Foundation of Scientific Computing

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Module No. # 01

Lecture No. # 10

Today, we will begin our discussion by talking about dispersion of wave phenomena specifically, with the example of surface gravity waves as we had briefly touched upon in the last class. We will talk further more on gravity waves forming over layers of variable depth and in the context we will bring about some properties of wave propagation namely, that is known in optics but, with the help of a wave system from mechanical sciences.

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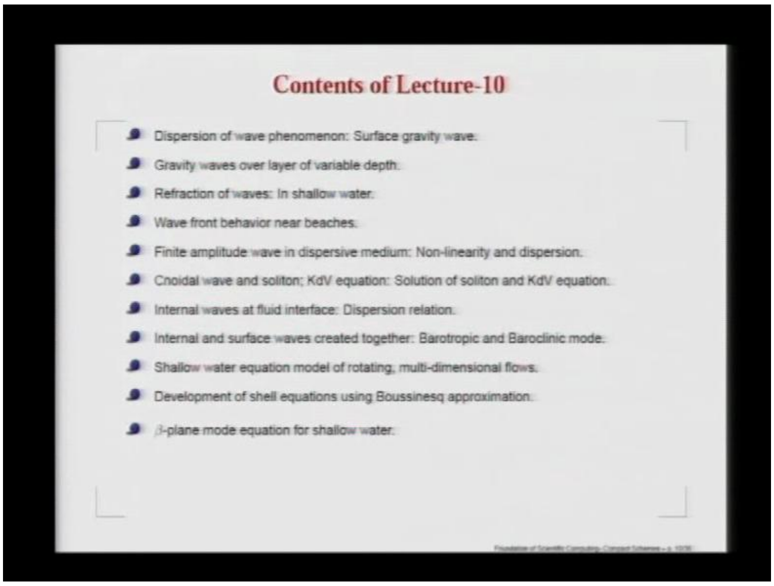

We are going to talk about refraction of waves that we see forming in shallow water that is given in terms of wave fronts and as noticed near beaches. We will notice that why always the wave approaches at right angle to the beach and we will also notice, how the waves turn around the islands that is also an example of refraction of waves.

So far we have been talking about waves of small amplitude so that the linearized analysis can work. But today, we will start this course talking about finite amplitude waves in dispersive media. Here, we are going to see two competing physical mechanism: One is the nonlinearity, which tries to amplify the amplitude of the waves and in contrast, we have already noticed that dispersion tries to reduce the individual wave number or frequency content.

We are going to talk about finite amplitude waves. As an example, we will see there are the possibilities of forming Cnoidal wave or Solitons, whose governing equations is given by Korteweg De Vries equation or KDV equation. Having discussed about how this cnoidal waves or solitons are created as equilibrium between nonlinearity and dispersion will also talk about, how we solve the problems involving solitons. This will require some exposure how KDV equations are solved.

Now coming back to our discussion on waves, we will finally leave behind the surface gravity waves and start talking about internal waves. These internal waves are formed at fluid interface and we will try to describe its dispersion relation. We will also talk about complex wave systems, where internal and surface waves can be formed together and this is very important in the context of oceanography and atmospheric science. We will see that presence of this kind of waves on the surface as well as on the interior gives rise to two different modes namely, the Barotropic and the Baroclinic modes, we will talk about them.

In the context of geophysical fluid dynamics will develop shallow water equation as form over the ocean or atmosphere that is constituted by rotating multi dimensional flows. In the process, we will develop shell equations using Boussinesq approximation for density variation or the heat transfer problems. We will finally, talk about a special simplification of the shell equation based on some variation with the latitude which is called as the beta plane mode. So, we will conclude our discussion today with that topic.

We have been talking about effect of dispersion as evidenced in various wave phenomena and looking at surface gravity wave on water, we are noted that not all the time you would have waves of single wave number and circular frequency and then, we decided to define an arbitrary disturbance in terms of this phase function theta.

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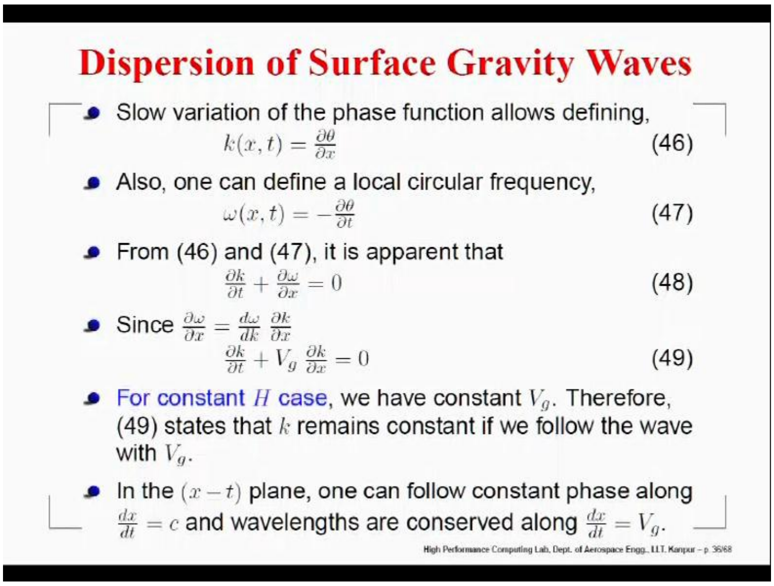

