Foundation of Scientific Computing

Prof. T. K. Sengupta

Department of Aerospace Engineering

Indian Institute of Technology, Kanpur

Module No. # 01

Lecture No. # 16

Today's discussion on lecture 16 begins once again, by our discussion on Spectral Analysis tool. As we mentioned before, we would like to develop an analysis tool which is applicable for the full domain that would have different types of discretization for different points. So, in this context, we have already discussed in last lecture about equivalent wave number.

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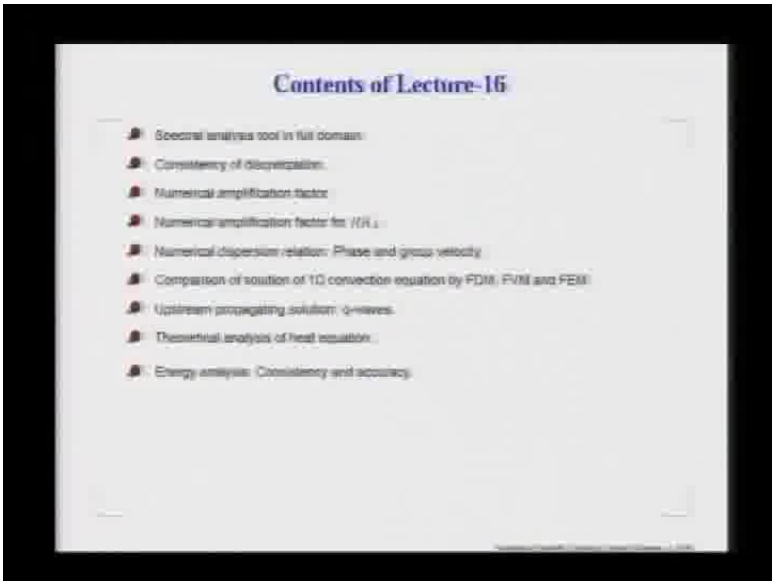

Once we have developed equivalent wave number, we are going to talk about consistency of discretization in terms of this equivalent wave number expression. Once we apply it to specific space time dependent equation, we can talk about numerical amplification factor.

We have seen already that Euler time discretization with various kinds of central schemes are unstable; that is why we would move over to numerical amplification factor for 4 stage Runge-Kutta method because we have already talked about the requirement of single step method as oppose to multiple step methods. Once we have talked about space and time discretization together, we are in a position to talk about numerical dispersion relation. Once we have the numerical dispersion relation, we will show how phase, phase speed, and group velocity could be computed from this numerical dispersion relation. We will do it specifically with the help of 1D convection equation and show the power of this analysis tool by comparing different finite difference, finite volume, and finite element methods, one by one.

We would like to bring one particular aspect of any discrete computing method. It is the existence of spurious upstream propagating solution and this is what has been called as Q-waves. So, this is something that we will be talking here, in great detail. This would basically conclude our discussion on discretization.

So, having finished our discussion on discretization, we will basically start our discussion on various solution methods. So, we will begin by solving parabolic partial differential equation. In this context, we will adopt the heat equation as an example of parabolic PDE.

We will begin our discussion of parabolic PDE solution method by theoretically analyzing the heat equation. Specifically, we would like to bring to your attention, the concept of physical instability versus numerical instability. That is why we need to have a former understanding of theoretical aspect of the solution. In this context, we may like to introduce an equivalent energy. Having done that, we will show that for a physically stable system, we cannot afford to accept numerical instability. In this context, we are also going to talk about consistency and accuracy of solution methods for PDEs. Then we should continue with our discussion.

We are actually in the process of developing an analysis tool which is in the spectral plane; that is why we call this as a spectral analysis tool.

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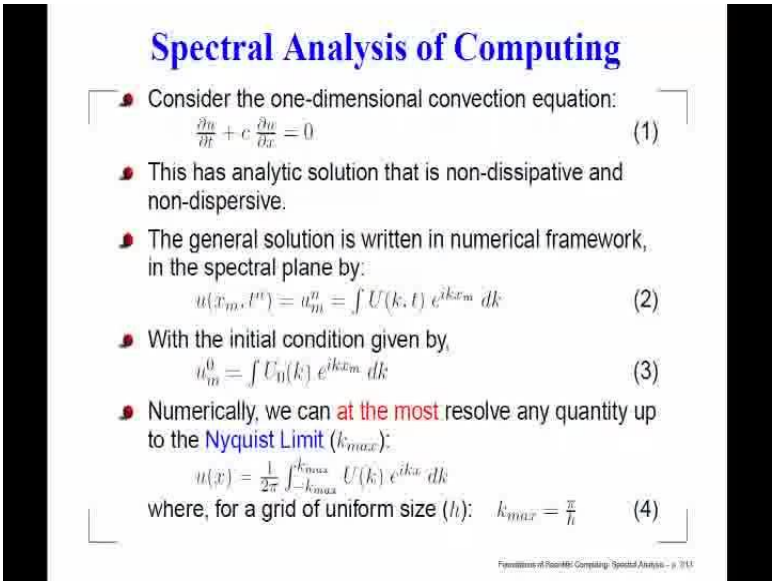

What we have been able to do is also analyze such a scheme in the full domain; that means, unlike what people have done earlier, people have developed methods where we could just simply look at the scheme - what happens in the interior. But you can notice here that we can find out effectiveness of discretization in terms of this k equivalent in a node-wise manner. So, for each and every j, I could evaluate this, the moment I decide to freeze upon the method of discretization through the choice of this C matrix. On the blackboard, we developed this.

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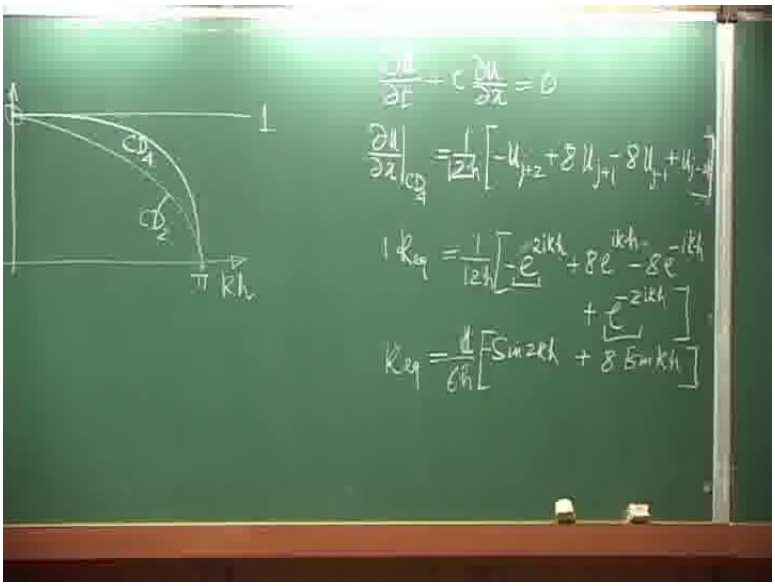

We showed that if we take a second order central differencing scheme that gives k equivalent as sin k h by h. Then, from there, actually we drew a portrait of this effectiveness and plotted it in the non-dimensional wave number, ranging between 0 and pi. On this side (Refer Slide Time: 05:33) we plotted k equivalent by k and what we notice that k h going to 0. What we are getting here? k equivalent by k here would be sin k h by k h (Refer Slide Time: 05:52). So, that is your sin x by x and we know the familiar property of the function that it just simply decays to 0 like this (Refer Slide Time: 06:03) and this is your value 1. So, ideally, what you would like to have is that all scales are resolved exactly, but discrete method shows that it is scale dependent. Depending on the value of k, you have different effectiveness.

What about this point? This point has to be equal to 1, why because this is the limit for which you are going from the discrete to continuum; h going to 0. If h goes to 0, then we reach this point (Refer Slide Time: 06:42). Then, by equivalent resolution, it should be exactly equal to the theoretical estimate. So, that is something that we have talked about.

