**Foundation of Scientific Computing**

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**Indian Institute of Technology, Kanpur**

**Module No. # 01**

**Lecture No. # 26**

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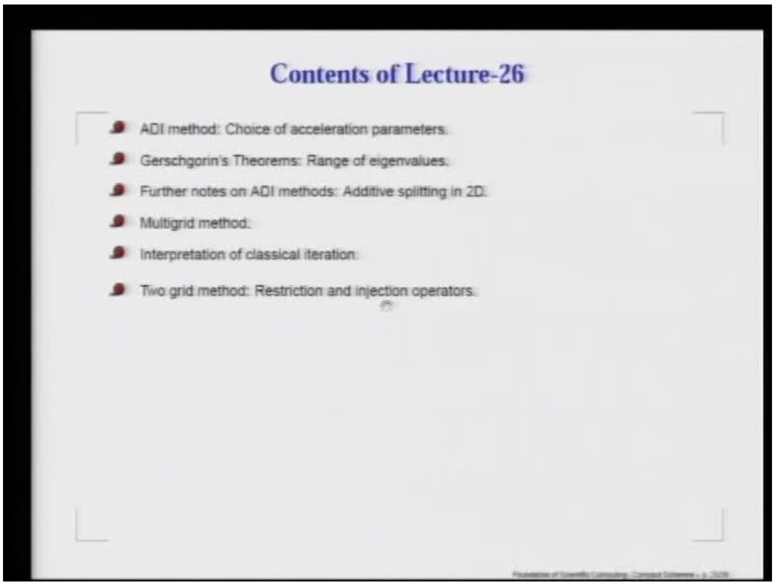

## Contents of Lecture-26

- ADI method: Choice of acceleration parameters.
- Gerschgorin's Theorems: Range of eigenvalues.
- Further notes on ADI methods: Additive splitting in 2D.
- Multigrid method.
- Interpretation of classical iteration.
- Two grid method: Restriction and injection operators.

In this lecture 26, we are going to begin with a review of the ADI method that we have discussed so far and talk about the choice of acceleration parameters, namely, the Peaceman-Rachford and Wachspress methods. We will talk about Gerschgorin's theorem that identifies various disks where the Eigen values reside. Once we know the Eigen spectrum, we can find out these acceleration parameters. In a further expansion of the idea of ADI method, we point out that this is essentially an additive splitting and works only for two dimension. When we go to three-dimensional problems, we do not have the ability to use the ADI method.

And then, also, we also refine that on many top problems ADI methods do not work and we move to a new class of method which is called the Multigrid method. Essentially, the idea here is to work on a set of grids, not necessarily on one grid like what we have done so far.

In the context of Multigrid method, we can fall back upon the classical iteration as the constituent of this Multigrid method, and that is why we reinterpret the classic iterative method in terms of the Eigen values and Eigen spectrum.

As an example, we begin the discussion with the two grid method; that is, essentially we are talking about one coarse grid and one fine grid, and introduce the various operators those are involved in solving Multigrid method namely - some of them are mentioned here restriction operator and the injection operators.

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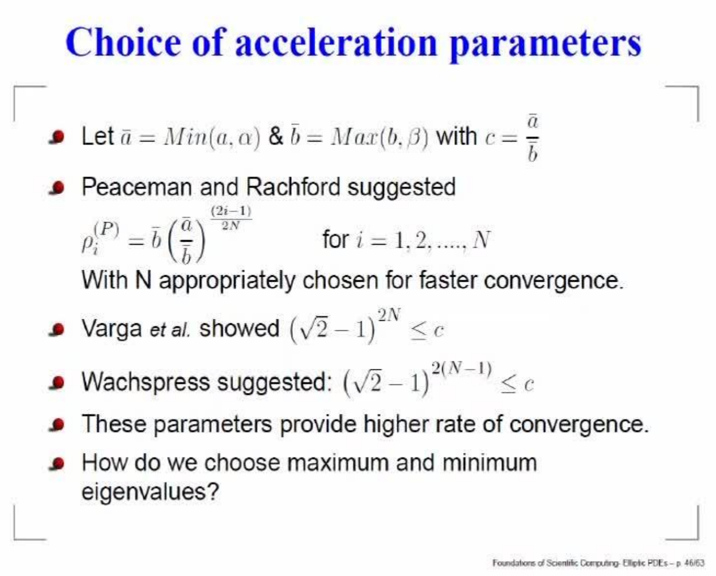

## Choice of acceleration parameters

- Let $\bar{a} = Min(a, \alpha)$ & $\bar{b} = Max(b, \beta)$ with $c = \frac{\bar{a}}{\bar{b}}$
- Peaceman and Rachford suggested
  $\rho_i^{(P)} = \bar{b}\left(\frac{\bar{a}}{\bar{b}}\right)^{\frac{(2i-1)}{2N}}$ for $i = 1, 2, ...., N$
  With N appropriately chosen for faster convergence.
- Varga *et al.* showed $(\sqrt{2}-1)^{2N} \leq c$
- Wachspress suggested: $(\sqrt{2}-1)^{2(N-1)} \leq c$
- These parameters provide higher rate of convergence.
- How do we choose maximum and minimum eigenvalues?

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We did talk about how operationally it is done. You do not need to look at all the Eigen values to reduce the error. What you do, instead, you strategically position some acceleration parameters between the minimum and maximum - a bar and b bar - and then this number of such strategically positioned acceleration parameters are obtained either by Varga’s suggestion or the one that is the given by Wachspress. We presume that these parameters would provide us with very high rate of convergence.

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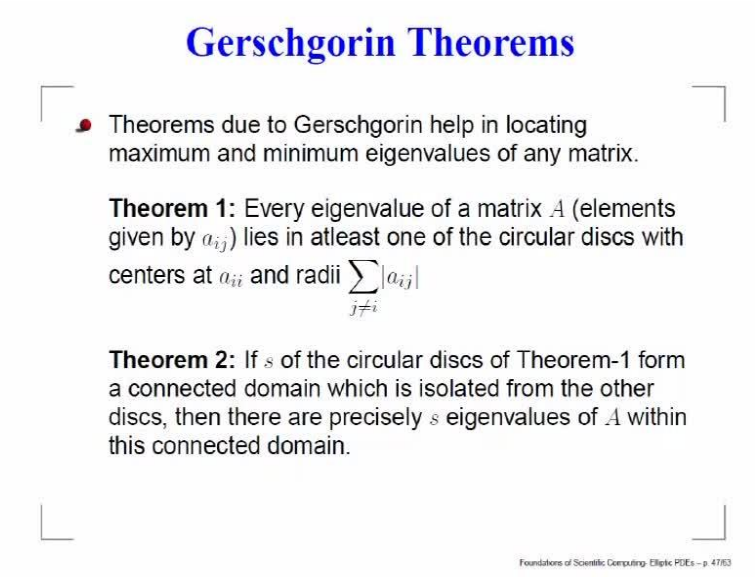

## Gerschgorin Theorems

- Theorems due to Gerschgorin help in locating maximum and minimum eigenvalues of any matrix.

**Theorem 1:** Every eigenvalue of a matrix $A$ (elements given by $a_{ij}$) lies in atleast one of the circular discs with centers at $a_{ii}$ and radii $\sum_{j \neq i} |a_{ij}|$

**Theorem 2:** If $s$ of the circular discs of Theorem-1 form a connected domain which is isolated from the other discs, then there are precisely $s$ eigenvalues of $A$ within this connected domain.

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We did talk about the theorem which happens to be a very handy tool in finding out the maximum and minimum of Eigen value of any matrix and that is due to Gerschgorin theorem. We actually use that Gerschgorin theorem 1 to obtain the disks which are nothing but symptomatic of the entries of the rows or columns of that matrix A.

What you do is that the disc has a center, which is given by the diagonal element and the radius of the disk is given by the sum of the modulus of all the half diagonal term in that row or column and you assemble all this disk - that gives you the kind of idea of the spread of Eigen values of that matrix.

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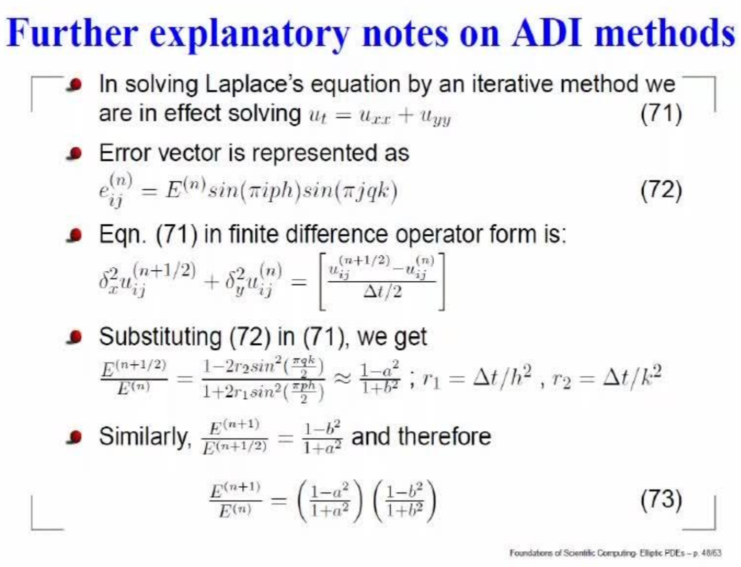

## Further explanatory notes on ADI methods

- In solving Laplace's equation by an iterative method we are in effect solving $u_t = u_{xx} + u_{yy}$ (71)
- Error vector is represented as
  $e_{ij}^{(n)} = E^{(n)} sin(\pi iph) sin(\pi jqk)$ (72)
- Eqn. (71) in finite difference operator form is:
  $\delta_x^2 u_{ij}^{(n+1/2)} + \delta_y^2 u_{ij}^{(n)} = \left[\frac{u_{ij}^{(n+1/2)} - u_{ij}^{(n)}}{\Delta t/2}\right]$
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- Similarly, $\frac{E^{(n+1)}}{E^{(n+1/2)}} = \frac{1-b^2}{1+a^2}$ and therefore

$$\frac{E^{(n+1)}}{E^{(n)}} = \left(\frac{1-a^2}{1+a^2}\right)\left(\frac{1-b^2}{1+b^2}\right) \quad (73)$$

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So, this is how one does and going with the theme of iterative solution of elliptic pde. Suppose, we are trying to solve and suppose say for the sake of understanding, assume that we are working on a Cartesian frame, and we have, let us say non-dimensionalize the problem in such a way that we can take the range of x and y from zero to pi. If we do that, then error vector could be written in terms of the Eigen functions taken from the Fourier modes.

That sin I is the x direction node number, j is the y direction node number and the amplitude is given in terms of capital E with a super script n inside the bracket. Then you realize that the step we followed in ADI method involved taking the solution at nth level.

Then just simply solving it with the h parameter and that would take us the solution to the half step n plus half, and this u t term would be nothing but then u n plus half minus u n by delta t by 2.