If we differentiate it with respect to space, we get the wave number. If we differentiate it with respect to time then, you get minus of that circular frequency that leads us to this identity. If we are looking at homogenous medium means where properties do not change with x then, what will happen is omega as a function of k would be really ordinary function, so that you can define an ordinary derivative here, so that this del omega del x could be written in terms of chain rule.

For such homogeneous case that has been H is constant; we would have constant Vg. This equation told us that if we fix our attention on a fixed k then, we ought to be moving at the constant group velocity. So in the xt plane, this showed us a couple of types of lines. The thin line corresponds to constant phase line, whereas the thick line corresponds to constant group velocity line and this was the scenario.

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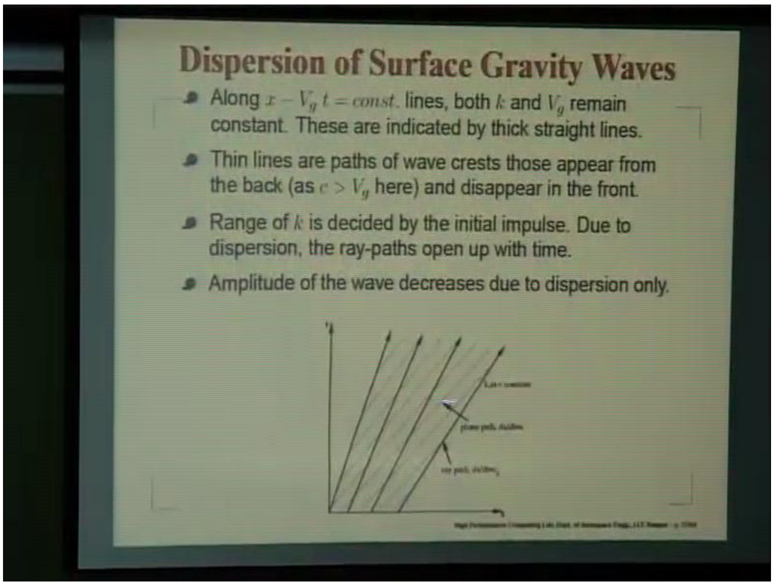

Let us say at t equal to 0 and as time progress because of dispersions it opens up and that is what we are seeing here, with time these waves are seen over a larger spatial dimension and the wave appears from the back and disappears at the front. This is one of the aspects of dispersion of surface gravity wave.

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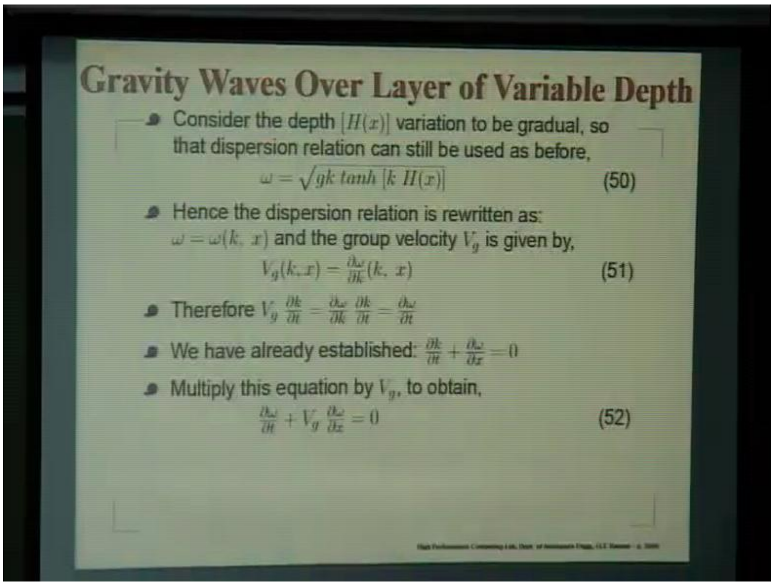

Now, if we look at gravity waves forming over variable depth fluid then, H itself would be a function of x. Let us also assume that this variation of depth with x is very gradual so that we still can use the same kind of dispersion relation that we had for constant H.

Only the fact is you will have to take the local value of the depth; that is an assumption and which will adopt it here.