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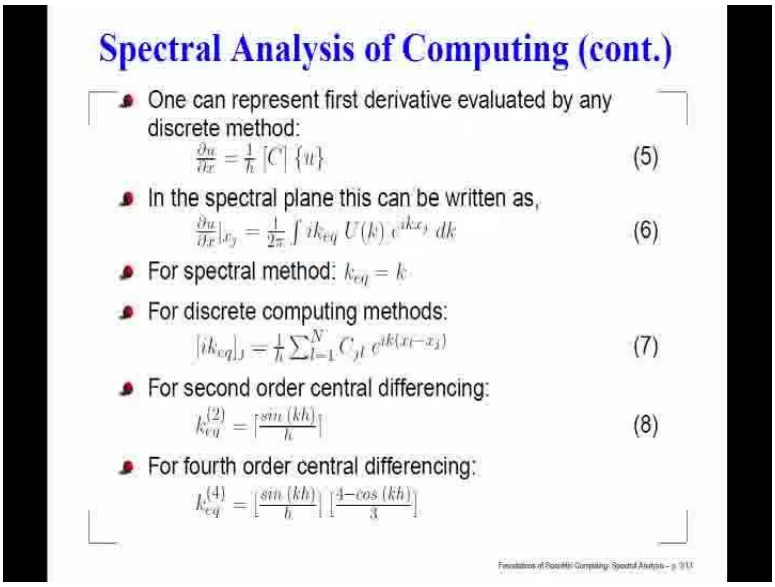

Now, suppose you take care of this discretization of the first derivative in that convection equation, basically investigating this simple equation, we have seen what the second ordered discretization does.

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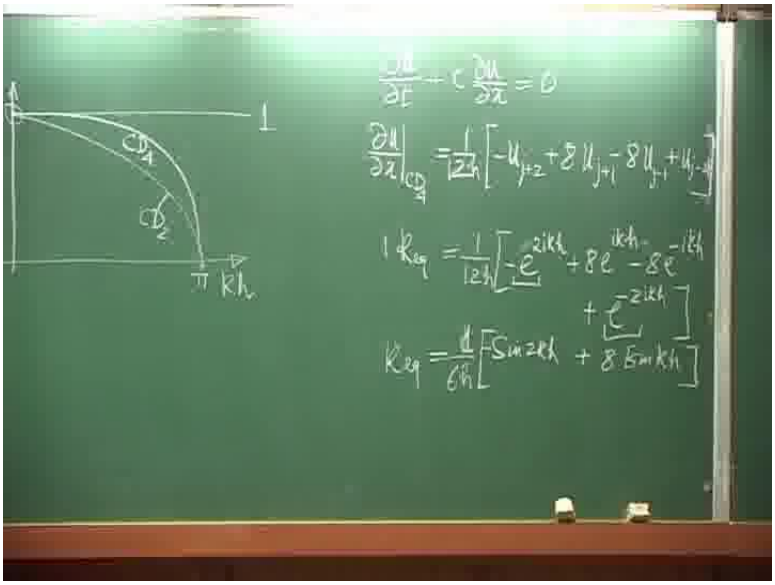

So, suppose you do it by fourth order central differencing scheme, that I think you would recall, this was the expression that we had written - u j plus 2 plus 8 u j plus 1 minus 8 u j minus 1 plus u j minus 2. So, once again we can calculate its k equivalent. You can very clearly tell me what this is going to be; k equivalent would be 1 over 12 h and from here I will get e to the power 2 i k h; so, I will get minus e to the power 2 i k h. From here I will get 8 e to the power i k h; from here, I will have e to the power minus i k h; the last one we will continue - e to the power minus 2 i k h.

So, you can see, things do happen pair-wise. So, you have here (Refer Slide Time: 08:45 to 09:36) appear with the opposite sign. So, you could club them together; each one of them will contribute 2. So, we could take this quantity out. So, this will be nothing but, I could also take i out. So, I will get here sin 2 k h from this and that and from there we will get 8 sin k h. So, this is the expression for i k equivalent. So, you can write k equivalent by omitting that i. So, you will get that and do a little bit of simplification and you will get this expression.

So, what you find that the fourth order differencing is equivalent to the second order differencing quantity multiplied by this factor. This factor has a role to scale it up. What I mean by that is - if this is the figure that I have got for C D 2, for C D 4 I will get something like this (Refer Slide Time: 10:05). So, it says that I have a much larger range of k h, over which my representation is more accurate.

So, this is the story with all explicit methods. As you keep on increasing its order, you keep seeing that this gives you better and better approximation. However, for all the cases, you would notice that this will go to 0 at the Nyquist limit.

[Noise] (Refer Slide Time: 10:36)

Yes. [Noise]

Pardon. Here? plus and that is why this has given us this (Refer Slide Time: 10:45) maybe I should have a minus sign here; so, I should have a minus sign there too. (Refer Slide Time: 10:57)

So, this is the story when it comes to discretization. So, what we are looking here is - what happens when we just discretize the special derivative term alone. Now, the story does not end there because what you end up doing is solving a particular equation where both space and time dependence come into play.

So, if I look at this equation, what we are going to do is again represent the unknown in terms of Fourier Laplace transform.

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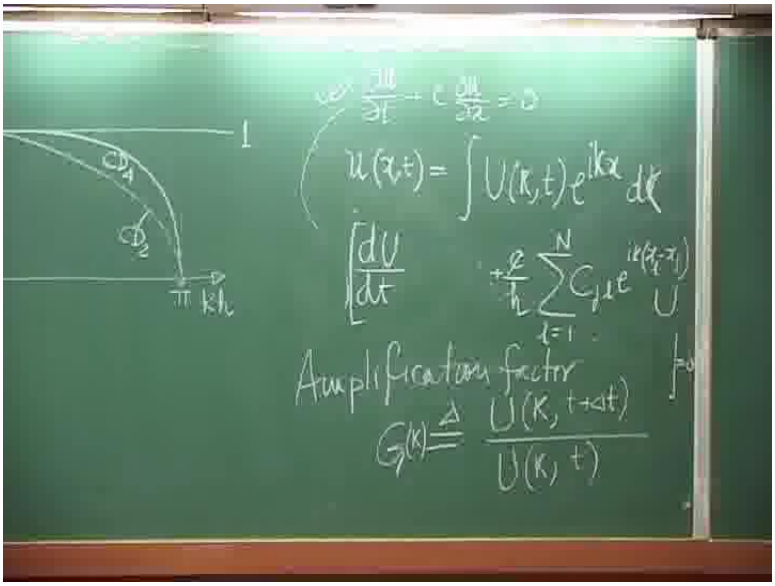

So, I will write like what I did yesterday; U of k and t into e to the power i k x d k. So, that is the way we are going to look at it. So, as you can see - this term (Refer Slide

Time: 12:20 to 13:20) will give you an integral dU by dt and this term remains as it is. This term will give us c by… this is a lower case c. So, I will write it like this. What did we write this as? In terms of if I am doing it for the jth point, I will write it C j l e to the power i k x l minus x j and l goes from 1 to n and this multiplied by u and the face part remains as it is. So, basically this is what we are getting from here to here via this spectral representation. Then, of course, it is true for the integrated quantity. So, the integrant itself must be equal to 0 and that is what you have it here.

The top equation is essentially what we have done - remove the phase part; remove the integral part; I have here is this (Refer Slide Time: 13:57). So, I have dU by d t plus c by h and the C matrix operating on the projection operator times U and so a little bit of a manipulation will you get you in this figure (( )).