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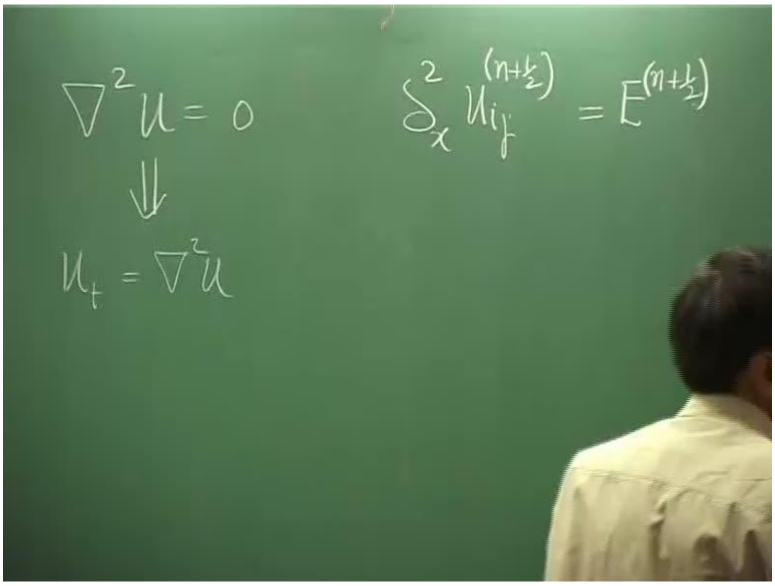

So what we can do is we can really substitute 72 in this equation. So, basically if you do that, what you are going to get? For example, if I write this down, that would be as given by 72. I would write it as E superscript of n plus half, that is the amplitude.

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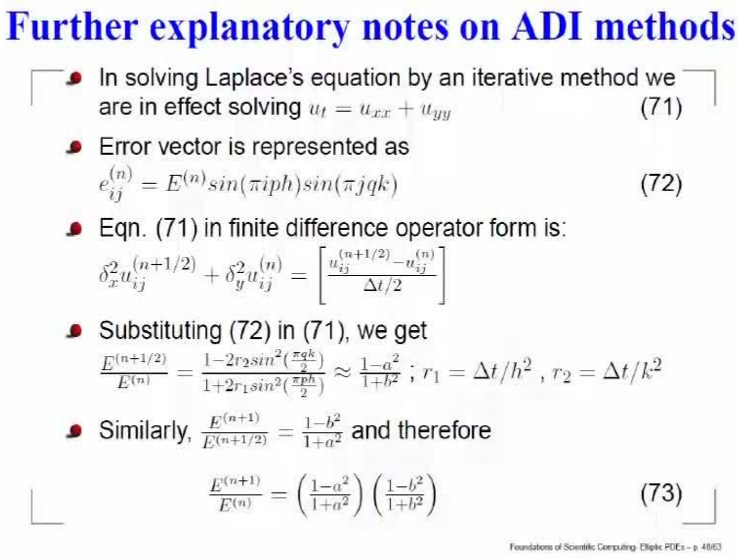

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And when I am doing the x derivative operation, the y dependent of the main frame, so this (( )) sin pi j q k would remain same. So, k is basically delta y, so, k is delta y and the h is delta x.

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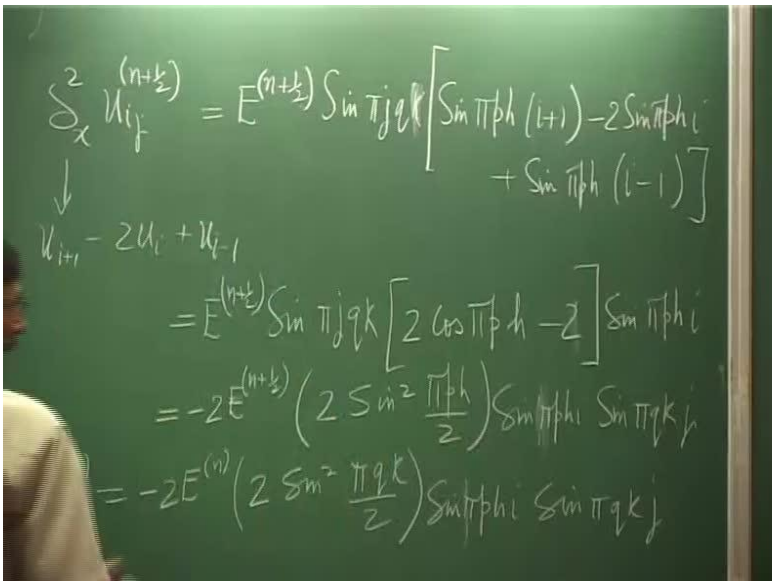

$$\delta_x^2 u_{ij}^{(n+\frac{1}{2})} = E^{(n+\frac{1}{2})} \sin \pi j q k \left[ \sin \pi p h (i+1) - 2 \sin \pi p h i + \sin \pi p h (i-1) \right]$$

$$\downarrow$$

$$u_{i+1} - 2u_i + u_{i-1}$$

$$= E^{(n+\frac{1}{2})} \sin \pi j q k \left[ 2 \cos \pi p h - 2 \right] \sin \pi p h i$$

$$= -2E^{(n+\frac{1}{2})} \left( 2 \sin^2 \frac{\pi p h}{2} \right) \sin \pi p h i \sin \pi q k j$$

$$= -2E^{(n)} \left( 2 \sin^2 \frac{\pi q k}{2} \right) \sin \pi p h i \sin \pi q k j$$

So, if I now write this, so this is essentially the operator is - we know it is something like: u i plus one minus two u i plus u i minus one. So, this is the formal of the function.

So that basically ((no audio 06:36 to 06:42)) of i plus one minus 2 sin of pi p h i plus sin pi p h into i plus one. So what happens then, this is, then would become n plus half and this (( )) as it is pi j q k and this two if I add it up, I will get two sin pi p h i (( )). So, basically I can pull out this factor and then I will get 2 cos pi p h minus one into minus 2 into sin pi p h i and of course, I could write it, somewhat, take that factor out. So, I get two E n plus half and this part, that part (Refer Slide time: 08:02) are synchronistic of what we began with. This will be one minus this; that will give us itself 2. That is why I had put a minus sign here to absorb the change in term. sin square pi p h by 2 into sin pi p h i sin pi q k j and that is this operator. So, this is what we have written it down; so that is this expression.

The second one, similarly, we could just simply write del y square u i j and that is n will give us something similar. So, I will write it as 2 E. Now, this will be at the nth square and y operation would give us 2 sin square pi q k by 2. This same factor.

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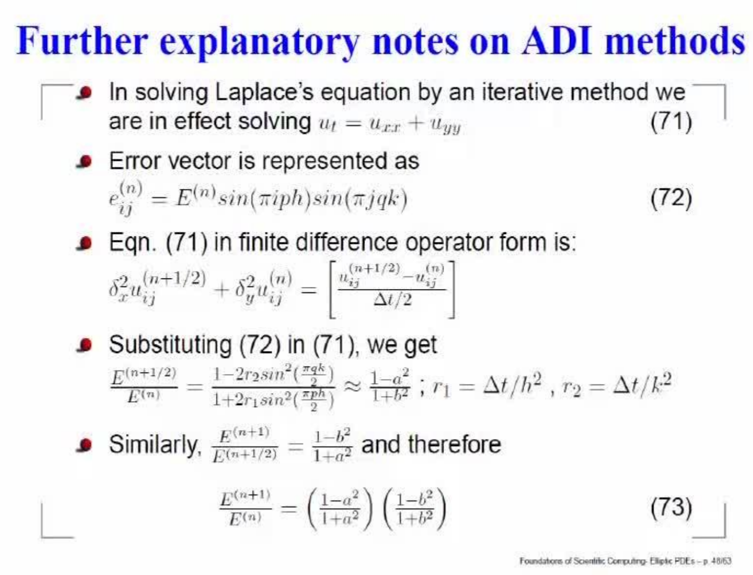

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The right hand side is easy for us to find out. That will be simply e n plus half minus e n like this and you can open it up and simplify it. This is what you are going to get.

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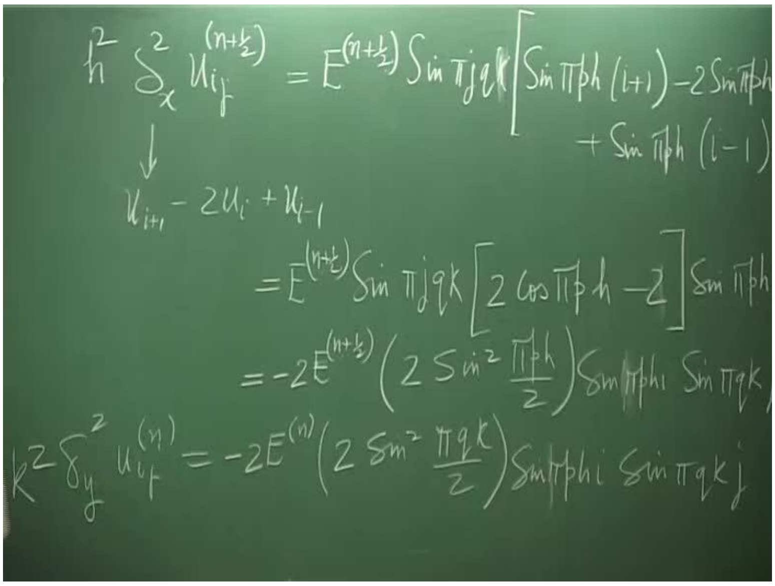

$$h^2 \delta_x^2 u_{ij}^{(n+\frac{1}{2})} = E^{(n+\frac{1}{2})} \sin \pi j q k \left[ \sin \pi p h (i+1) - 2 \sin \pi p h \right.$$
$$+ \sin \pi p h (i-1)$$
$$\downarrow$$
$$u_{i+1} - 2u_i + u_{i-1}$$
$$= E^{(n+\frac{1}{2})} \sin \pi j q k \left[ 2 \cos \pi p h - 2 \right] \sin \pi p h$$
$$= -2E^{(n+\frac{1}{2})} \left( 2 \sin^2 \frac{\pi p h}{2} \right) \sin \pi p h i \sin \pi q k$$
$$k^2 \delta_y^2 u_{ij}^{(n)} = -2E^{(n)} \left( 2 \sin^2 \frac{\pi q k}{2} \right) \sin \pi p h i \sin \pi q k j$$

So from here, we, of course, I have not written down that part; I have written it with h square. So, and this is which k square I have just written. So, I have just taken down that it's divided by (( )), so this is divided by h square (( )).

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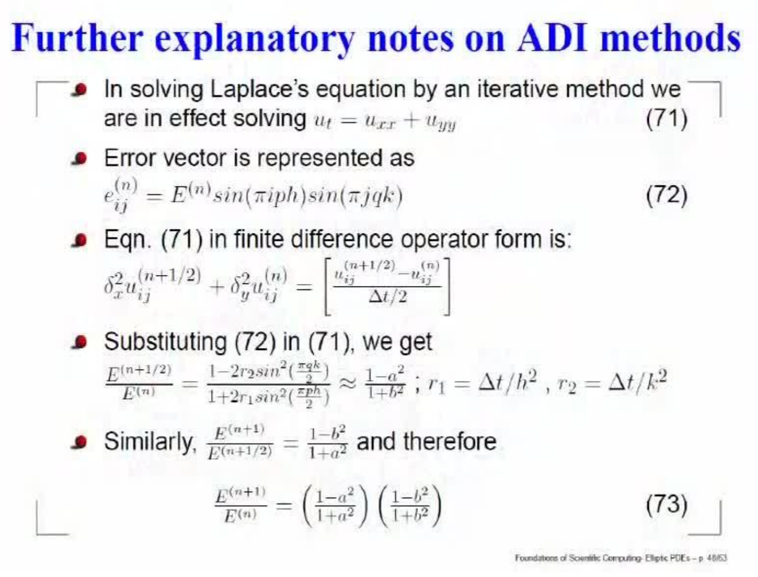

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So I will just simply write it like this, h square into this and k square into this (Refer Slide Time: 10:14) and I will substitute in this equation, simplify, this is what you are going to get. So, basically r 1 and r 2 are something like our Peclect number in the x and y direction.