So, once we have the dispersion relation, we can calculate a group velocity but, as we have said it is going to be a variable depth case. So, what will happen here? The Vg will be function of k as well as x, because it is an inhomogeneous case, your depth is changing and of course, the group velocity is also going to be a function of x. So, this is the difference from the previous case just now we looked at.

Now, If I look at this product Vg times del k del t by the chain rule, we can see this works out to del omega del t. We have already established this continuity relation between k and omega. Now, what we do is multiply this equation by Vg then the first term Vg times del k del t is del omega del t and this is what we are getting.

Now, what is happening in this case when you are tracking the waves over liquid of variable depth? What you notice is that omega remains constant here as a function of x and t. If we track it with the constant Vg but, the Vg is not constant itself because that is inhomogeneous, it changes from location to location.

So, what happened is, it is a tricky bit if you are trying to track a constant circular frequency; you will also have to change your speed of observation, you will have to track it. This is not very straight forward as it was the previous case, where we just simply had to track the crest and then we could have seen what was happening.

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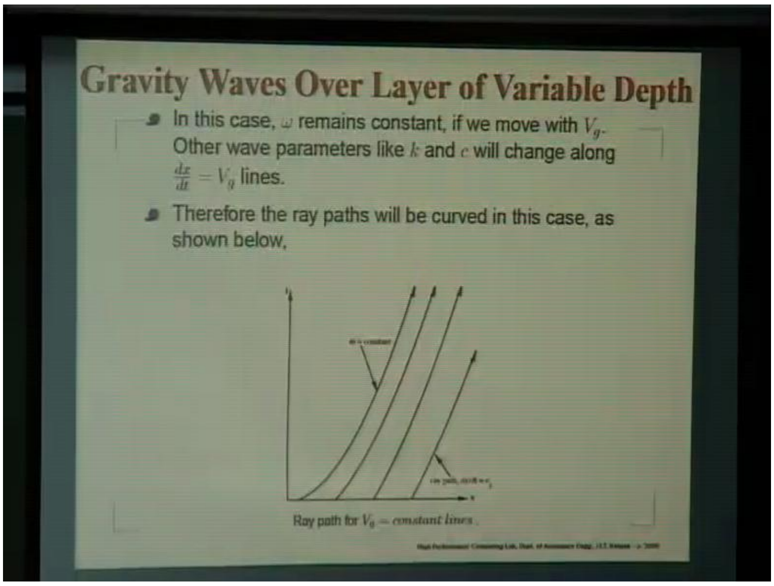

So, what happens is in this case, omega remain constant if we move with the variable Vg. Another wave parameter like k and c, they will all change with x. The ray path now will not necessarily be straight, they will be all curved. What we are seeing then this curve path corresponds to omega equal to constant, whereas the ray path - these are the path which defines a group velocity by its slope. So, you can see it keeps changing with the position.

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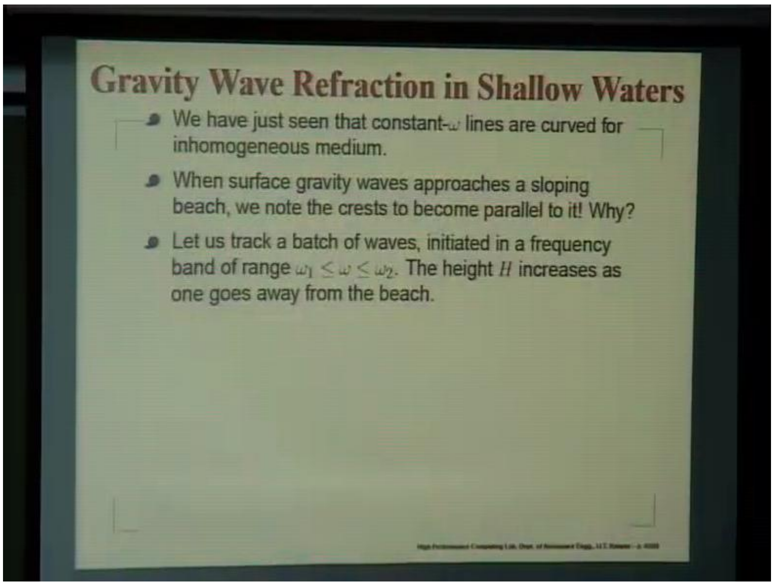

Now, this was some refraction, when we look at gravity waves forming over shallow water. We have seen that if we are looking at inhomogeneous medium then constant omega lines are curved. Now, if I approach a beach - sloping beach - so that the depth is increasing as I go away from the beach then, we need to find out why the crest eventually becomes parallel? What exactly I mean is given here; that if I look at some waves far away from the beach, they would be at an angle.