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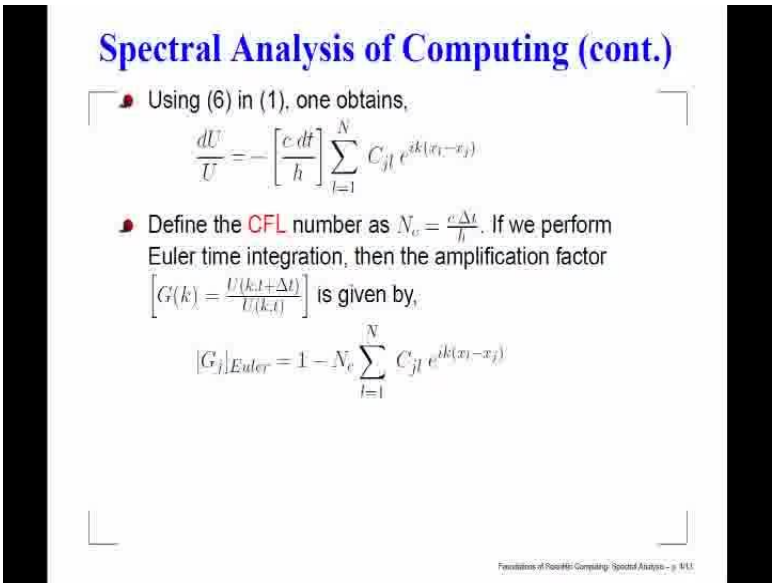

Now, in most of your computational activities wherever convection is involved, you will always notice appearance of this parameter which is shown here in that square bracket; this is a non-dimensional quantity. This is what is called as the Courant-Friedrichs-Lewy number or CFL number. So, just simply remember it as a CFL number. What it basically tells you is, a kind of a non-dimensional quantity N c which we will write and we will see that this is a fundamental independent variable that determines the property of the method.

Now, having defined the variable in terms of its Fourier Laplace transform like this, we can define what we will call it as an amplification factor.

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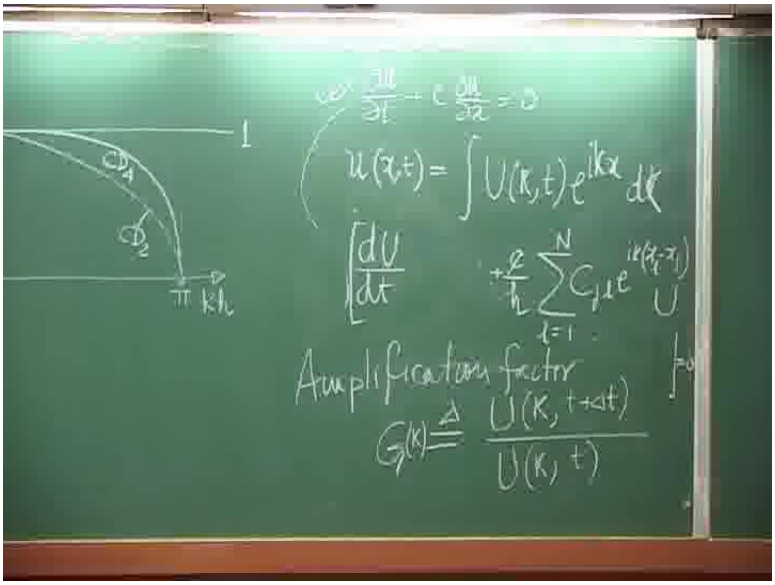

We will define it as U for that particular k, where we are looking at the solution at the advance time step divided by the solution in a k space at the old time step. So, in a sense, this is going to be a function of k.

You can very clearly see that in the limit of continuum when we take delta x equal to 0 delta t equal to 0, this G should be equal to 1; see easily, if delta t goes to 0, this limit goes to 1. What happens is a different story; we do a finite time step calculations and the moment we do that we deviate from its ideal value of 1.

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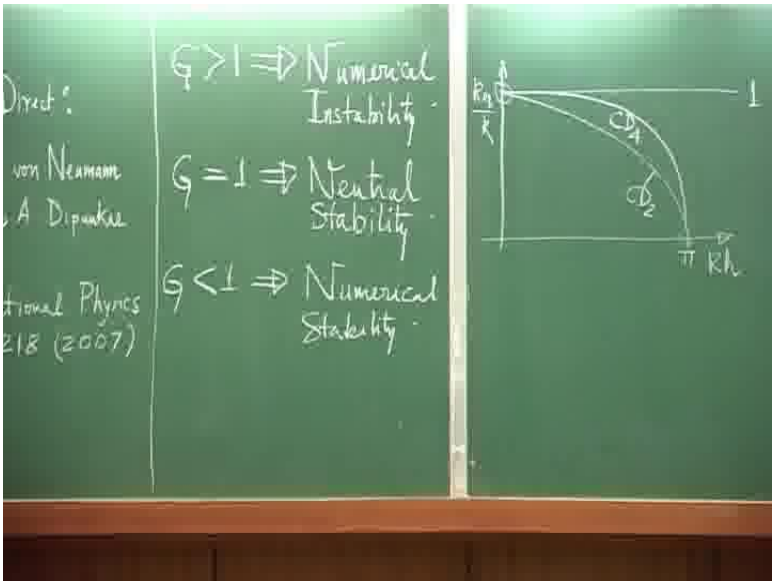

So, in computation, irrespective of equation you will always expect G should be as close to 1 as possible. What does this mean? Look, if I have G greater than 1, what does it imply? It implies the solution is growing with time. So, I will call that as instability. Since this is an action of a numerical activity, I will call it as numerical instability. If G is equal to 1, then it is neither growing not decaying; I will call that as neutral stability. When G is less than 1, we call this as numerical stability. So, with time, the Fourier Laplace amplitude will keep decaying.

There seems to be lot of misconception among for the practitioners of computing in the CFD community. I have noticed, time and again people tend to always think that you must have a stable of algorithm; nothing can far from truth. As you can see from the definition here, G should be equal to 1 and it does not matter what equation you are looking at. When I come to discussing parabolic partial differential equation, I will specifically pose a physical problem and I will talk about its physical instability and then relate that with the numeric.

However, irrespective of any equation that you are looking at, this is what we want. We should always aim for neutral stability; that is our ideal limits. Please do download this paper; this will have all those discussions given little more in detail (Refer Slide Time: 18:56).

So, what happens is - once I have written it down like this, now suppose I perform a Euler time integration, so far we have been silent about what we are doing with the time integration; let us say I am performing a Euler time integration on this term.

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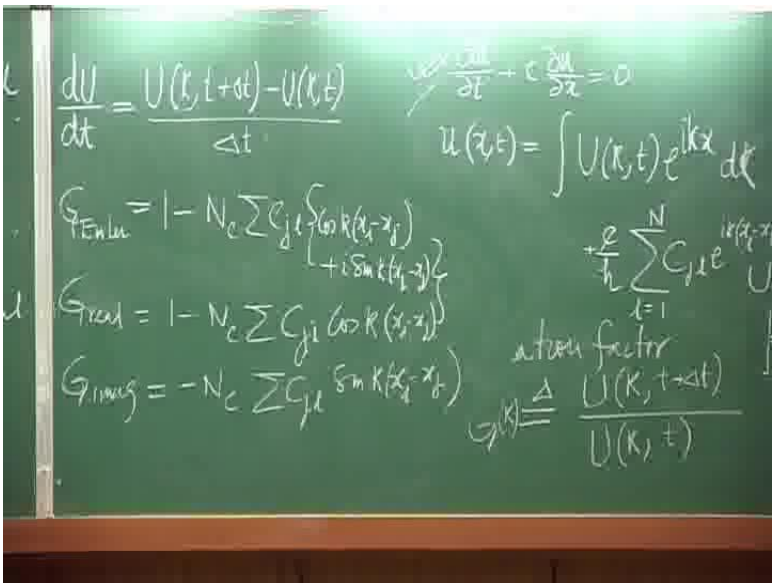

So, what I would do? I would write it as u of k t plus delta t minus u of k and t divided by delta t.

Then, you can see, this is the outcome because there is a U sitting out there. So, I could pull it out and I get dU by U equal to minus of this N c times this summation over this factor (Refer Slide Time: 19:49 to 20:18). What happens as a consequence, if I divide by U so this divided by U of k and t will give me G; so G minus 1 will be equal to this factor; so G will be equal to 1 minus N c into this factor.

What does it tell us? I have been telling you for a long time that this is a potentially a bad method to do Euler time integration. Why? You see, the C matrix is going to be of real entries. The way we discretize, you have noted various methods; some of them you have seen. C is a real matrix, but this phase function is complex. So, what we can do is I can take a modulus of this G and immediately you will notice that this is greater than 1. So, this modulus of G is greater than 1. What does it mean? That is an unconditionally unstable method.