That is what we are seeing that r 1 is the Peclect number in x direction and r 2 is the Peclect number in the y direction and you find that the amplitude, by which, the amplitude changes in the half step is given by this factor: 1 minus 2 r 2 sin squared by q k by two divided by one plus two r one times sin square by pi p h by two.

Now, this has a form of this fact - 1 minus a square divided by 1 minus 1 plus b square and since a and b are this similar, so there is no way you can make any observation about the value.

It could be less than 1; it could be greater than 1; it could be anything. This factor that we have written here, it could be anything. However, in the next half step, you would be doing something like this.

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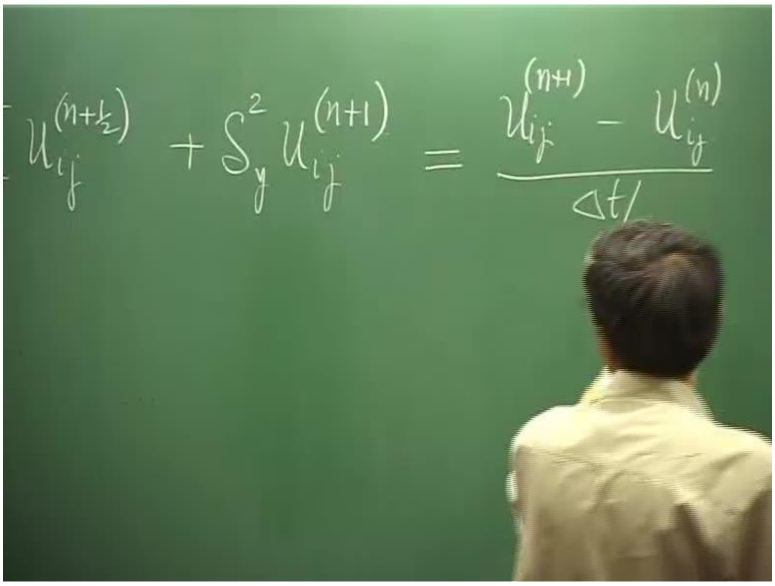

$$u_{ij}^{(n+\frac{1}{2})} + \delta_y^2 u_{ij}^{(n+1)} = \frac{u_{ij}^{(n+1)} - u_{ij}^{(n)}}{\Delta t/}$$

Delta x square u i j, we will take it as the previous step, but now we are going to do the next half step, that will be taking us to an end of the step, that is what we are getting. This side we get u i j n plus 1 minus u i j n and that will be divided by delta t by 2.

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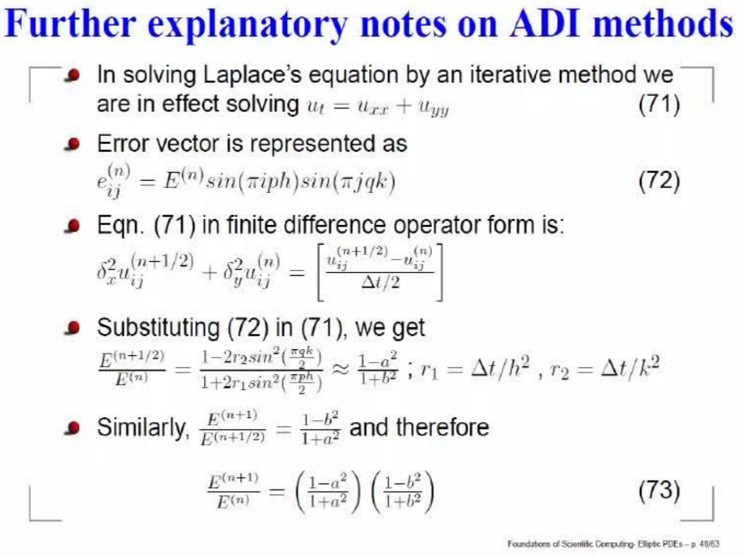

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So, that is the second half step that we have not written it down, but that is required. You go through the same kind of substitution of the error vector that you have provided here and do the calculation.

You would see with the amplitude ratio, in this half step works out to be on the strategical in nature. It looks like 1 minus b square divided by 1 plus a square. So, over all actually that total step would take you from e n to e n plus 1.

So, simply multiply these 2 factors and you get this. This is exactly like what we say about the similarity transformation that Eigen value does not change and in the end what you get that individually this factors are not guaranteed to be less than 1 but the compound step, the product of the two gives you this factor - 1 minus a square divided by 1 plus a square and this second is similarly 1 minus b square.

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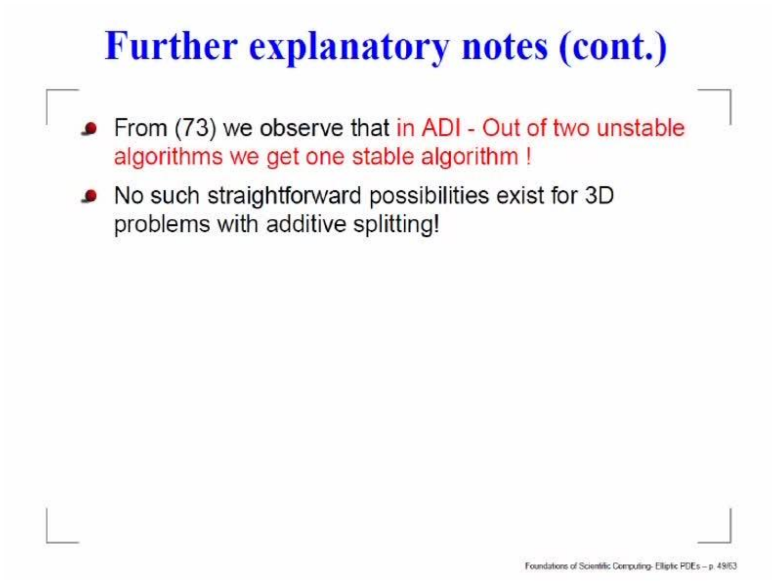

Further explanatory notes (cont.)

- From (73) we observe that in ADI - Out of two unstable algorithms we get one stable algorithm !
- No such straightforward possibilities exist for 3D problems with additive splitting!

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So, basically this is what we (( )) conclude that in ADI what we are doing that individually the steps could be unstable, like what we just now explained. However, when we put them together, we get a stable algorithm. So, that is the key to success of ADI method.

Now this was possible because we are looking at a 2 dimensional primer and the operator that we had, we could split it additively and we could do those things; so, symmetrically the error cancelled out into associative method.

For 3 D problem, we really can have the same version of straight forward applying this procedure; you may have to do another additional manipulation. So, additive splitting does not really work out as easily as it was for 2 D.

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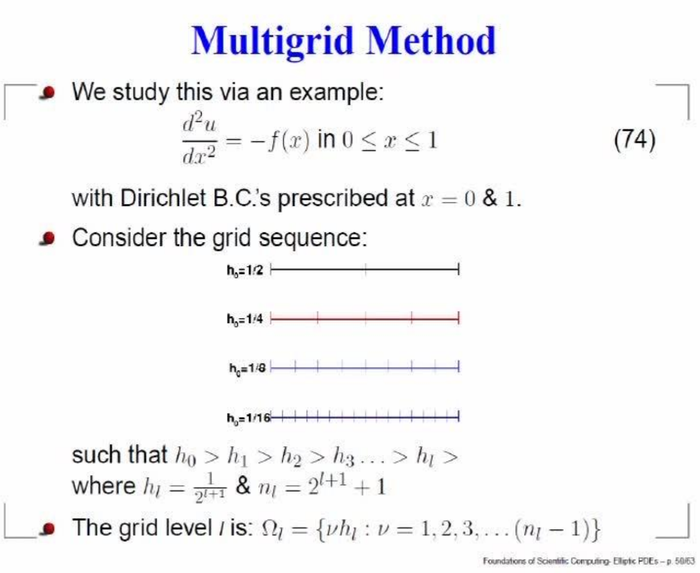

## Multigrid Method

- We study this via an example:

$$\frac{d^2u}{dx^2} = -f(x) \text{ in } 0 \leq x \leq 1 \qquad (74)$$

with Dirichlet B.C.'s prescribed at $x = 0$ & $1$.

- Consider the grid sequence:

$h_0=1/2$

$h_0=1/4$

$h_0=1/8$

$h_0=1/16$

such that $h_0 > h_1 > h_2 > h_3 \ldots > h_l >$
where $h_l = \frac{1}{2^{l+1}}$ & $n_l = 2^{l+1} + 1$

- The grid level *l* is: $\Omega_l = \{\nu h_l : \nu = 1, 2, 3, \ldots (n_l - 1)\}$

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That is one thing that we observed in our ADI and there is another thing that you have noticed that, when we work with ADI method, it shows those acceleration parameters and once we choose those acceleration parameters, we state with it in the same grid.

We kept on working with the same grid, but we kept on changing the acceleration parameter to reduce the error. Now, there is a parallel work that was done in Russia in the late 50's by a mathematician named Federinko.

Federinko actually smart enough to notice that when we are doing all this - not I mean iterations - like if you recall, we worked out (( )) expressions for g 1 g 2 g 3 g 4 g 5, what did we find? That as we went wrong, it becomes to basically worst and worst.

In the fifth step, actually the ratios at over shot by 1. So, Federinko's observation was that, if we keep doing any kind of iterative operation, what happens is the first few steps are pretty much optimistic step; it reduces the error very rapidly. Then it stops, why does it stop? If you now connect that point d i vector, you would see that as you went along we kept on working on different Eigen values.

Your iteration number increases means your Eigen values are going up in the list. So, what happens? Initially you work with the error which have much larger wavelength, they are controlled easily, g 2 also is the next harmonic and g 3 will be with the next harmonic and so on, so forth, but by the time we have come to the fifth harmonic, things have started misbehaving; that is exactly what you notice. So the idea was that it is probably, then not worthwhile to keep on struggling with the same method which is know is doomed to not lead any good dividend.

It is going to get worser and worser. So, how do we improve? So, this observation was that the errors those which reduced had some relationship with the size of the domain and this wave we discretized. Those errors which are of the size of the domain, they decrease faster; that is what we saw in g 1 g 2 etcetera.

But the moment we keep on going into higher harmonics, there it was no more, good. So, this was noted very smartly by Federinko and he suggested what is now called as Multigrid Method.