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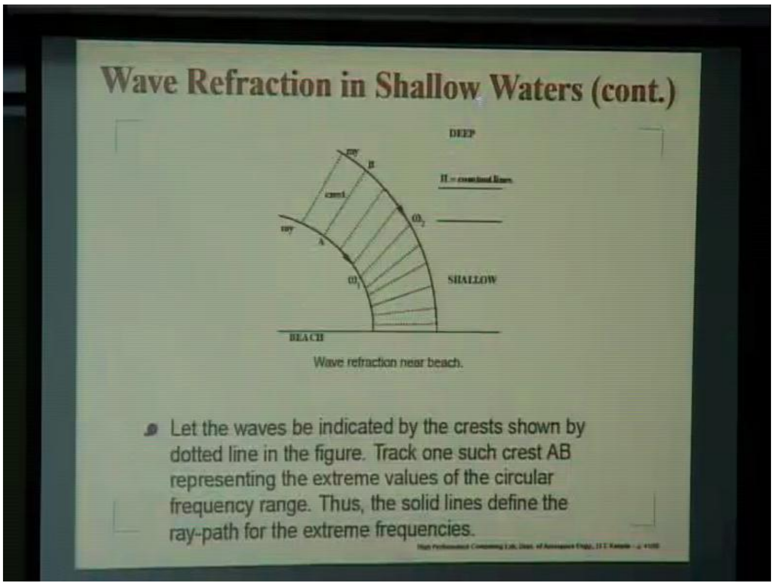

So, these dotted lines indicate the locations of crest. As this crest keep approaching towards the beach, you notice that they so move around in clockwise manner and when it hit the beach, it actually become parallel to the beach. So this is something we can define it in terms of dispersion property. This sort of phenomena, where the waves turn in an inhomogeneous media is called refraction - you have seen it in optics.

So, here also what we are seeing in a mechanical system a wave refraction and what happens is that suppose, I am looking at a batch of waves characterized by frequency range between omega 1 to omega 2 then, these two curve lines indicate those limit of omega. Let us say the inner one corresponds to omega 1, the outer ray path actually corresponds to omega 2 and as I told you the crests are given by the dotted line.

Now, let us look at what is happening. As we have said it is a slopping beach, so as we go away H increases. So near the beach, we have the shallow part; as we go out, we are reaching the deep part. So, what is going to happen? The deeper part, if you recall the expression for C; C would be omega by k then, you will see that higher the value of H, you will get a higher speed. So, what happens is if I look at an individual crest like AB, the point B will move faster compared to point A.

So, what will happen as a consequence that the point B will move at longer distance compared to A and slowly, it will turn in a clockwise manner and of course, when you reach very near vicinity of the beach it becomes parallel. So of course, both of them - both the extremities - move with the same speed and once it becomes parallel it remains parallel. In fact I have not shown it but, you can take a look at similar thing happening in a say hypothetical case.

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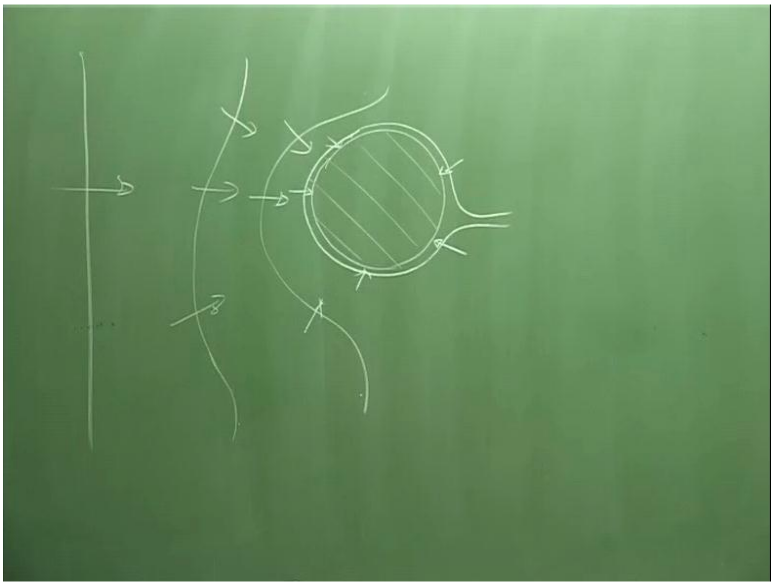

So, let us say this is some kind of an island and the waves are coming like this. What will happen? As you can see that again, if we consider that the depth is increasing in the radial direction. Then, what will happen here is that as it comes closer, it will start bending and when you are in the very near vicinity of the beach, you are going to see the crest going like this. So what will happen here? You would be getting the crest directions like this and here, you can see that this is going to happen.

So, this is a somewhat of a very counter intuitive situation that even though there is a mean convection from left to right but, at the back of the island you would see again the waves will approach towards the beach. So, this is a very interesting phenomenon where you can see how refraction can explain some commonly observed phenomena.