So, you look worried. Tell me, you have any confusion? Let us work it out. So, I have 1 minus N c and this C j l and this will be cosine k x l minus x j plus i sin k x l minus x j.

You can see this. So, you can see the real part 1 minus N c (Refer Slide Time: 22:17) and you have the imaginary part which will be nothing but…(Refer Slide Time: 22:28). Now can you see what I said?

So, you are now convinced that this modulus will be greater than 1 and that makes Euler time integration very undesirable; it will lead to instability. So this is that. What are the other better methods that we have?

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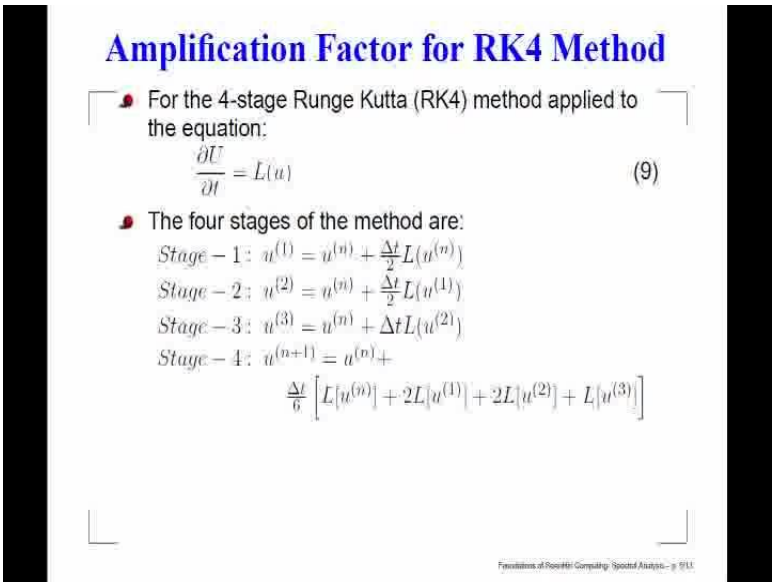

Let me tell you for some of the time integration methods that we have investigated, we have developed ourselves, we find this is a prime candidate which gives excellent property and this is your 4 stage 2 time level Runge-Kutta method.

So, let me explain how this method works and how this method is better in terms of numerical amplification factor. Suppose I have a space time dependent equation. So, I do all kinds of spatial discretization; put all those terms on the right hand side and call it as a L operator; so, that determines all yours spatial independence. Then, we have this kind of an evolution equation del U by del t. Well, please forgive me, this should be a lower case u; this is not that capital u; it should be lower case u.

So, del u by del t is equal to L of u. By now, all of you are familiar; we have already done it when we were looking at solution of ODEs. In the 4 stage Runge-Kutta method we performed these 4 stages, having started with the solution at the nth level. We find out an intermediate stage solution which we call as u superscript 1. Having obtained that use that to calculate this function L of u here, and then, from there we calculate the second stage function U2; then we have the U3; finally, we collate all these intermediate stages into the next step solution that is u n plus 1. This is all there in your notes.

So, in fact, you can notice that one of the brackets has gone wrong, up in the stage 2; anyway, 2 and 3 there is something wrong. So, basically, let us see what happens when we in corporate our spectral description and try to get the value of G for this particular time integration method.

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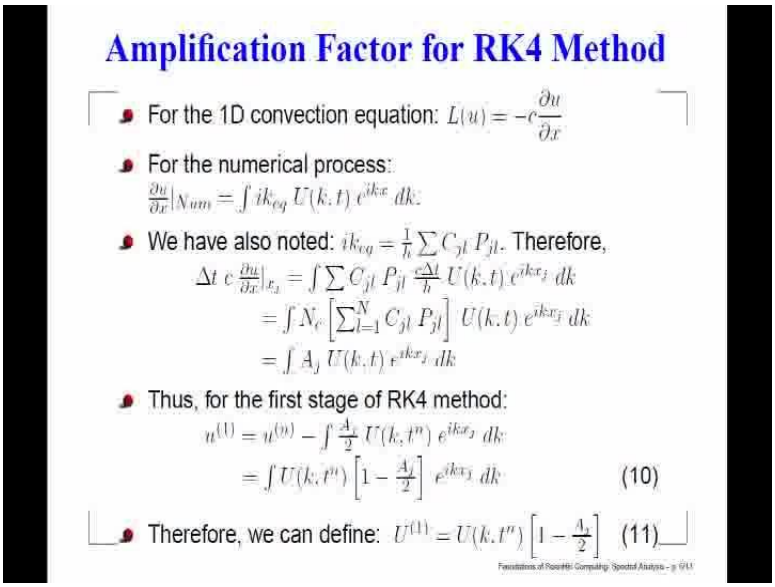

So, coming back to your 1D convection equation here (Refer Slide Time: 25:34), I can put this on the left hand side. If I do this, this quantity is nothing but your L of u that is your L of u for this. Now, we have also said that numerical description of the derivative with respect to x; we could write it like this. So, if I write delta t times, there is a c out there and times del u del x, we are going to get this. C is there, delta t comes when I multiply. As you can see in the previous stage, every stage I need to multiply by delta t here. So, here you can see, there is a delta t by 2, delta t by 2, and so on and so forth.

So, delta t is part of the story. So, we get once again that factor c delta t by h; that is what we called as the CFL number or N c. So, c delta t by h, we keep it up front here as N c. Then, this is what we have done. This P j l is nothing but this quantity e to the power i k x l minus h j; I have just simply economized on space by writing that. Then you have to have the Fourier Laplace amplitude U k of t and integrate over all possible case that is what you get. So, again let us economize in expression and call this whole thing here N c times this summation C j l P j l; let me call that as A of j.

What does the subscript j imply? j implies that we are looking at the phenomena at the jth node. So, that is what we are doing. Having done that this is our first stage. u of 1 is obtained in terms of the starting A point u n times this part. That is what we have to do and that happens to be minus A j y 2 and this quantity. So, u n itself is U k t n e to the power i k x j. So, the whole thing can be written like this. So, I could write u of 1 as some kind of Fourier amplitude, capital U of 1. That capital U of 1 is nothing but evaluated at t n times this 1 minus A j by 2.

So, this is the way that I will describe the first intermediate solution either in terms of 10, or in terms of its Fourier amplitude by this expression level.

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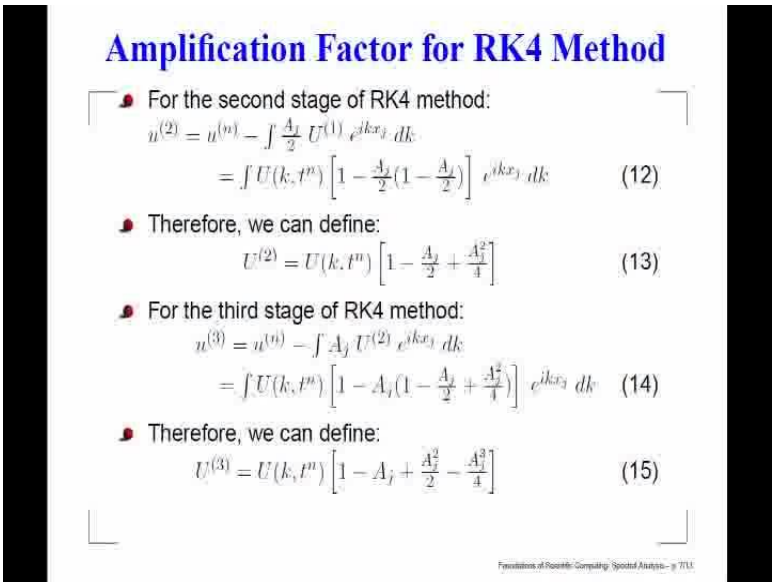

Now, we go to the next step. The next step follows in a similar manner because what do we do there? We take u n minus delta t by 2 into L of u evaluated at the previous intermediate stage U of 1.