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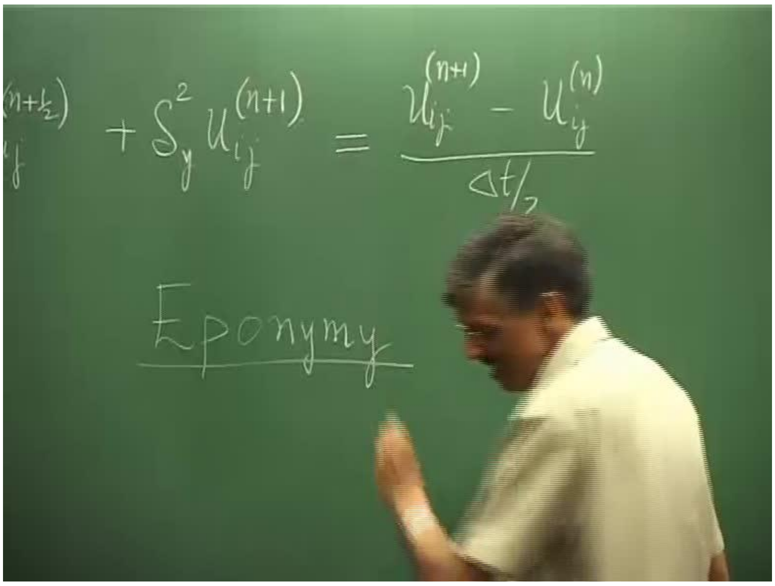
$+\delta_y^2 u_{ij}^{(n+1)} = \frac{u_{ij}^{(n+1)} - u_{ij}^{(n)}}{\Delta t/2}$
Eponymy

Now, I have often told you about this, well, - I thought you never note it down - Eponymy. (( )) somebody does the work, somebody gets the credit. Federinko did it and wrote it in his thesis; nobody knew about it. Then there was a smart American mathematician Archie Brand, he comes out in 80s. He notices all this, he gets a similar idea.

He says we have a new method called multigrid method. So, in the multigrid method the idea is as follows; that is, what we are trying to show you by an example. Let us say, we are trying to solve a trivial problem, of course, we can analytically solve it.

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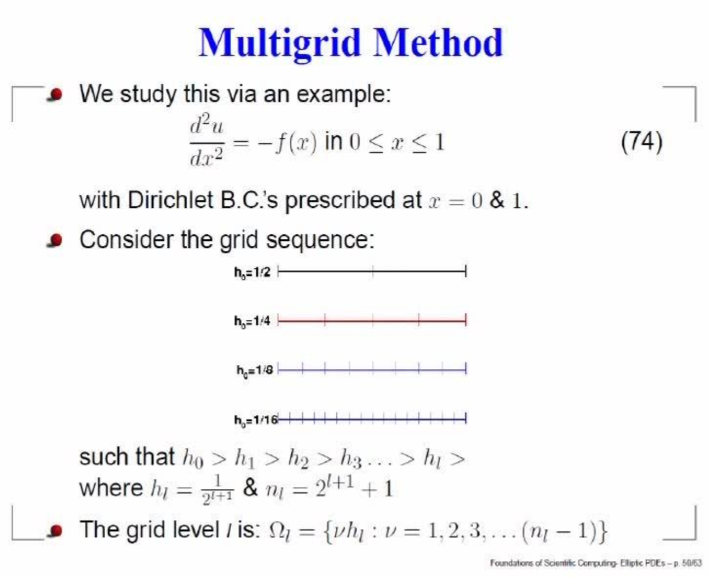

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  where $h_l = \frac{1}{2^{l+1}}$ & $n_l = 2^{l+1} + 1$
- The grid level *l* is: $\Omega_l = \{\nu h_l : \nu = 1, 2, 3, \ldots (n_l - 1)\}$

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But I am just using a just gem stick how it works. What you do is, suppose I have to solve this oddly know about an initial problem, in a smaller, in a domain, non-dimensional domain with a 0 1.

And that is it. We have Dirichlet condition provided at the end of the domain. Then what we do is we keep on working on different types of grids because every time I work on a particular grid, I know which are the error components, which are going to be damped.

So, the moment I exhaust that possibility, I go to a different grid and there I will start working on a different wavelength of error component. What is advantageous in this sense? It is advantageous in the sense, that if I am working on a coarser grid - of course I am doing less work, lesser number of points - but the moment I decide to like in ADI, I kept on working on the finest possible grid, but this is where the distinction came in multigrid method.

That you were start off with a fine grid, but then you can keep on migrating to coarser and coarser grid and then you can go through this process in a sequence, in an iterative sequence. Every time you exhaust, the possibility of error being reduced in that particular grid, then you go to another grid.

That is have the name multigrid comes from. You worked with multiple grids. So, of course, we know it already is the story that Federinko did the work, but it flourished in eighties with a lot of effort from Archie Brand and many other mathematicians followed their work.

So, what we do actually in lecture? We have the domains between 0 and 1. So, the least possible discretization that we could do is this right, I mean one term and the wave. So then, the spacing is 1 by 2.

I think there we did mistake; this should be h 1. So this is how the initial grid h 0 is half, that spacing is half and half. Then we review this spacing that factor of 2 and then I get the spacing will be one forth.

Then I go to the third one that will be one eight, one sixteenth and so on, so forth. So this is the way, - there is a typing mistake, so this should be h 1 h 2 h 3 - so, we take this sequence of grid such that the spacing has this hierarchy.

And the numbers of points, of course, have been taken to be given by h l. Let us say lth level grid, we are looking at the number of points is given by 2 to the power l plus 1 plus 1 that is what we are getting - h 0 h 1, you can see that and then, of course, the spacing is not over 2 to the power l plus 1.

So, at any lth level, we can talk about a grid given by the set. We will write it as omega l would be nothing but that nu times h l nu is the counter of the parts. So, if I have the spacing as h l, I will have h l 2 h l 3 h l all the way up to n l minus 1 into h l. So this is the idea. Now, let us understand why we need to work on multiple grids. So, what I do is I just simply discretize that equation that we started off in solving.

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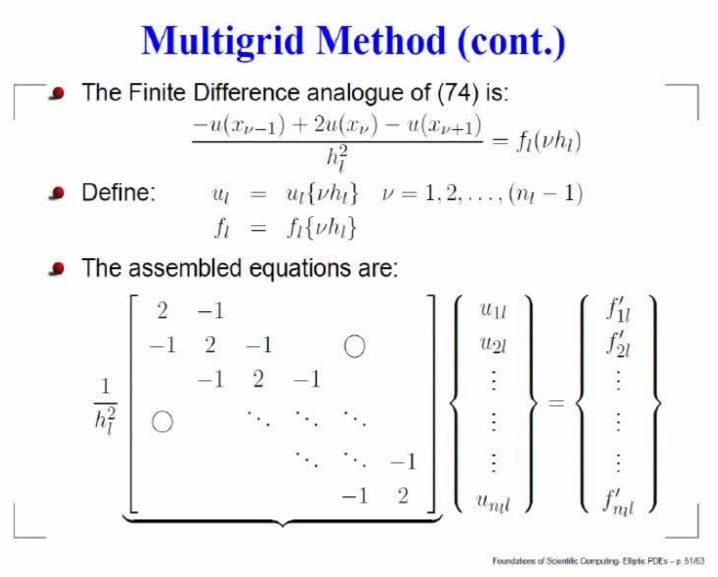

## Multigrid Method (cont.)

- The Finite Difference analogue of (74) is:

$$\frac{-u(x_{\nu-1}) + 2u(x_\nu) - u(x_{\nu+1})}{h_l^2} = f_l(\nu h_l)$$

- Define:

$$\begin{aligned} u_l &= u_l\{\nu h_l\} \quad \nu = 1, 2, \ldots, (n_l - 1) \\ f_l &= f_l\{\nu h_l\} \end{aligned}$$

- The assembled equations are:

$$\frac{1}{h_l^2} \underbrace{\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & \bigcirc & \\ & -1 & 2 & -1 & & \\ \bigcirc & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}}_{} \left\{ \begin{matrix} u_{1l} \\ u_{2l} \\ \vdots \\ \vdots \\ \vdots \\ u_{n_l l} \end{matrix} \right\} = \left\{ \begin{matrix} f'_{1l} \\ f'_{2l} \\ \vdots \\ \vdots \\ \vdots \\ f'_{n_l l} \end{matrix} \right\}$$

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So, what we get is basically a second derivative of u v plus with respect to x; so we write it like this. Now if I define this, the grids at the lth level, let us say we are working on the lth grid. So, that is why we are using the subscript l there and is the forcing function l. Then this is the discrete equation would purposely chose because it is easy to follow, its gives you a tridiagonal matrix, you can strictly diagonal dominant property.

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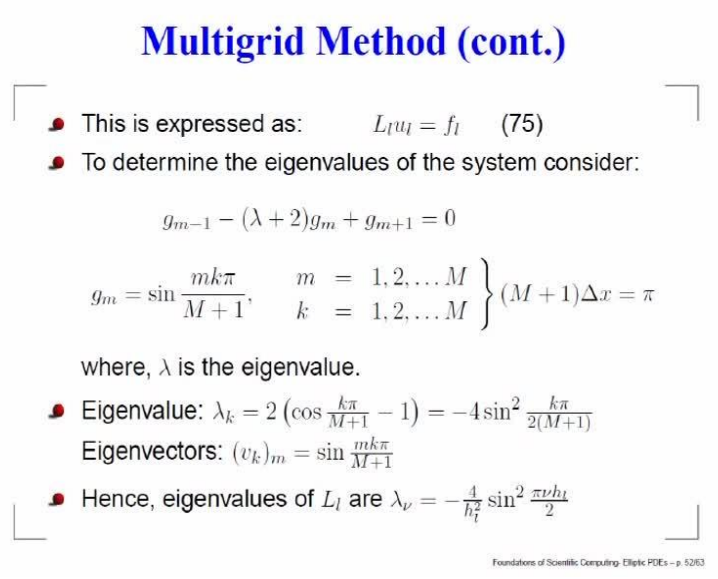

## Multigrid Method (cont.)

- This is expressed as: $L_l u_l = f_l$ (75)
- To determine the eigenvalues of the system consider:

$$g_{m-1} - (\lambda + 2) g_m + g_{m+1} = 0$$

$$g_m = \sin \frac{mk\pi}{M+1}, \quad \left. \begin{aligned} m &= 1, 2, \ldots M \\ k &= 1, 2, \ldots M \end{aligned} \right\} (M+1)\Delta x = \pi$$

where, $\lambda$ is the eigenvalue.

- Eigenvalue: $\lambda_k = 2\left(\cos \frac{k\pi}{M+1} - 1\right) = -4 \sin^2 \frac{k\pi}{2(M+1)}$

  Eigenvectors: $(v_k)_m = \sin \frac{mk\pi}{M+1}$
- Hence, eigenvalues of $L_l$ are $\lambda_\nu = -\frac{4}{h_l^2} \sin^2 \frac{\pi \nu h_l}{2}$

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And what we do is try to understand what it is. So, basically the discrete equation that we would have could be something linear operator, operating on the u at the lth grid level, forced by this forcing term f l.

And then, if I write, if I look at any one line in this discrete equation, so, if I write the corresponding Eigen vectors of g, so, this is nothing but minus g. So, the diagonal term will be g n. So, it is a 2 g n and then minus g m minus 1 and this will be minus g m plus 1.

And Since we are trying to find out the Eigen value of this matrix, so, that would be nothing but this Eigen value equation - k x equal to lambda x, that is what we are looking here. So that is what we are written - a x equal to lambda x and then we wrote any one of the line that would be this.

Now you understand the diagonal with M. So, we have a super diagonal plus 1, sub diagonal plus 1 and the diagonal itself is minus 2 because we have on the right hand side minus lambda into j; so, put it on this side you will get this. We have a periodic problem that is all we say.

So if I am (( )) dirichlet condition problem, so what I could do is, I could write down any arbitrary solution in terms of a complete set and let that complete set be a fourier set.

So, if I do that, I can take this as the basis function, sin of this. So, now I am actually looking at the number of points in m direction is given by capital M. So, that is how I am doing; I am dividing phi by a cross one and then, this 2 indices m and k will identify the modes. So, we are looking at the mth row; so that is why that m is there and k is the one that we can even for that single row, I can have all possible Eigen value combinations for k ranging from 1 to n.