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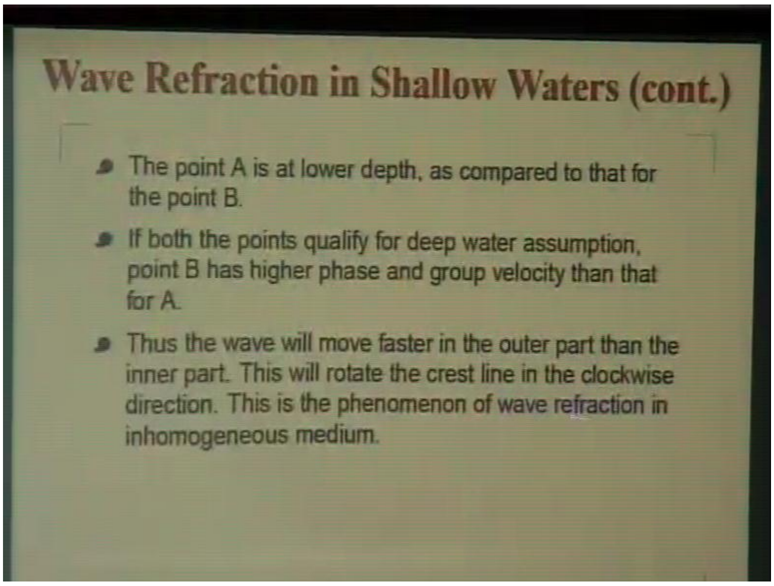

Now this is what we have already explained; that in the outer part, we will have a higher velocity. The waves will move faster there compare to the inner part. This rotates the crest line in the clockwise direction and this is the phenomena of wave refraction for inhomogeneous medium.

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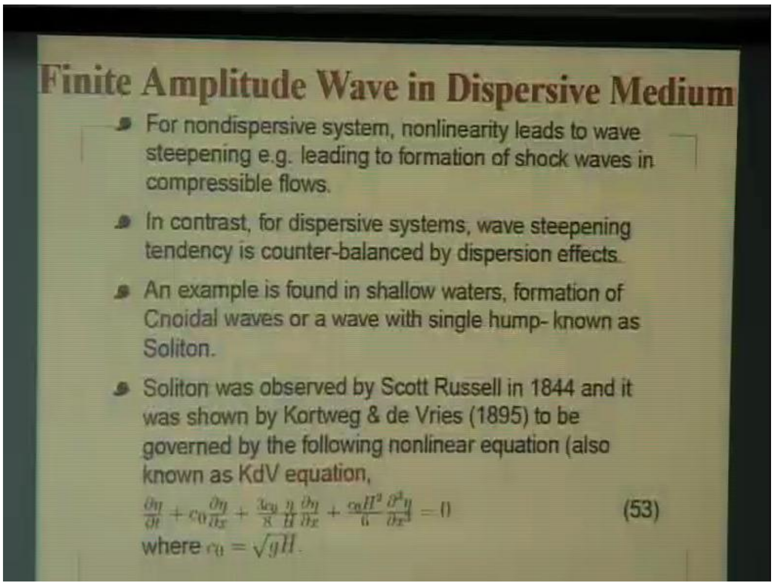

Now, we come to another aspect of wave motion. This is related that what happens when the wave amplitude is finite? When wave amplitude is finite in a dispersive medium then, we can expect to see some kind of a nonlinear fact. Let us now try to understand what nonlinearity does that if I start off with a wave front like this - waves like this – then, what will happen is if I create some kind of a disturbance locally then what happens to this disturbance? We are going to see the same thing that we have just now talked about.

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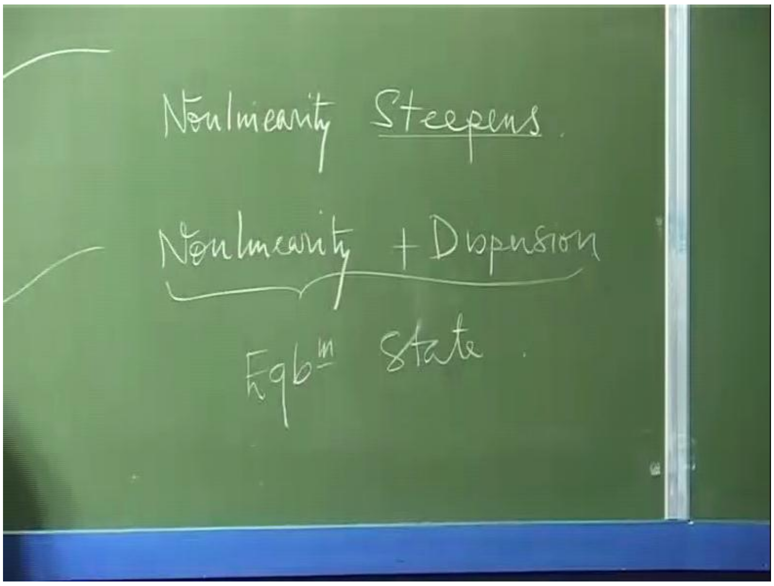

The speed of propagation would be given by a local wave speed times the convection speed of the wave I mean, this part we had not talked about but, this you already know; this is something like your Doppler effect. If I have a motion of the medium so that motion also adds up to the wave speed and gives you the net resultant wave speed.