So, that is what I could do. I could write it in terms of its Fourier amplitude, capital U of 1. Again, this gives me capital U - this quantity, but U of 1 we have already written down as U into 1 minus A j by 2, but there, this is up front factor A j by 2; so, that comes in here. So, the whole thing works out like U of k comma t n 1 minus A j by 2 into 1 minus A j by 2. So, basically this whole quantity minus this space path is the Fourier amplitude for the second intermediate solution.

We proceed and obtain the third quantity, the third intermediate stage that is u n minus again this is delta t. See, in U1 and U2, we have delta t by 2; U3 we have delta t; so that is why we have just A j; otherwise, previously you were getting A j by 2. So, U3 is u of n minus A j into U of 2 into this. U of 2 is in front; you plug it in there; this is what you get. So, that explains to you what this U of 3 is. So, this is an expression that helps you explain everything in the k plane.

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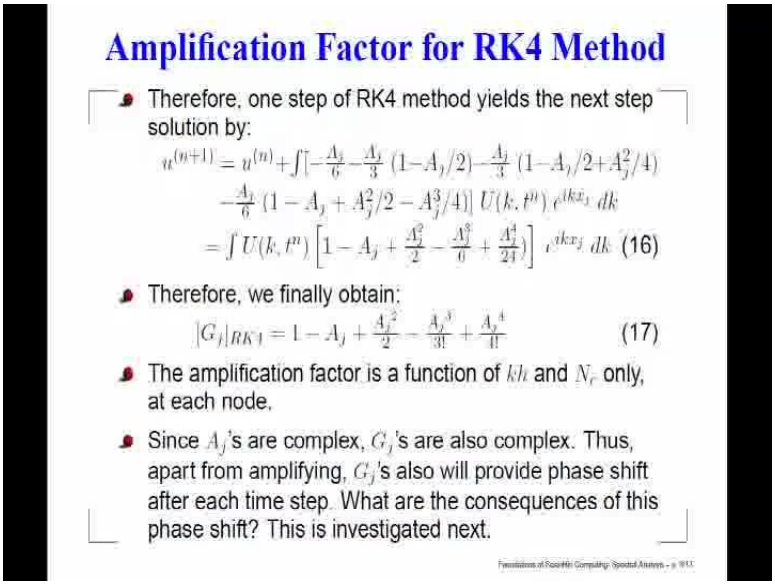

Now, having obtained all these quantities U1 U2 U3 etcetera, you put it in the final collative stage where you get the solution at the new level in terms of the older value;

this is what you get; do a little bit of algebra and you get this. So, we get 1 minus A j plus A j square by 2 and so

So, what we get here for this RK4 method is this and as you can see, this A j itself was a function of N c and those P j l etcetera. So, that brings in the non-dimensional wave number, here k h and N c; so, this G j is going to be a function of k h and N c. Please do remember that A j's themselves are complex; so, G j also will be complex. What it does? This sort of operation with G j will not only amplify or attenuate, but it also will provide you with a phase shift. So, if I have a real quantity and an imaginary quantity, I can write it in terms of modular time step e to the power i phase. So, basically every operation or every time step would be equivalent to multiplying this previous time step solution with the amplitude plus a phase shift.

Now, you would be interested to know that what kind of a phase shift that you are getting? Because in solving this equation that we have started with, mainly the 1D convection equation, the solution is very straight forward.

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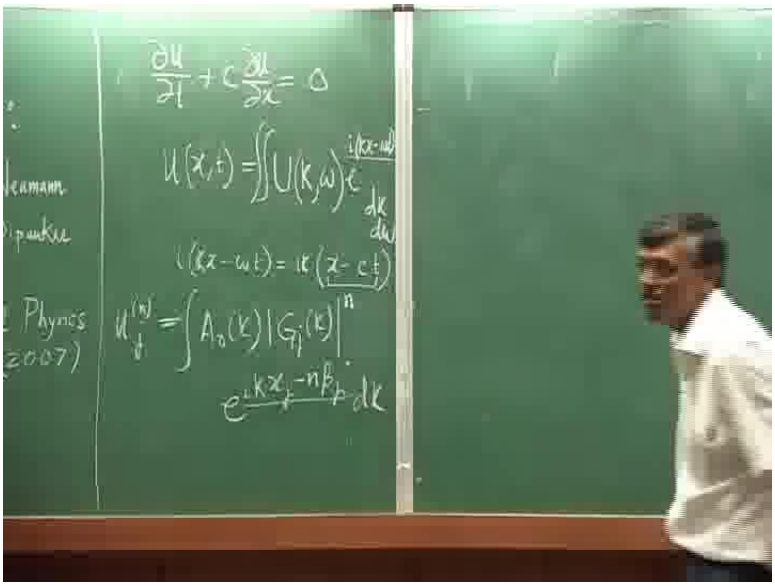

The solution is, as if you recall, we would write it like this - if I write in terms of… not in terms of k and t, but let us say, now in terms of the frequency itself if I write, if I introduce frequency, then what will happen? Then, I will have e to the power i k x minus omega t and d k d omega.

So, that is what I will get and you can see the phase path. The phase path is i k x minus omega t. So, if I take k out, I will get x minus c t. So, here actual solution shows the phase 2 change by this expression x minus c t. if I give you a solution at t equal to 0, at a subsequent time, you have the same solution, but it is shifted by c t to the x minus c t. So, this is a kind of a phase shift.

Now, I want to know this G that we have uncovered here, for this RK4 method in 17.

What does it do? How is it related to the c that we are looking for in the exact solution?

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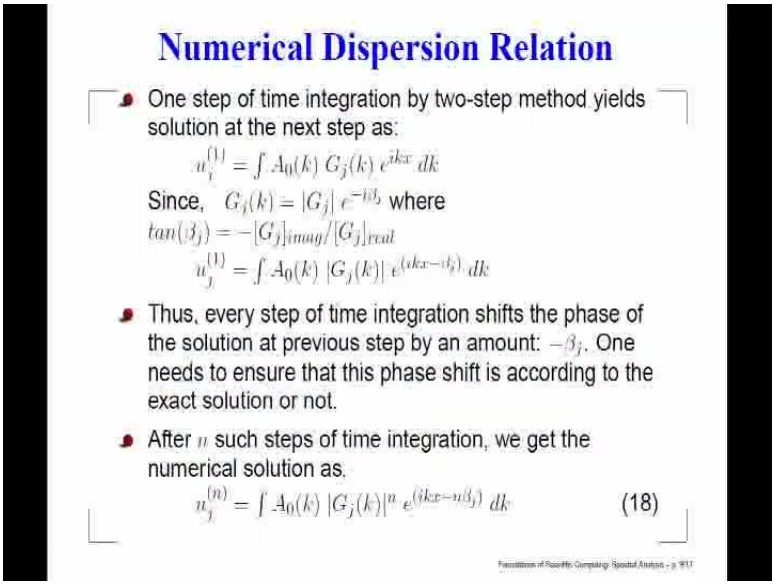

To understand this, I have demonstrated here, what happens. Suppose, I start from the initial solution which is given by the initial spectrum, A naught of k times e to the power i k x d k, then, I am looking for the solution at the first time step delta t; that I will call, u of 1; please do not confuse it with the first stage of RK4; this is what I am talking about time integration going from u 0 to u delta t; so this is your u delta t. That would be equivalent to multiplying by G of k; that is the definition of our amplification factor.

Now, this G of k itself I have written here in terms of modulus and time, say phase shift. What is this phase shift? It is nothing but tan inverse of G imaginary by G real with a minus sign upfront.

So what happens is, I could then write this u, the solution at delta t would be the initial spectrum times the modulus of the amplification and e to the power i k x minus beta j.

Why do I write j? Because each and every point will have a different phase shift; that is what we have developed through this matrix theory that we can obtain this point wise. So, that is what we conclude here, that every step of time integration shift the phase of the solution obtained at the previous step, by this amount minus beta j.

One needs to ensure that this phase shift is according to the exact solution; we need to do that. So, suppose I perform n such steps, I have arrived at the solution at the nth time step; that would be again given by my initial spectrum.

There is a mistake. This G j of k, there should be a power n because every time I get 1, if I am doing n steps, there is n missing here. Please do understand that there is a mistake. Please do not quote me later that your note was wrong and that is why we have done it wrong.