Now, you can substitute this and work out; you will find the Eigen value is exactly like almost similar to what we have done here; we write down, substitute here, and simplify, you find out that the Eigen values are this for this equation.

The Eigen vectors, of course, is simply chosen Eigen vector that we have. So, if I now look at this equation, where we were look into yourself that had to one over h m square outside also. That is what the Eigen values of the l m matrix is given by minus 4 by h square into sine square pi nu h l by 2.

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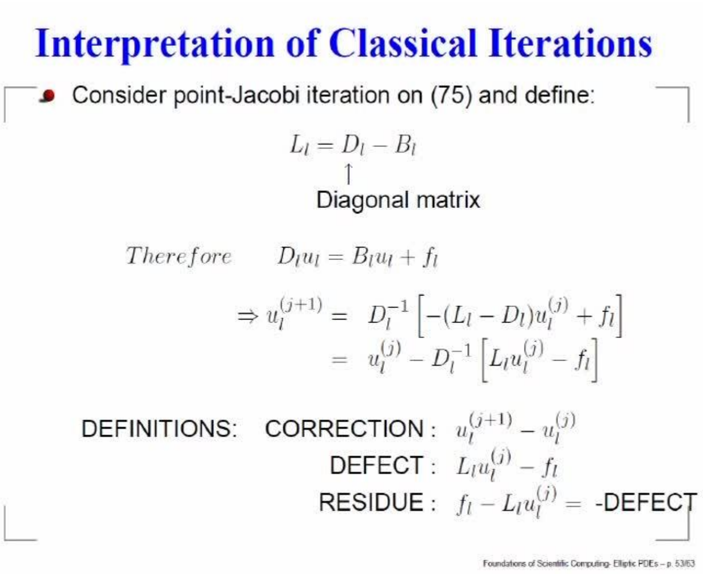

## Interpretation of Classical Iterations

- Consider point-Jacobi iteration on (75) and define:

$$L_l = \underset{\uparrow}{D_l} - B_l$$

Diagonal matrix

$$\mathit{Therefore} \qquad D_l u_l = B_l u_l + f_l$$

$$\begin{aligned} \Rightarrow u_l^{(j+1)} &= D_l^{-1}\left[-(L_l - D_l)u_l^{(j)} + f_l\right] \\ &= u_l^{(j)} - D_l^{-1}\left[L_l u_l^{(j)} - f_l\right] \end{aligned}$$

DEFINITIONS: CORRECTION : $u_l^{(j+1)} - u_l^{(j)}$

DEFECT : $L_l u_l^{(j)} - f_l$

RESIDUE : $f_l - L_l u_l^{(j)} =$ -DEFECT

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Now, if I think of very simple, let us say Jacobi diagonal iteration, then what I do usually is, I write down the l operator in terms of a diagonal part and anything that is, that will be diagonal entry is written as u of l.

So, if I substitute this into l l u l equal to f l, this is what we get. So, in the actual iterative sequence, what we do is, basically we put the left hand side with the new level j plus one th level of iteration, and this right hand side, we keep it as the previous level j. Then, the Jacobi iteration will imply that I have taken it as simply D inverse.

The diagonal entry, that is what we have done also something wants to example of c by d matrix because that is the way we do. So, Jacobi iteration would be nothing but k equal to diagonal part of and using that, if you find that m matrix we talked about. So, you actually write it down - b l inverse and b l itself is nothing but minus of l l into l l minus d l. So, if i carry out this operation d l inverse inside, I will see that this is the way actually point Jacobi iteration works.

That, whatever we have at the new iteration value is nothing but the old iteration level value - minus d l inverse. So, this is your l inverse into the defects, this quantity, we call it as defect.

We have also defined something called residue which is nothing but minus of defect .So it is all that. So, basically what we are seeing that at every row we end up obtaining a correction. Correction is u j plus 1 minus u j.

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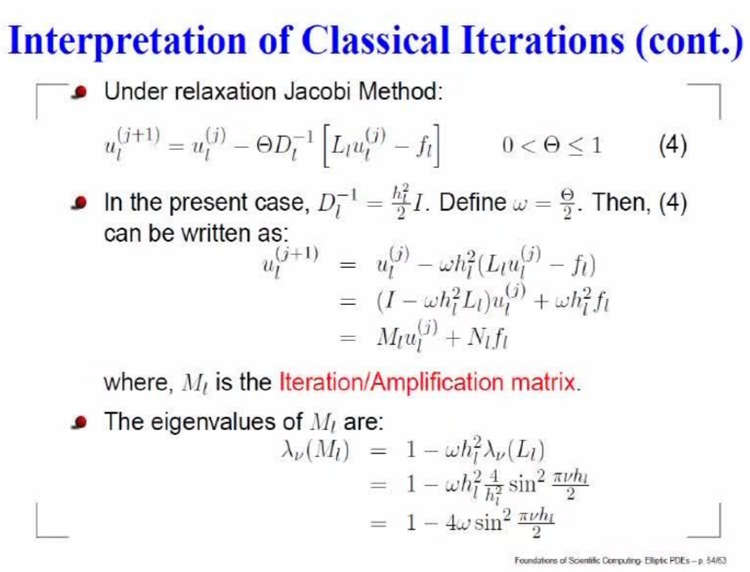

## Interpretation of Classical Iterations (cont.)

- Under relaxation Jacobi Method:

$$u_l^{(j+1)} = u_l^{(j)} - \Theta D_l^{-1}\left[L_l u_l^{(j)} - f_l\right] \qquad 0 < \Theta \leq 1 \qquad (4)$$

- In the present case, $D_l^{-1} = \frac{h_l^2}{2}I$. Define $\omega = \frac{\Theta}{2}$. Then, (4) can be written as:

$$\begin{aligned} u_l^{(j+1)} &= u_l^{(j)} - \omega h_l^2(L_l u_l^{(j)} - f_l) \\ &= (I - \omega h_l^2 L_l)u_l^{(j)} + \omega h_l^2 f_l \\ &= M_l u_l^{(j)} + N_l f_l \end{aligned}$$

  where, $M_l$ is the Iteration/Amplification matrix.

- The eigenvalues of $M_l$ are:

$$\begin{aligned} \lambda_\nu(M_l) &= 1 - \omega h_l^2 \lambda_\nu(L_l) \\ &= 1 - \omega h_l^2 \frac{4}{h_l^2}\sin^2\frac{\pi\nu h_l}{2} \\ &= 1 - 4\omega \sin^2\frac{\pi\nu h_l}{2} \end{aligned}$$

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That is given by the defect given by the choice of diagonal matrix; that is how we do. This is something very known to us. So, what we also have seen that sometimes, we could actually increase the convergence rate by under relaxation into using this factor capital theta here. Since it is under relaxation, we keep it between 0 and 1. That is how definition of under relaxation of this class of methods.

And for the problem, that we have chosen D inverse is nothing but h l square by 2 into i h s. That is what we do because the diagonal entry is minus 2. So, if I take d inverse that will be minus 2 by h m square already is there. So, that goes in the numerator; so, I get d inverse as h l square by 2 into Eigen matrix.

And so this has a half factor here; so theta by 2 If I define as omega, so I will write down this under relaxed Jacobi method, as the new iterate is equal to the old iterate minus omega h l square into this f l.

So, that could be written by in this particular (( )) M l by this row old depict. So, M l is nothing but our iterative or the amplification matrix that you have already defined. So, how well this method is going to converge? **It** will depend upon the Eigen spectrum of this m matrix.

And M matrix is given by i minus this. So we have already found out the Eigen value of l l matrix; so, it is not at all difficult to find out a l matrix, Eigen value, that will be nothing but one coming from this i matrix minus this one omega h l square and the lambda of the l l matrix, Which itself is four by h l square sin squared pi v h l cube. So, basically what we are finding **is** the Eigen values are given in terms of this and this we have often commenting that we choose theta capital theta or this case omega.

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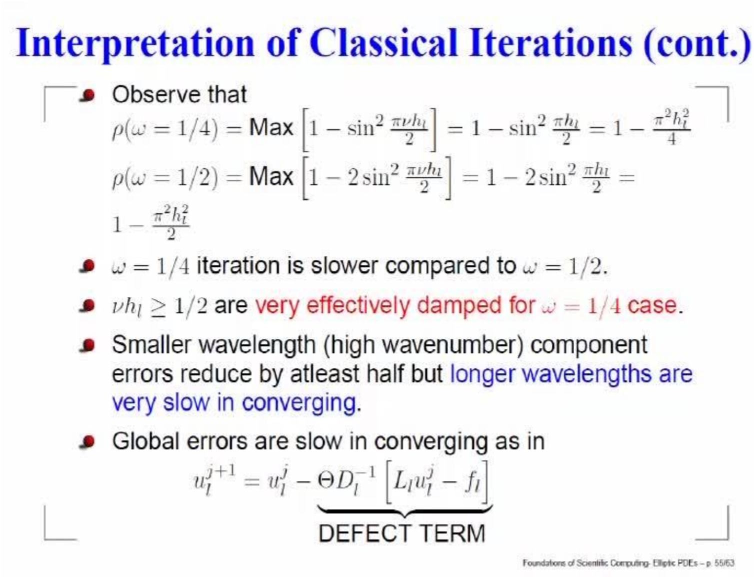

## Interpretation of Classical Iterations (cont.)

- Observe that
  $\rho(\omega = 1/4) = \text{Max}\left[1 - \sin^2 \frac{\pi\nu h_l}{2}\right] = 1 - \sin^2 \frac{\pi h_l}{2} = 1 - \frac{\pi^2 h_l^2}{4}$
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So that, to hasten our convergence and so if I choose, let us say theta is equal to half, so theta equal to half would give me omega as 1 4 and if I take theta equal to one means no relaxation at all, that will give me omega equal to half because that is theta by 2, am I right?

So, what happens is the Eigen value of the matrix, the spectral radius as we called by rho. So rho for theta equal to half will be nothing but maximum of this sequence 1 minus sine square pi nu h l by 2.

So, what will be the maximum? The maximum will be, when this is going to be the least sin square part, will be least, that will correspond to nu equal to 1. So we get this 1 minus sin square by h l by 2 and if h l is small, I could see that this is the spectral radius.

So this is what we have commented upon earlier also that the under relaxed Jacobi method. Actually, the spectral radius - where you did h l square? So, what does it mean? That it means that you can solve the problem with a coarser grid.

The moment you go over to a finite grid, what happens? It shall comes down and a row comes closer to one that means your error reduction ability comes down. So if you work on a finite grid, you will find it much tougher to convert.

So, this so happens that people have historically found out. Initially you know with the limited computing ability, we used to take 20 30 points and show some good results and more than if you have a better computer, is instead of taking 20 30 points ,you take 200 300 points.

And immediately you see that in the convergence history as become very worst, so that comes out from this observation, we can see because the spectral radius is directly proportional to h l square and if h l becomes smaller, so, h l square becomes even smaller.

So we have spectral radius is very close to one and we do not get any benefit even for this modern simple equation that you have and whereas, if you look at the case where you do not have any under relaxation, if it is 1 minus 2 sin square pi h l h.