What you are going to see that with time, the presence of the nonlinearity is going to make this part of the wave which has a positive displacement; there the velocity would be what? That C prime as we have seen, it is a directly proportional to square root of H. We will have a higher velocity compared to this part. So what happens is, with the passage of time, you are going to see something like this (Refer Slide Time: 16:10).

So this part will move faster compared to this, so there would be a kind of a steepening of the wave. This is the normal attribute of nonlinearity; nonlinearity steepens by itself. Now, what happens when we have nonlinearity as well as dispersion - these are just the opposite end of the spectrum. Nonlinearity tries to steepen and - we have talked about in couple of last classes that - dispersion tries to disperse it, the amplitude comes down and so what can happen is; this happens without dispersion and what you would find that nonlinearity plus dispersion actually can take us to an equilibrium state.

The equilibrium would be that nonlinearity would try to steepen; dispersion will try to attenuate and thereby, we can get a kind of purely periodic behavior. Whenever you see that in shallow water they have called the Cnoidal waves. I will probably be able to show you there is the example taken it from the web here, what you are seeing is waves forming of the coast. This is in the shallow water.

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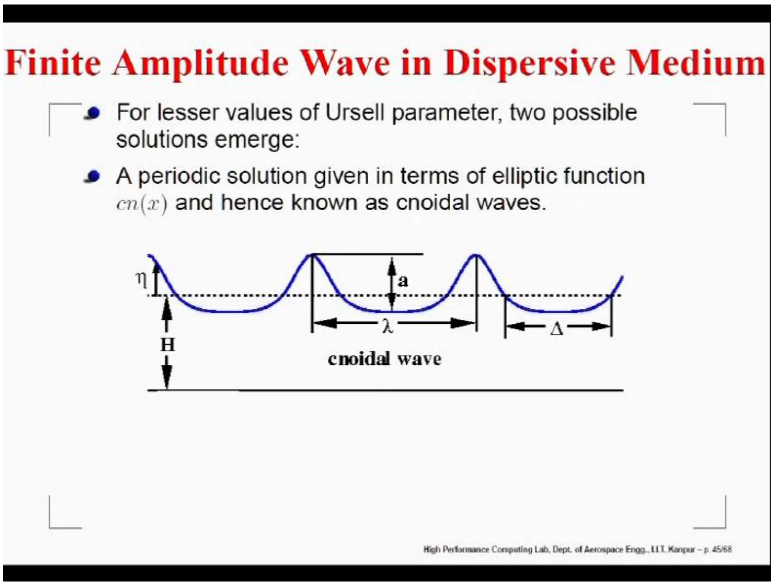

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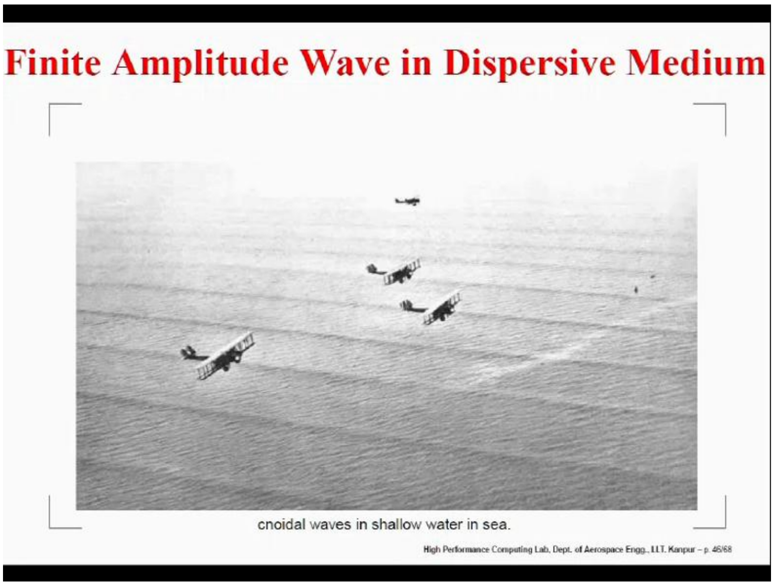

What you are noticing that these are not sinusoidal, these are actually of this type (refer Slide Time: 18:08), so you have a steepened crest and a flattened trough; these are not pure sinusoidal waves. This is what you actually see nature forming and this is an example of what we call as the cnoidal wave.