So, what I am saying that you will have G j of K this has been operated n times. So, this exponent (Refer Slide Time: 36:27), this is not a superscript and then I have the phase e k. So, if I am doing this, every step I am getting beta j; so, I am going to get minus beta j this times d k. So, this n is missing there. So, please do note there correctly that this is incorrect variable.

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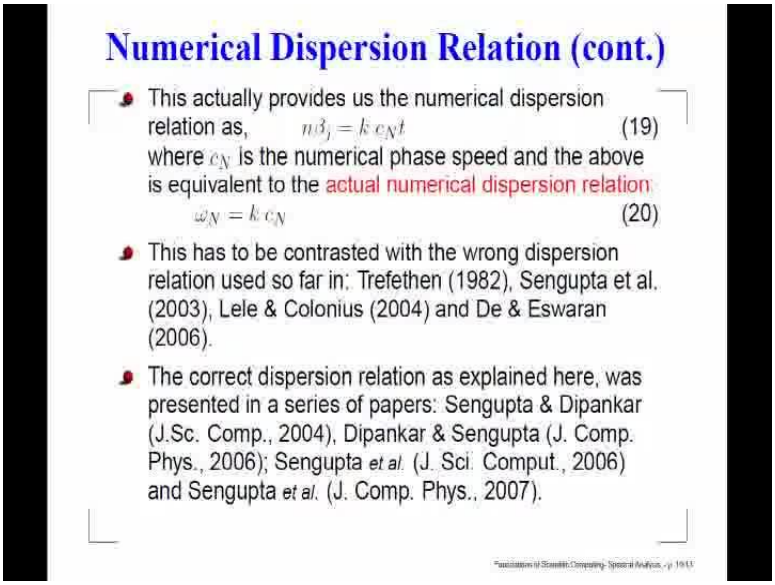

Then what happens? Now, having obtained this expression, you notice - compare this with this here (Refer Slide Time: 37:10). If I look at this phase relationship, with this phase relationship, I can see something emerging; this beta j is somewhat related to this omega here.

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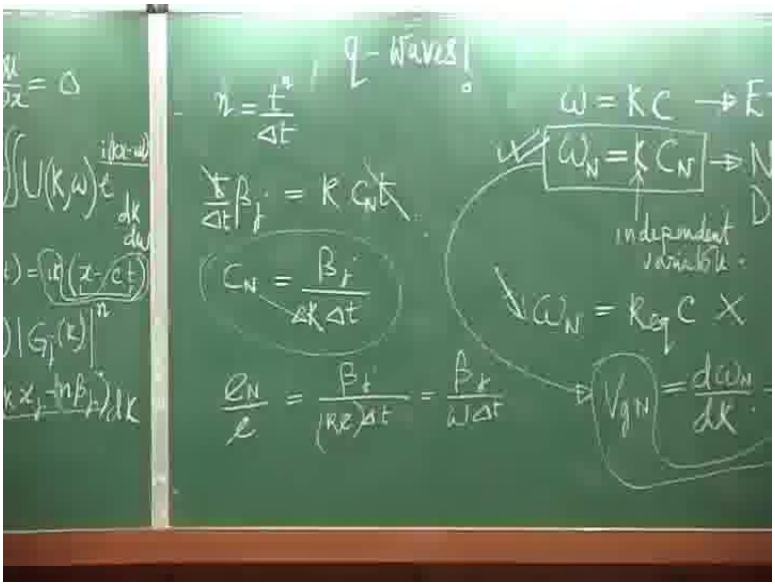

What is n? n is nothing but the time step. So, that will be like your t n by delta t. So, what happens then? n times beta j that is what we are seeing here,; that is going to give me something like this - i k c t.

So, I will write that as, there is an i here also and n beta j should be equal to k C t, if everything was fine and nice. However, we have already seen that doing discreet computation means sacrificing something. What is that something? k is still our independent variable. So, we are keeping that as our point of reference. We are going to see that this C which was to be a constant in the exact solution numerically does not remain so; it does not remain so.

So, if I replace this n by what I have written there - t by delta t, then you can see this t will cancel and what we are finding here? An expression for C N - that is nothing but beta j by k delta t. From the exact solution, we have noted omega equal to k C. So, this is our exact solution. But numerically what we are doing? k still is the independent

variable; C does not remain the same; this also, this is your numerical dispersion relation. See, we are relating omega with k. That is what we define.

At this stage it may appear it is a very simple thing. Tell you what that people have been doing things wrongly for decades? Including us, as you can see here, it started with professor Trefethen's work in 1982; then Lele Colonius also and this professor Eswaran from mechanical copied our work wrongly and they also ended up doing wrongly.

So, what you understand here is that your independent variable is k. What was the mistake people we are doing wrongly before? You know what people we are doing wrongly before; they were just simply writing omega N as what we have done in the beginning of the class, change k to k equivalent, and then say multiply by C. So, this is wrong whereas, this is the correct way of expressing the dispersion relation (Refer Slide Time: 41:31).

What does it mean? I mean are we just simply nit picking? No. It is very profound because what we are seeing here is that this C N is now a function of k, whereas in your exact solution, C was a constant. That is why we talked so glowingly about that say non dispersive, non dissipative solution. Here, what we are seeing is an act of discreet computing. We are getting a numerical phase speed C N which is a function of wave number. So, that means what? Different k component will send their crest at different rates. So, this is really a big development which was first pointed out here. Well, we have tried to convince people with most of publications there, as you have noticed.

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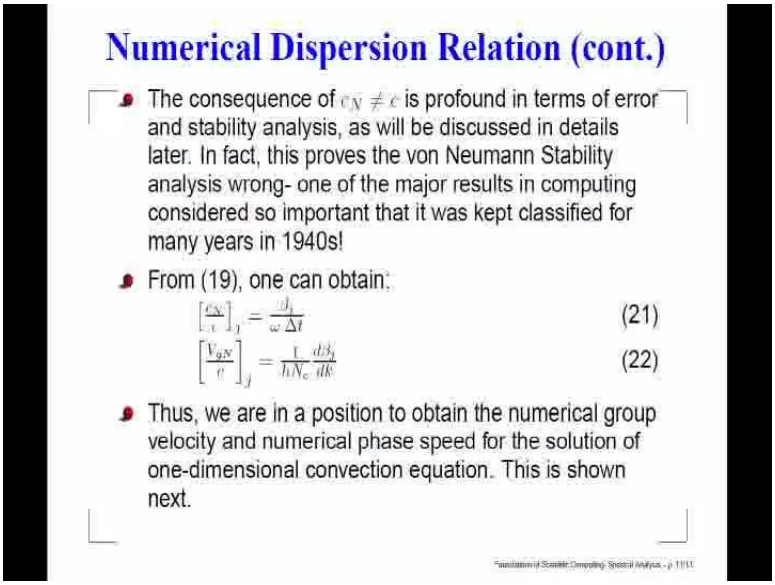

Continuing about discussion that C N is not equal to C is really profound in terms of error and stability analysis; there is a nice history about this. When during the Second World War, this group of people was in the Manhatten project in New Mexico developing the atom bomb. One of their main Stalwart mathematicians helping them was Von Neumann.

Von Neumann actually developed an error analysis or stability analysis. This was based on this assumption, a wrong one (Refer Slide Time: 43:30). At that time during the war, people are secretive about was goes on in research front; they classified the work; so, people did not know what was the work, but everybody knew that Von Neumann has done something which revolutionizes computing; explains a lot of features of computing. So, it was only I think in 1947 and 49, some papers started coming out, but as you can realize, that work was wrong and we actually first brought it out. We corrected Von Neumann's error; that is the main thing about this work.

What we notice out of all these exercise? I have got an expression for C N. So, what I could do? I could define a non-dimensional quantity which I will call c N by c. So, that would be beta j and I will have k c into delta t. k c is omega; so, beta j by omega delta t. So, that is what we have written here in 21. So, this is your consequence of numerical activity that you do not see c N by c equal to 1, but it becomes beta j by omega delta t. So, what happens?