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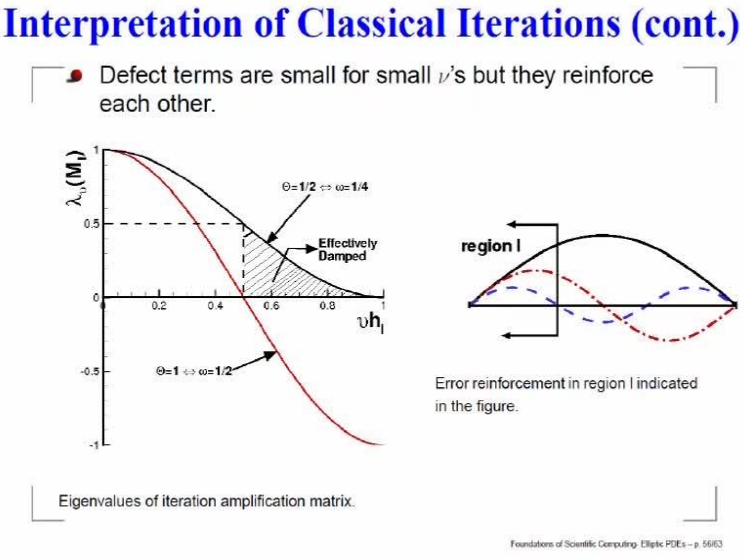

## Interpretation of Classical Iterations (cont.)

- Defect terms are small for small $\nu$'s but they reinforce each other.

Error reinforcement in region I indicated in the figure.

Eigenvalues of iteration amplification matrix.

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So, if I, instead of reading all that lets go over to this next page and then we will see what we are talking about, like when we take theta equal to half, omega equal to one fourth, my Eigen values are given like this.

I could plot on this axis nu times h l. So, if nu times h l is zero, so then I will **have** maximum case sin square 0. So, I will have the Eigen value as 1 but as the Eigen value number index increases which keeps on the changing it by its value and we have seen its 1 minus sin square nu pi h l by 2. So, when I am going to **put** 1 here, nu h equal to one; so, that will be pi by 2; so, 1 minus 1 will be 0.

So what happens, that that may show here. Say, if I am working on this particular level of grid, then the different Eigen values corresponding to the error behave differently because of the associated Eigen values that I have seen. The ones those which are small, they do not change very much.

Whereas the Eigen values, which are to the right extremes, they are going to be damped very strongly, whether coming close to 0. So what happens is, I could mark out a space where I say, this part, the shaded part is the area (Refer Slide Time: 34:38) where I get a very strong damping starting form half to 0.

So, every time I do that, my error gets magnified by this Eigen value. So, this part struggles; it does not change very much, but this part attenuates at a quite rapid rate. What does the large value of nu h l imply in terms of wavelength?

These are very small wavelength. So, what happens is, whenever we do this classical iteration, in the initial few steps we keep on removing this small-scale error very happily because of this nature.

However, if you look at the case where you actually did not do any under relaxation that corresponding lambda would be this, given by this red line. Why it is 1 minus 2 sin square pi nu h l by 2?

So what will be the range? It will go from plus 1 to minus 1 and that is what you have seen. So, if you do not under relax, then what you have seen that it is only some intermediate stage where error is reduced. Even this side, you know, the error would not be reduced, why because it is close to minus 1. See, your ability to reduce error will depend on how far away you are from one close to 0 in magnitude.

So, what happens is that, this is the funny case that you are not under relaxing. So, in **this** process what we do is, we can control errors which have large wavelength size of the domain given by this side.

At the same time we are not able to control the errors. (( )) which are the slopes? I have given domain and I have given grid; it will depend on the grid spacing. We have already talked about Nyquist limit. So, we know that moment I choose h l which is the maximum wave number that I can control is given by the Nyquist limit.

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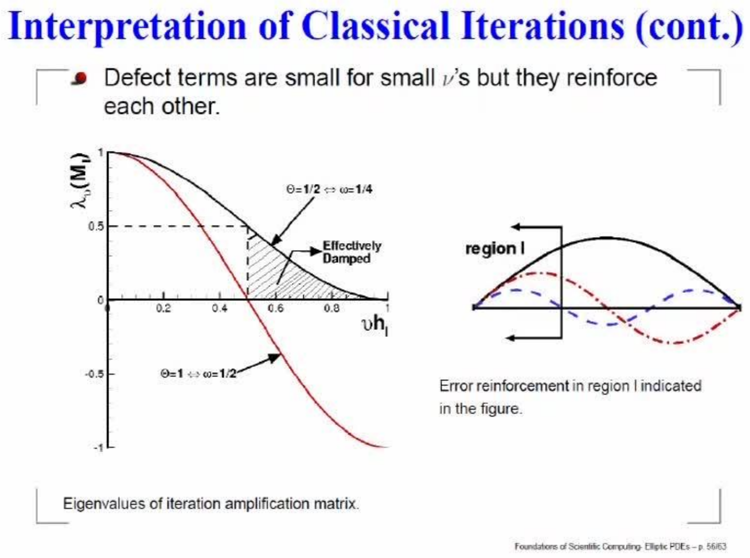

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So, if I do not do this under relaxation, I end up reducing error in the middle by high wave number and low wave number error remain sort of reconstruction they does refuse to change and whether you are under relaxing or not.

In both the cases you see, what is the troublesome area?  Troublesome area is this end; this end corresponds to the largest wavelength of a problem that you have; the domain such that is what this corresponds to.

So, that variation, so, if I talk about my domain by this horizontal line, those ones, those Eigen values here corresponds to I have shown you by three possibilities - the black line corresponds to the largest wavelength that you can handle with a grid, that is half the wavelength. The next one, of course, is the red line that is a next harmonic, that stands the whole domain by a single wave and the blue line is something like a sub harmonic, right - it’s 3 by 2.

Where, I could also have drawn the second harmonic and so and so forth. Now, if you look at the error components drawn as what we have done here, now, what we are seeing that, in this part you note this black error which is a positive could be cancelled by this red component, but that is not so in this range to the region I have shown - region one to the left of that vertical line is where all this error components are conspiring together to add, they are all edema.

So, it is a very interesting situation that why we do not want to do this kind of normal mode analysis and feel good. I am taking care of one harmonic at a time but that is not the way error behaves, error appears together.

So, if I would have taken any arbitrary error after say few (( )) of iteration, what I find that in this region, in this part all of them add up, so, I will get a kind of the errors picking up in this region and the other part, it has some mutual cancellations and it does that.

So, even though we are talking about a linear problem, even though we are talking about a linear problem where I would legitimately demand that I could superpose the solution, but unfortunately the error shows that its distribution, spatial distribution changes because of this kind of a property.

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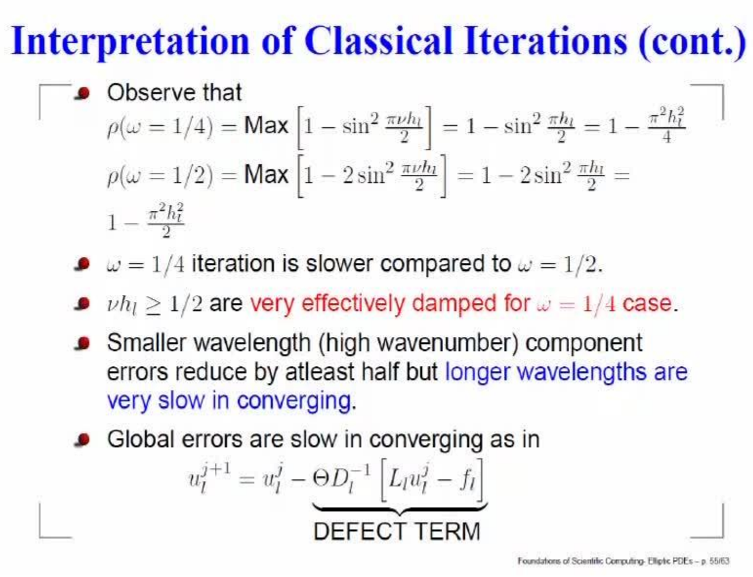

## Interpretation of Classical Iterations (cont.)

- Observe that
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So, error reinforcement in region one is very much of a possibility. So what we confine is that we can draw this conclusion that we have written, but compared to omega equal to one fourth, omega equal to half has a better convergence.

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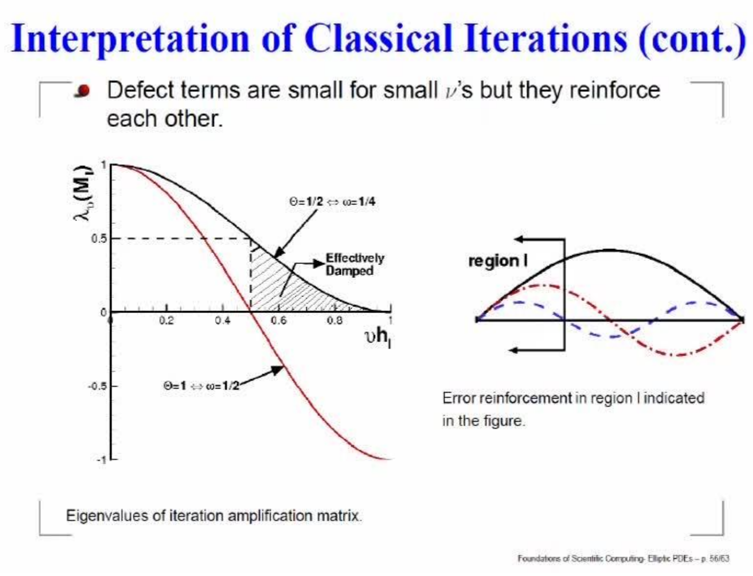

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Error reinforcement in region I indicated in the figure.

Eigenvalues of iteration amplification matrix.

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Why did I said that, scroll, go back to the figure, what is the most difficult thing is to control? The most difficult thing to control is in this region; in this region you can see that red is better than black, is not it and so red is what that is without any under relaxation.

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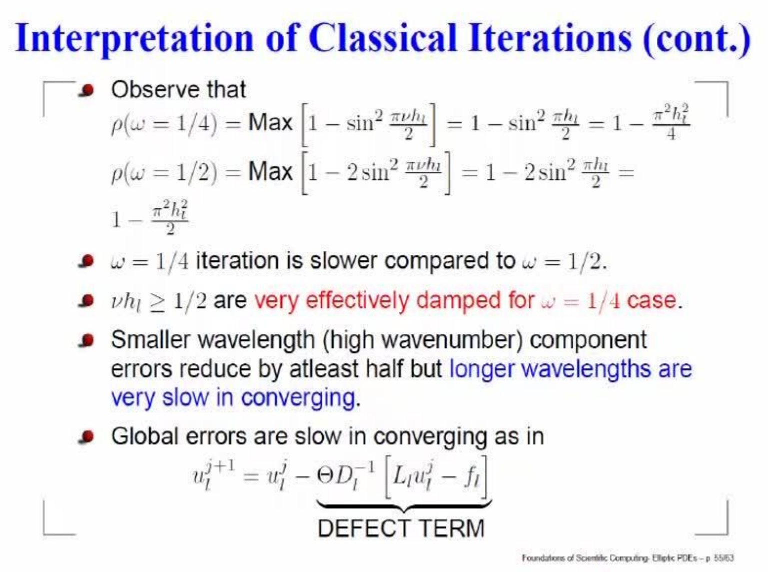

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That is what we say that when we look at the convergence history, we might see the solution without any under relaxation. To do better than this, they want than with this.