Here, we expect to get some kind of a perfect balance and when these waves have variable wavelength and this wavelength could be very large compared to the depth.

Cnoidal waves are typical waves where the wavelength may be 5 or more times the depth, the lambda is greater than 5H. That is one kind of waves where you see that.

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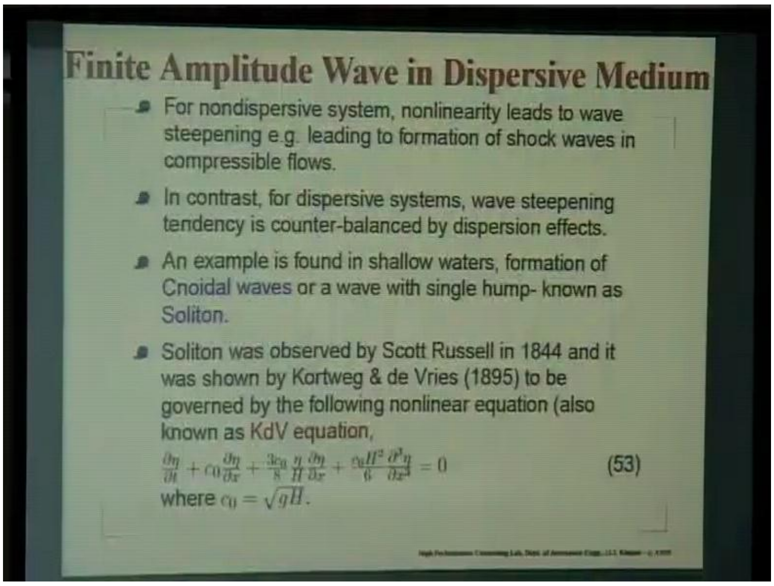

Suppose, the wavelength of this cnoidal wave is very large and you end up getting only a single wave that is what is called as Soliton. Soliton was accidently observed by Scott Russell, when he was noticing the behavior of the water in a canal, what he found that having dropped a big object on the water, a wave was created, which was exactly like solitary wave. I think I have a picture here for you; this will be like this, so you get a wave like this and this keeps moving.

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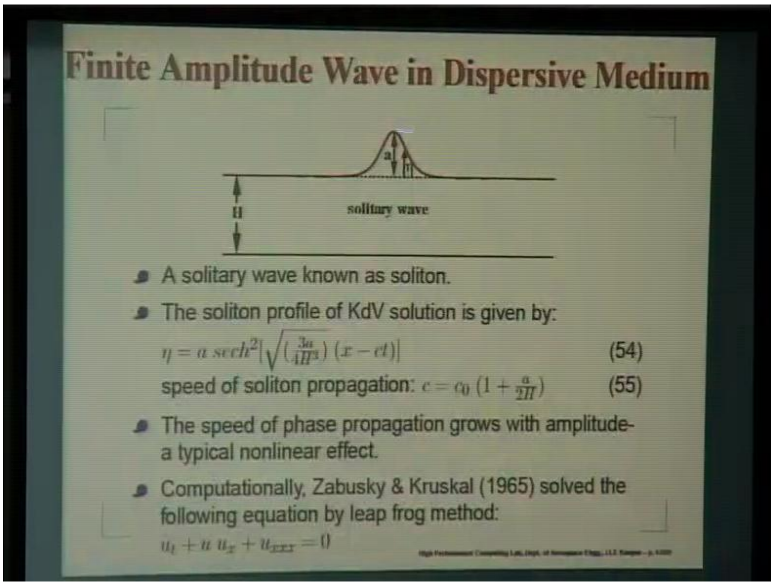

In fact, you know these were the times in UK where they were lot of interest in transporting goods by canals. Actually that was one of the reason at the early network of canals are made in UK. This interested him so much that he followed it on a horse back and he could see this solitary wave was there for almost about couple of kilometers.

So of course, Scott Russell went back and did some experiments in the lab and seen it. However, its theoretical explanation came much later with the publication of this paper by Korteweg and De Vries. This was basically thesis of De Vries and they established an equation for these phenomena which is now called as the KDV equation. There are initials of those two gentlemen. What you notice in this equation, which is called the KDV equation, is the first two terms are quite familiar with us. This is your one de convection equation and this is something your nonlinear term.