So, you choose a method for spatial discretization. You choose a method for temporal discretization. You obtain the value of G which has a real path which has an imaginary path; you find out what is the imposed phase shift from this G, and that determines how far it is for one. Having given you this expression here (Refer Slide Time: 45:33 to 46:08), you can immediately calculate its numerical group velocity, which I will call as V g N, which will be nothing but d omega N d k. So, if I plug that in there, I will get this equal to c N plus k d c N by d k. If everything was nice and fine, you should have seen that these two should have been the same; V g equal to c that is an exact solution.

But the very fact that numerical calculation makes C N non constant function of k, adds on this part that is the source of numerical dispersion. In fact, lot of calculation goes on in the literature, where people do claim that they have done this and that; they are essentially source of this Fourier's dispersion; that is inherent with all numerical methods; you cannot just simply wash them away until unless you choose your methods and parameter very carefully. That also tells you that at this point in time with the type of computing power that we have, we cannot solve the equation in a direct sense. We will always have to leave some kind of error.

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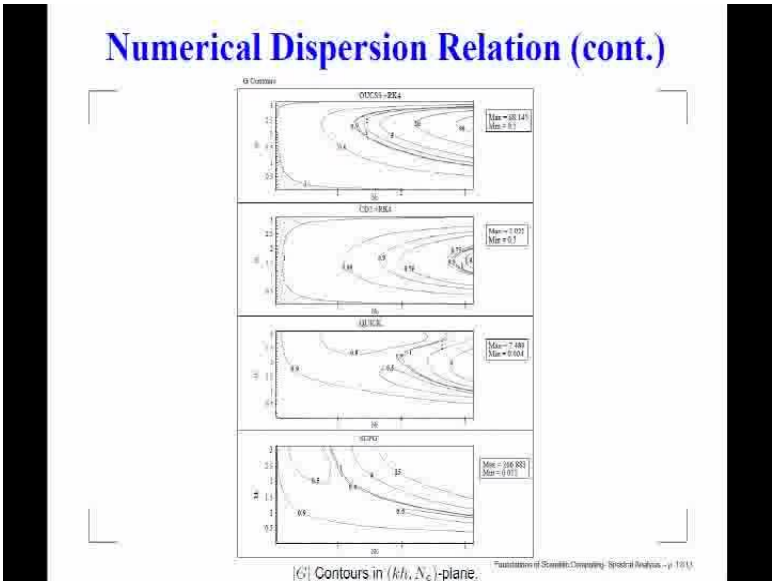

What is the error? How this error is contributed? As we go along, we will explain more. At this point in time, I will show you some results.

This figure may not make tremendous sense, except that in this figure what we have done? We have plotted these G contours; that modulus G that we talked about; the method in the second figure that we have is a CD2 method along with RK4. We have marked the region here with dash line where, your G is mod G is 1.

Recall, I wrote down that to do a correct calculation we must have neutral stability. So, This have been plotted in this plane; on the x axis I have N c - the CFL number, on the y axis I have k h. It tells you that to do an error free calculation coming from numerical amplification consideration, you need to keep your delta t very small; that means N c very small, so that you remain in the numerically neutrally stable region.

There are other methods. I will talk about these methods, but just simply know that this is the finite volume method, this is the finite element method, and unfortunately, neither of these two methods which are very popular very much in use; may be tens of, thousands of people use.

Then, as you can see, they do not have any G equal to 1 region. So, all they get is lot of… well you have solutions here; this is G equal to 1 line here; on this side, you have totally damp solution. So, as you keep integrating, your solution amplitude will come down; on this side, you have unstable path - this pocket. The same thing happens here with this finite element method called Petrov-Galerkin method; some of you may have taken a course; you know what it is; what you notice that G equal to 1 line is here. On this side, it is a completely damped solution, and on this side you have amplified solution. So, this is about the story of G.

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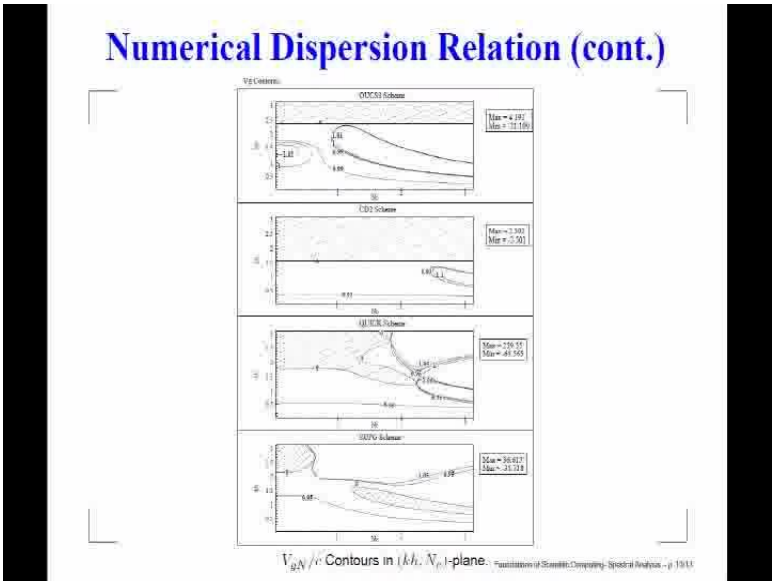

You need to also know, what the numerical dispersion is. We have talked about this; b G n by c; so, if we plot that again in N c at k h plane, we notice some very interesting feature. Let us keep our attention focus once again on this second figure because that is what we are quite familiar with now. We noticed that that V g N by c - this contour line, I suppose this line is 0.98. So, basically, even if you look at very small range of k h, here you are already started getting 2 percent dispersion; instead of 1, it is 0.98. This line here corresponds to 0.98, but interestingly enough, look at this line that is a very fascinating and interesting line that is V g N by c equal to 0.

What does it mean? Below that line, you have V g N by c positive; above it is negative. The equation that we started solving, solution should have propagated from left; if c is positive it goes from left to right. If my numerical dispersion relation is such and I am in this part, solution will go in the wrong direction.

This feature is vaguely understood, but this is where we have actually quantified it and put them across for different types of methods currently in use. This is one of the methods that we have developed. You can see that this value is actually pi by 2. So, for any k h value which is greater than pi by 2, they will go in the wrong direction. Since these are not physical waves, somehow this nomenclature has stuck this type of spurious solutions are not called p-waves, but q-waves. So, q-waves means spurious upstream propagating solutions.

As you can see, almost every numerical method has this kind of a feature. So, you cannot just simply say that I have 1 method superior to other; in fact, the case for finite volume and finite element is really pathetic because they not only attenuate the solution, their dispersion relation property is also equally bad.

So, I think we would conclude here. I still have not told you about the error analysis path which will come little later.

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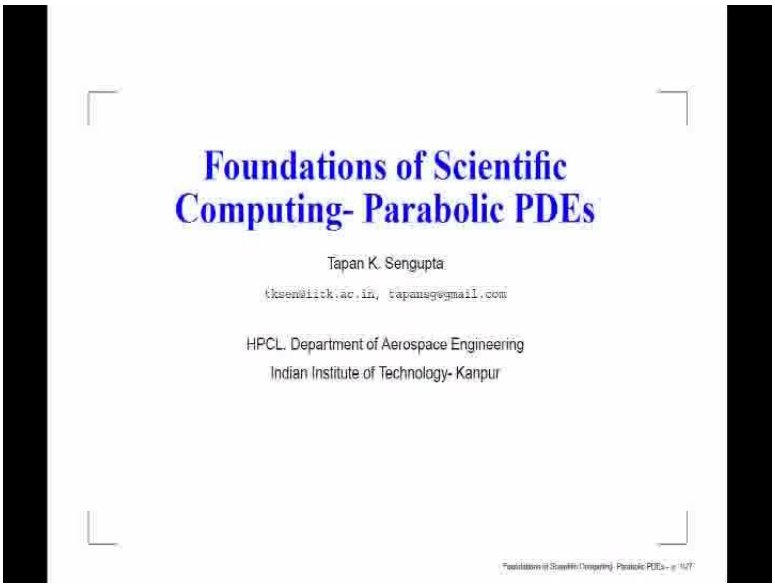

You please download this paper and then I will come back shortly, but let me now go to the next topic that we would like to discuss.

