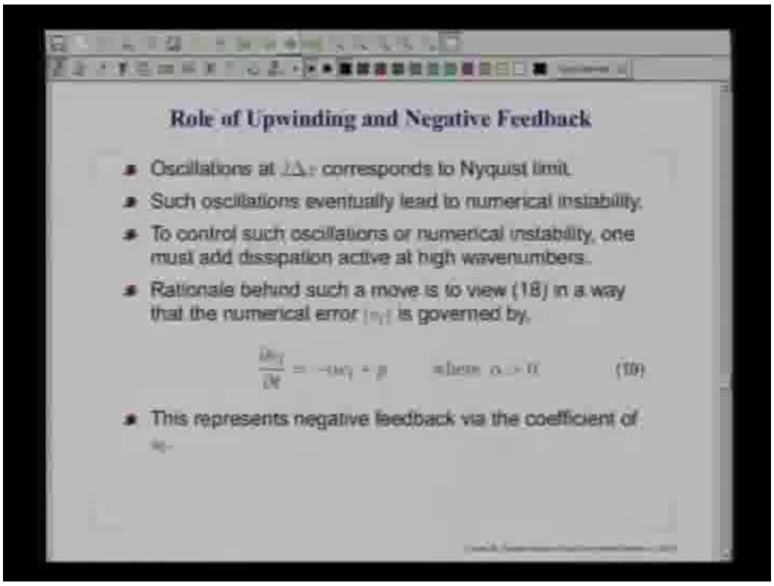

Now, if I write down the corresponding error equation, which is not very trivial, to control this oscillation at 2delta x what we really understand is that the central schemes are insensitive to the node point we are looking at.

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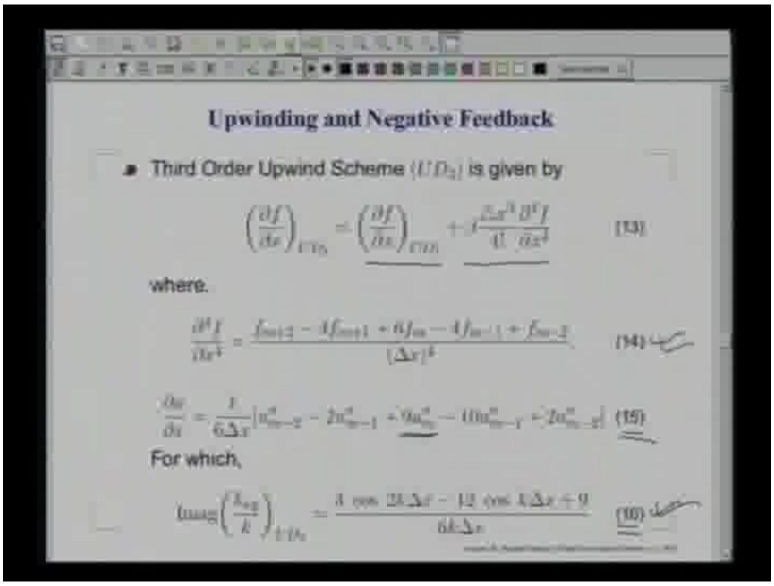

However, if I do upwinding link, I notice that the schemes as we have seen here, if I am looking at the n th point, there is a term here, which involves the n th point itself, which is not there in central schemes.

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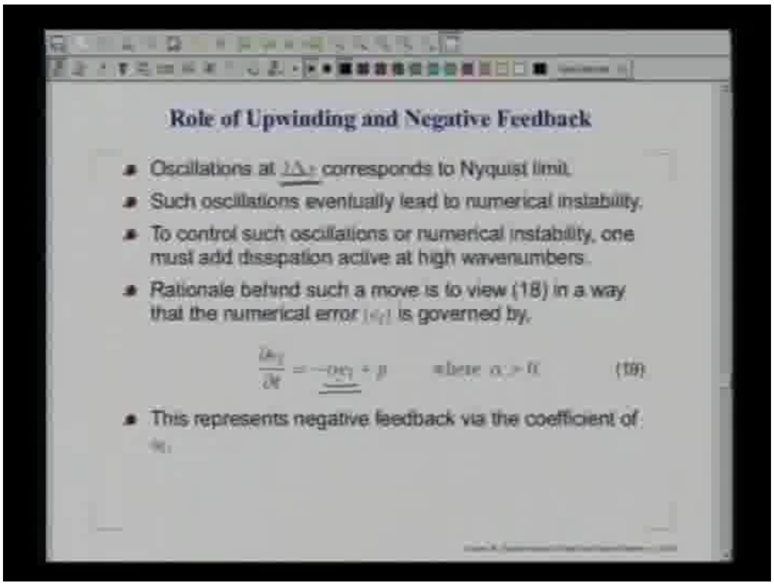