Whereas, we are also noticed that with the under relaxation what we are able to do - we could remove the error component which was at the highest wave number h. So, what happens is, this is the cracks of the second point that for nu h l greater than equal to half.

The error is very effectively done for the under relaxation case. So, that is what brings us to the dilemma, you know, again we cannot afford to stay in one grid and then keep on working with different theta.

That is what it means; see the difference between these two, tells us about how different theta does and that is exactly what you are doing in ADI. In ADI, we are not keeping a fixed theta, we are those are acceleration parameters.

We kept on changing them and we are trying to see what happens; and this is what, I all this, as I jokingly say it is a poor man’s blanket, I mean you try to control the error on the high wave number. If a low wave number is not controlled and if you try to control your low wave number, you cannot do it if you decide to stay in the same grid.

Now, this is what we learn. Then, that is smaller wavelength, that is high wave length or component error reduces by at least half, but longer wavelength are very slow in converging.

And this is a practical experience. If you go ahead and try to solve the elliptic equation, we will always come back after some attempt and say initially things look very good, error reduced, but then, after sometime it stopped changing at all and that is the story; that is what we narrated there.

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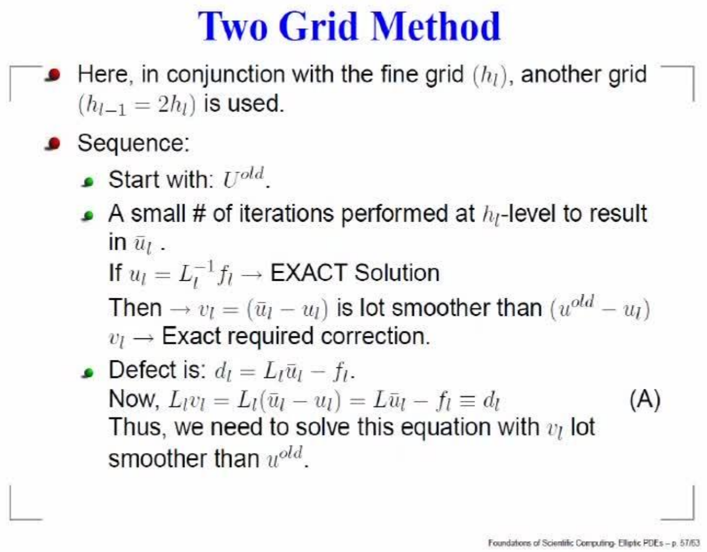

## Two Grid Method

- Here, in conjunction with the fine grid $(h_l)$, another grid $(h_{l-1} = 2h_l)$ is used.
- Sequence:
  - Start with: $U^{old}$.
  - A small # of iterations performed at $h_l$-level to result in $\bar{u}_l$ .
    If $u_l = L_l^{-1} f_l \rightarrow$ EXACT Solution
    Then $\rightarrow v_l = (\bar{u}_l - u_l)$ is lot smoother than $(u^{old} - u_l)$
    $v_l \rightarrow$ Exact required correction.
  - Defect is: $d_l = L_l \bar{u}_l - f_l$.
    Now, $L_l v_l = L_l(\bar{u}_l - u_l) = L\bar{u}_l - f_l \equiv d_l$ (A)
    Thus, we need to solve this equation with $v_l$ lot smoother than $u^{old}$.

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So, basically the error, the iteration improvement that we are given by the last equation seems this as actually dictated up on by this different term. Now, so what we can do is, instead we decide to work on multiple grids.

So, let me try to explain the whole thing with the help of only two grids. So one grid, let us say has a spacing h l, that I will call as the fine grid and another grid, which is requires the spacing; so that will be a coarser grid I will call it as h l minus 1.

So, to give that, we would be working on to give the starting some kind of old guess that we are calling as u of superscript old. Then what will you do? You will do some small number of iterations in the finest grid itself at h l level itself.

What will happen that we have seen, that we could, if we do under relaxation, then we can remove those high wave number component. What does the high wave number component do? So, if I have a solution and if you look at the low wave number component, it goes like this; the high wave number components will be riding on it. and If they are small like this, then we are going to get something like this.

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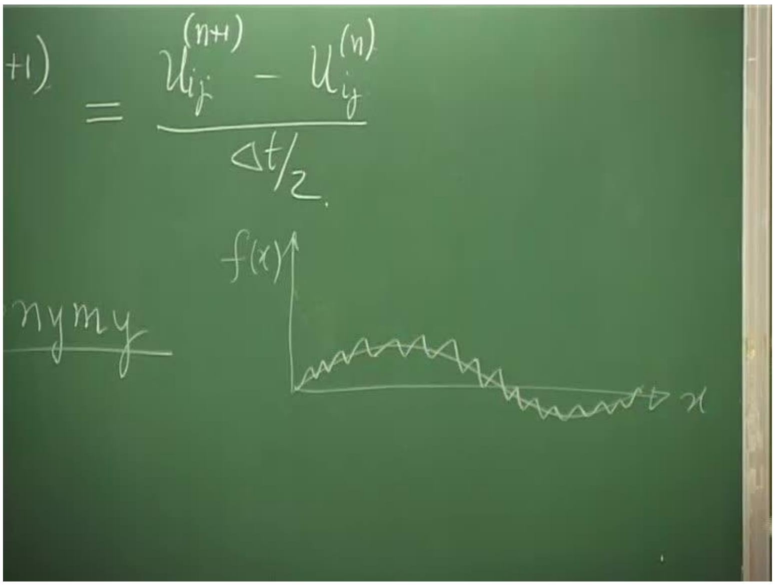

So, this is, let us say the total error would be given by this jagged line; so, that jagged line implies that you have low wave number as well as high wave number component. Now, the moment if you recall what you did here, moment we did few iterations in that grid, we are removing this part of the error.

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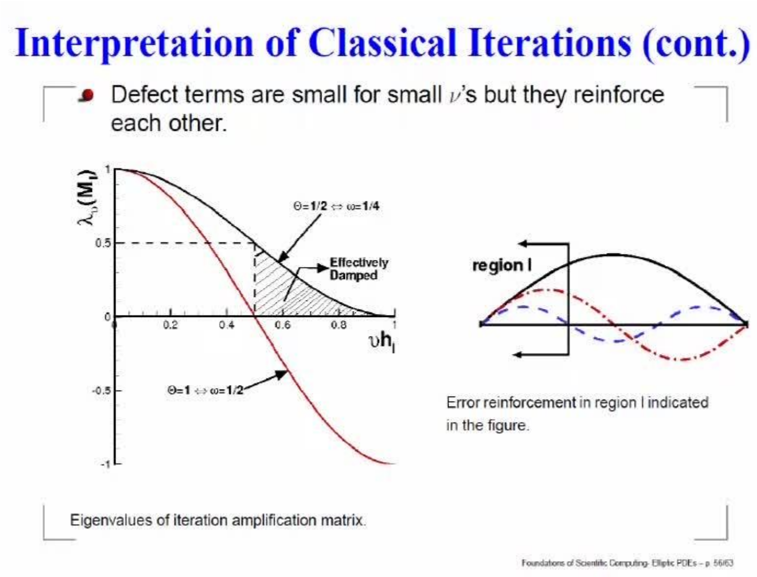

## Interpretation of Classical Iterations (cont.)

- Defect terms are small for small $\nu$'s but they reinforce each other.

Error reinforcement in region I indicated in the figure.

Eigenvalues of iteration amplification matrix.

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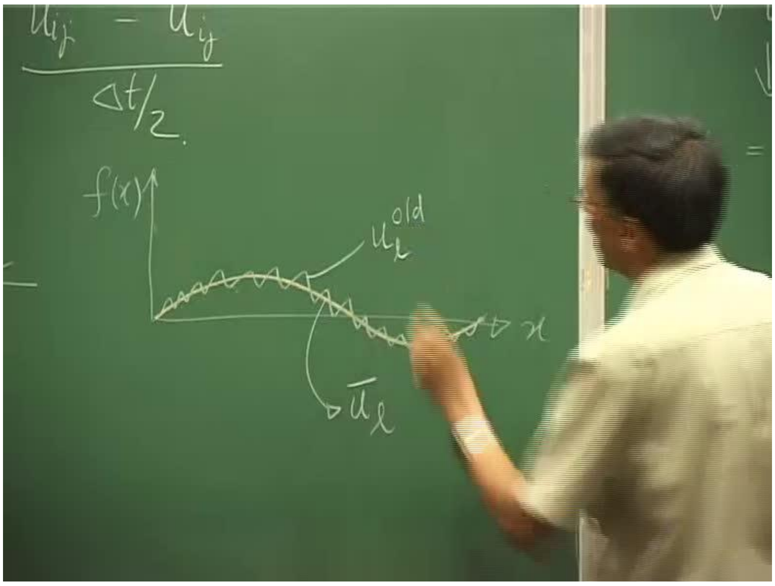

So, what will happen? This will actually after a few levels of solution at h l this variation, this high wave number variation will go away and I will get a rather a kind of a solution like this. So, this, I could call it by say - u old and this is what let me call it as u bar.

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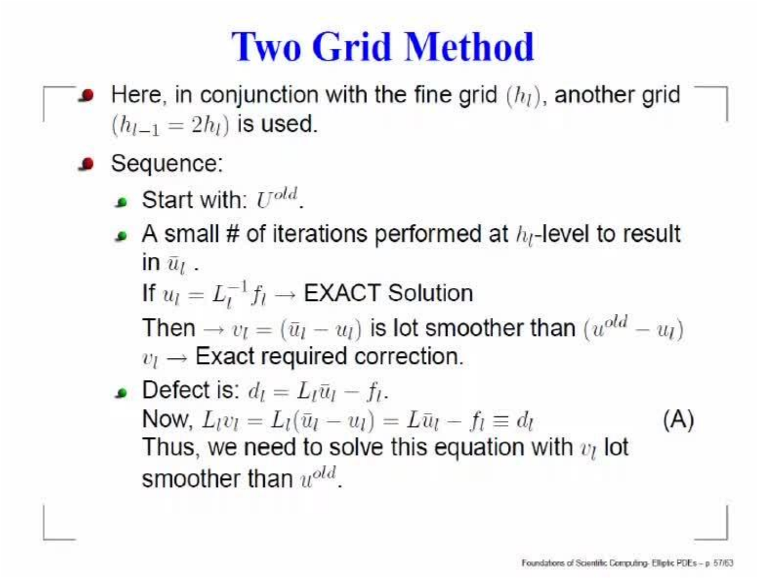

## Two Grid Method

- Here, in conjunction with the fine grid $(h_l)$, another grid $(h_{l-1} = 2h_l)$ is used.
- Sequence:
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    Thus, we need to solve this equation with $v_l$ lot smoother than $u^{old}$.

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This is at the lth level. So this is what we have in domain layer that we start off with some initial guess perform some small number of iterations at h l level and then we get a smoother solution.