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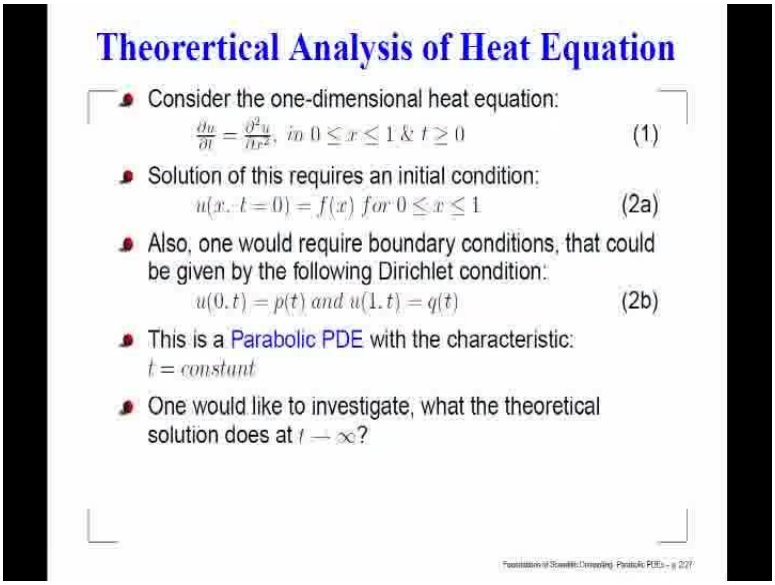

That is basically going back to classical thing that any computing course tries to teach you - how to solve different types of PDE's. So, let us begin with parabolic PDE's because that is how historically it all began.

Well, once again with our formal practice, we start with some given equation. So, let us look at say 1 dimensional heat equation in a domain, finite domain x, varying between 0 and 1, and for all time we want to get the solution. It is a space time independent solution; so, you require an initial solution here; that is given by this function f of x; in addition, it is a bounded domain problem between 0 and 1; so, you need to prescribe boundary conditions; the boundary conditions could be time dependent; so, that is why we have written them as p of t and q of t.

Now, we have already studied that. We know that the characteristic is t equal to constant; that is how the information propagates.

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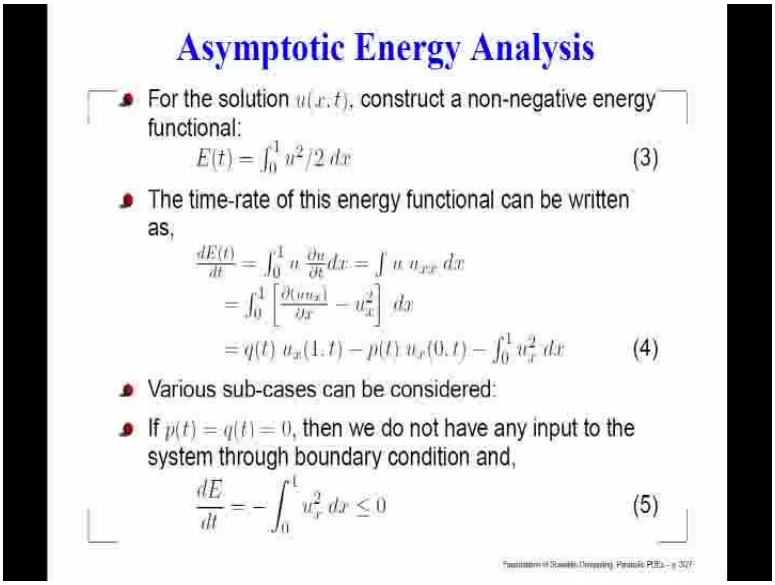

Now, we would like to first discuss, what the property of the physical solution itself is for very large time because if I do not know that then I do not know what I am computing; so, first and foremost, I would like to know what the solution is doing. To understand that we define a quantity - a non-negative, functional, which I represent as energy. So, let us call this capital u of t as u square dx. So, this is a positive function.

We want to find out how this quantity changes with time. So, what you do is you have the definition, differentiate it with respect to time; then you will get u; del u del t is u x x; so I have got this; then I can do it a little jocularly here and that is what I am going to get del x of u u x minus x square. So, if I do that this is exact differential. So, I can integrate it out. With the help of those boundary conditions at x equal to 1 and x equal to 0, the first part gives me these two solutions; whereas, the last path, I keep it as it is; that is minus of u x square dx.

Now, many a times, most of the times when you have thought this, any of these equations, especially the heat equation, you have most of the time told - let us look at something; we give some initial temperature distribution and then see what happens. So, there is nothing from the boundary. So, if I do that this p of t and q of t are 0; then this part is not there; then what happens? I have dE by dt is equal to minus of dx. So, it is a strictly negative quantity. So, what does it say? That if I do not do anything through the boundary, then the energy is going to decay with time.

So, that is a very nice feature of a physical solution. We do not like to consider a physical case where energy grows unbounded. There would be such problems of physical instability, but that is not what we have talked about. Here, we are talking about a benign case where, solutions do not block. What it shows? If I have a rod, I create some kind of a heat; heat distribution at t equal to 0.

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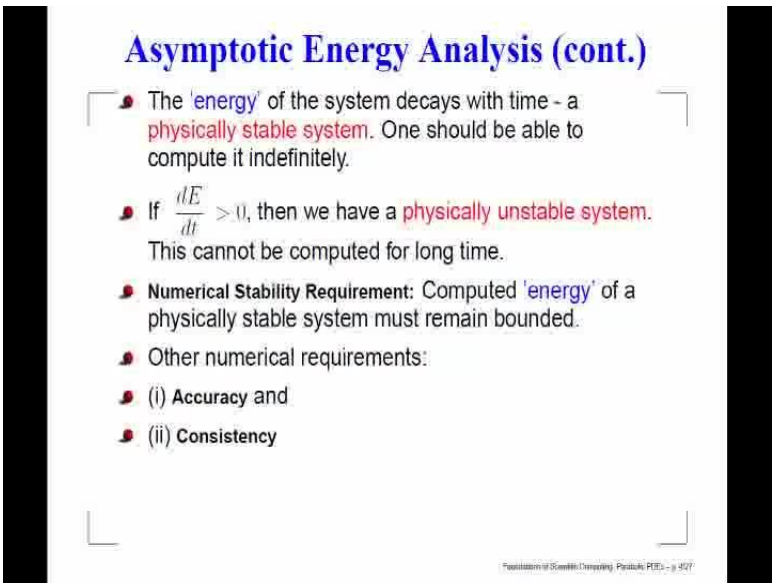

What happens subsequently? It says, where you look at the solution, the energy will continuously keep coming down. However, you can realize by a judicious choice of this function p and t and q and t, we can do lot of interesting things.

So, in the second part, I will show you some interesting thing that we have done very recently. To show you that even for this parabolic equation, heat equation, you can actually generate wave solutions; we will do that but later.

So, we have realized that when we do not have any boundary excitation, the energy of the system decays with time. So, it is a basically physically stable system; then one should be able to compute it indefinitely; there should not be any problem until unless your method is wrong.

If your energy increases with time, then we have physically unstable system; you cannot compute it. You will see that other physical processes will intervene and you will never get a situation where solution goes unbounded because some of the energy has to come.

Various processes like what we studied in case of solitare, you realize that there was this competition between focusing and dispersion - that got us a steady state solution.

So literally speaking, in physics, you will never come across a continuously unstable system; it will be unstable, but then it eventually saturates because of other processes. However, coming to our numerical stability requirement, we need the energy of the physical system to remain bounded; not only that we also need these two quantities which would make sense; we need the solution to be accurate and we need the method to be consistent.

See, we have defined something like this - the energy is going to decay with time. I will introduce you to a method which was introduce it lot of fanfare and people actually used for nearly 30-40 years; even you can go to the search engine and find that there are still some people using it this method called Du Fort-Frankel method.

What happens? It does not follow the principle of the physical solution. Such methods are called inconsistent methods. So, we must be concerned about consistency. So, I think it is a nice place to stop here.