Whenever you do upwinding you get this term, when you get this term, you actually can write down the error equation, which will involve this term. So that means, the error at the l th node is coupled to the error at the l th point itself. If suppose p is 0, I could integrate it and I can show that e of l goes as e to the power alpha t minus exponent. If alpha is positive, then this is going to show that the error is going to decay with time. So, that is the whole key to stabilizing schemes. You need to add a term in such a way that it provides this minus alpha of this kind. This is what exactly you decide by this.

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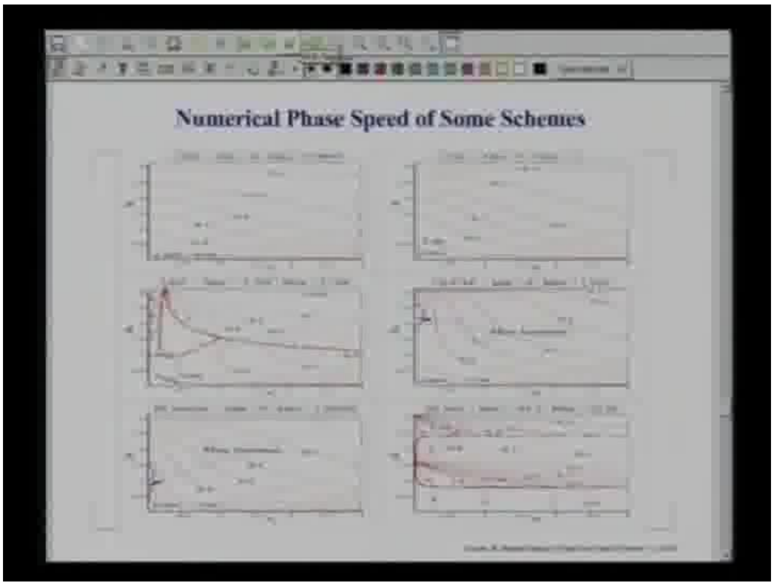

So, I suppose you can get some of this plots as it is shown here, numerical amplification factor for some of the schemes CD2, CD4, UD3, this SS schemes that we have talked about. Same way, we can work out the expression for C N by C. We can work out the expression for V g N by c. So, this is your phase speed representation. Here, you have the group velocity representation. What you notice is, certain properties which we will be exploring once again, when we talk about compact scheme later time.

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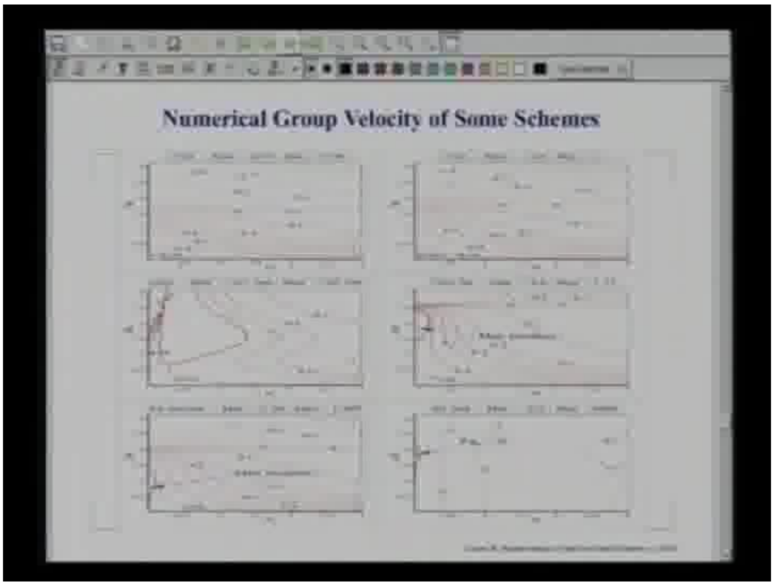

So, I think, I will just stop at this point. We will get back to a framework where we will be talking about compact schemes, which are essentially implicit schemes, which perform some of this task that we have set up today in a better manner; that is what we will be taken up from next lecture onwards.