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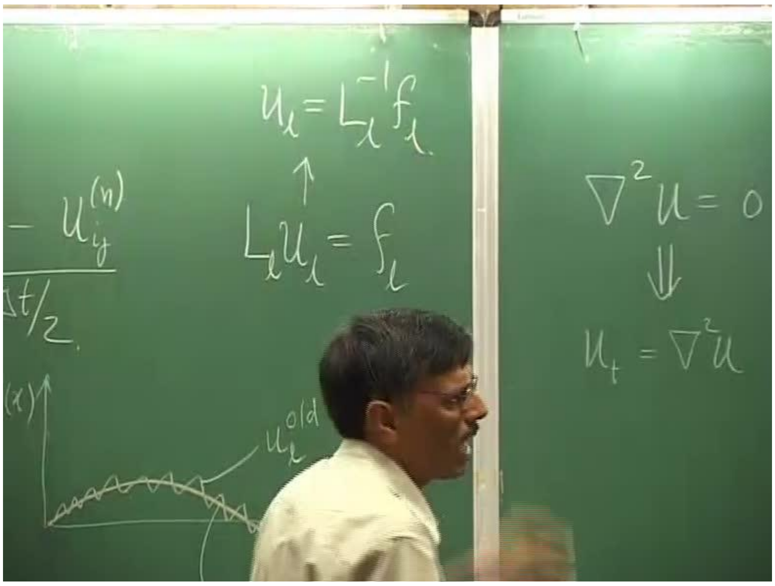

So, now you can understand this u l bar that we have obtained; it has smear out on these high wave number variations. So, what happens is, if I know what we are trying to solve is, we are trying to solve L of u l, well, this is also grid dependent; the operators also depend on a grid; that is what we are trying to do.

So, its exact solution; we will be able to write it as L l inverse; so that is what we had. So, the exact solution is given here in this fashion.

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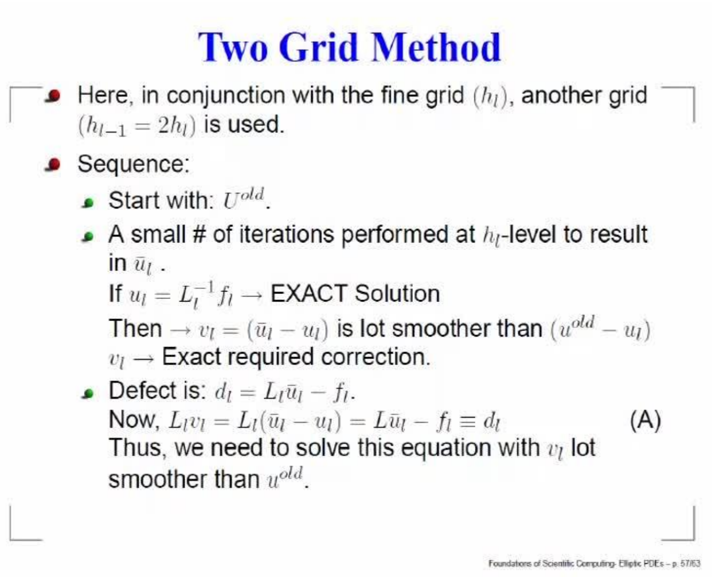

## Two Grid Method

- Here, in conjunction with the fine grid $(h_l)$, another grid $(h_{l-1} = 2h_l)$ is used.
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    Thus, we need to solve this equation with $v_l$ lot smoother than $u^{old}$.

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And then, what I could do is, I could construct a departure vector or error - I call it as v l; v l is nothing but the smooth curve solution minus non-linear solution. So what happens is, if I have done that operation, then this v l is not smoother than what we had began. Because now having done on this iteration iterative method, we have removed the higher frequency; so it becomes smooth; that is the old partners. Now having obtained the u l bar, I can calculate the defect, because if you recall, the defect is what drives the solution.

That is what we wrote in the last term. The defect term actually drive the solution. So what happens is we calculate the defect and then we try to solve for the new unknown v l. In that, let us say the finest grid itself, then this is like this.

So, v l itself is u l bar minus u l; so that is nothing but a defect. So what happens is, now we need to solve actually this equation for real with a defect which is lot more smoother than what we had corresponding to u old.

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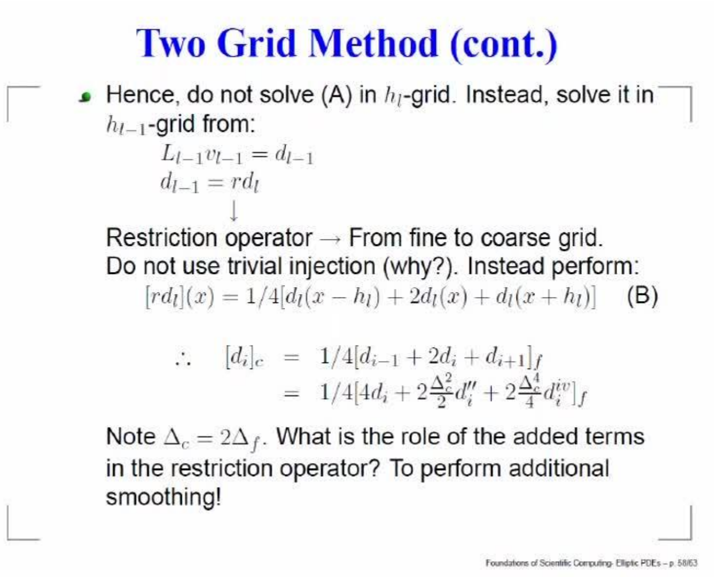

## Two Grid Method (cont.)

- Hence, do not solve (A) in $h_l$-grid. Instead, solve it in $h_{l-1}$-grid from:

  $$L_{l-1}v_{l-1} = d_{l-1}$$
  $$d_{l-1} = rd_l$$
  $$\downarrow$$

  Restriction operator → From fine to coarse grid.
  Do not use trivial injection (why?). Instead perform:

  $$[rd_l](x) = 1/4[d_l(x-h_l) + 2d_l(x) + d_l(x+h_l)] \quad \text{(B)}$$

  $$\begin{aligned} \therefore \quad [d_i]_c &= 1/4[d_{i-1} + 2d_i + d_{i+1}]_f \\ &= 1/4[4d_i + 2\tfrac{\Delta_c^2}{2}d_i'' + 2\tfrac{\Delta_c^4}{4}d_i^{iv}]_f \end{aligned}$$

  Note $\Delta_c = 2\Delta_f$. What is the role of the added terms in the restriction operator? To perform additional smoothing!

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Now, if it is already smooth and we have already in the fine grid, doing anything more is not going to help you. What I could do actually is instead, now I could go over to a coarser grid. See basically if this kind of smoothing at this scale, say, if I look at this scale, this scale corresponds to what: h l. See, this is corresponds to in that figure, we show that the right extreme points - those are the maximum wave numbers which we could handle given this grid size.

So, that is what we are suggesting now that look, we are exhausted on the possibility what we have be cooled in the h l grid. So, now, let us do that instead of solving l v equal to d in the l grid, we go over to a coarser grid.

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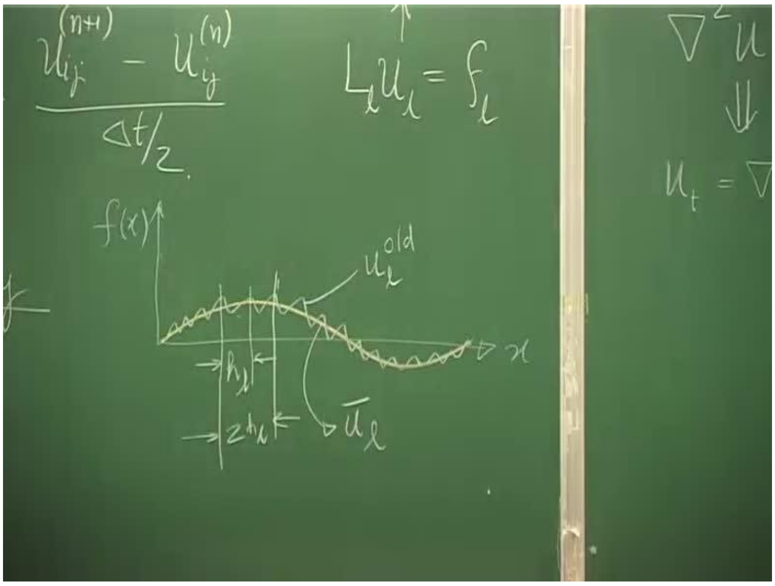

So what happens is, now, I will try to reduce error, in what, in this weight where the grid spacing is 2 times h l right, that is about coarser grid definition. So, what happens, say, I am exhausted both errors at the smallest possible level to the finest grid; my grid to the next coarser level grid and that is what is given here.

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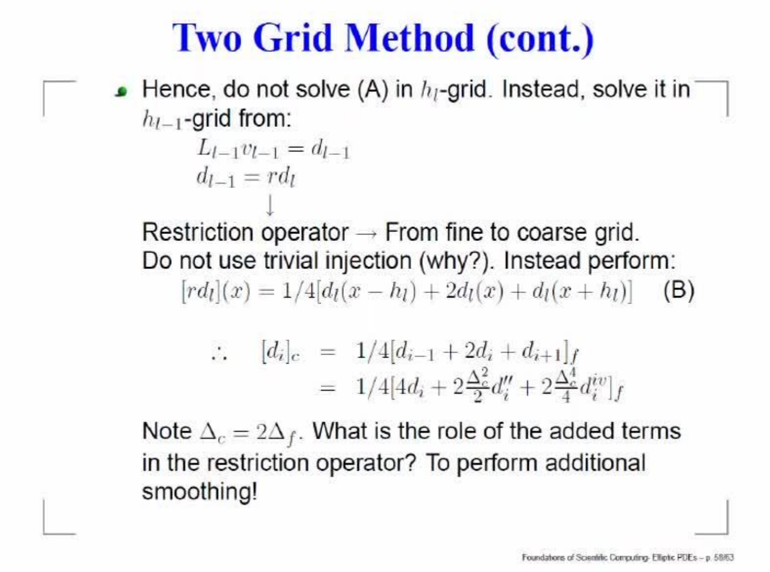

## Two Grid Method (cont.)

- Hence, do not solve (A) in $h_l$-grid. Instead, solve it in $h_{l-1}$-grid from:

$$L_{l-1} v_{l-1} = d_{l-1}$$
$$d_{l-1} = r d_l$$
$$\downarrow$$

Restriction operator → From fine to coarse grid.
Do not use trivial injection (why?). Instead perform:

$$[r d_l](x) = 1/4[d_l(x - h_l) + 2 d_l(x) + d_l(x + h_l)] \quad \text{(B)}$$

$$\begin{aligned} \therefore \quad [d_i]_c &= 1/4[d_{i-1} + 2d_i + d_{i+1}]_f \\ &= 1/4[4d_i + 2\frac{\Delta_c^2}{2} d_i'' + 2\frac{\Delta_c^4}{4} d_i^{iv}]_f \end{aligned}$$

Note $\Delta_c = 2\Delta_f$. What is the role of the added terms in the restriction operator? To perform additional smoothing!

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Now, how do we solve this equation? Solving this equation requires knowledge of this defect at the coarser grid. Now what we do is we define the defect in the coarser grid is equal to r times d l.

What is r? r is some kind of a operator. So, I have the defect in the fine grid d l. I could have, just because the points are common between the fine grid and the coarse grid when from a perspective of the coarser grid, the points is common with the fine grid.

Then I could just simply pick it up; if I do that, that is what in mathematical Galvin is called trivial injection. I have simply take it and (( )) you know, things are very simple; we try to confirm everyone around by jagged.

So, this is mathematical version and they will say do not use trivial injection. I will tell you why, but instead they say that you do this, instead what you do is, whatever may be the error that you have obtained in the fine grid.

You take a combination of that at say x location by picking up the value there, add up its two neighbors and scale it; so that you get equal to one and you know what has happened. Now, I think we will discuss it tomorrow and then we will see what actually we did.