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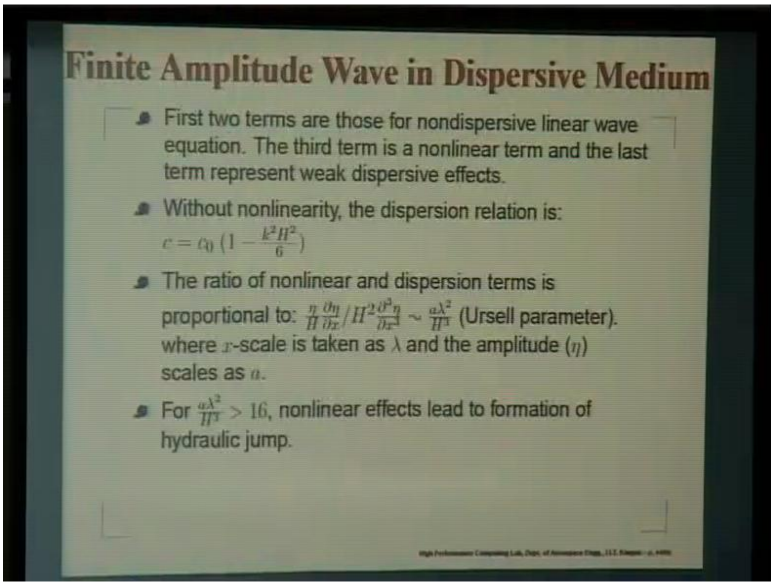

This the third term represents the nonlinearity because there is a pears in eta square and the last term or the fourth term in this equation is due to some dispersion. What do we mean by dispersion? By now we are familiar, dispersion comes about as a consequence of odd derivative terms; the even derivative terms in the differential equation gives you dissipation. So, it gives you dissipation.

So, any odd derivative here in this case, you can see this is a third derivative; first derivative here itself can give you a dispersion relation. So, that gets reinforced by the presence of the third term. However of course, you cannot write down the dispersion relation that easily because of the nonlinearity. So if you knock off the nonlinear term the dispersion relation looks like this (Refer Slide Time: 21:54).

This is your first two terms; if I write down the dispersion relation we have obtained for surface gravity wave, expand it in a power series and just return the first two terms that is what we get. Now, I told you that this kind of periodic behavior is observed when the nonlinearity and the dispersions play opposite roles. So, what happens is, we try to figure out what this ratio of this nonlinear and dispersion terms are.

The nonlinear terms is eta by h del eta del x and this is the third derivative dispersive term, the ratio of this is written like this. So, what we have done? H is the vertical direction but, x let us say length scale. We associate it with lambda and eta, the wave amplitude which we can associate with say a. Then, you can look at an order of magnitude analysis this will give you something like a times a and there is a downstairs so, that is why you get only a.

Here this two coming together will give you H cube and then you have these terms. There is H square and there is this. So overall you are going to get a parameter which is called the ursell parameter given by a lambda square by H cube.

Now, when this parameter is greater than 16, implying the nonlinearity is quite strong then, you get what is known as a hydraulic jump. For lower values of Ursell parameter two possible solutions are seen to occur, a periodic solution in terms of Jacobi elliptic function called C n of x.

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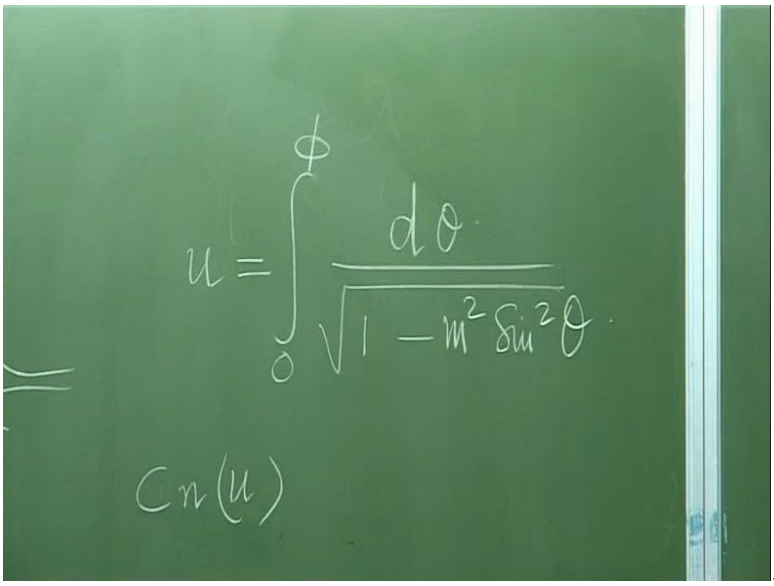

Let me, tell you what this is. Suppose, I write u as this Jacobian elliptic integral equation of course, time. So, if I have this integral equation then C n of u would be what we are calling as the cnoidal wave. There is enough material for one can look at - one can convert this equation that I have written in the previous slide here. We can convert it into an ODE and we can solve it. So, that is possible but, people also solve this as PDEs.

What happens is, this cnoidal waves are characterized by this wavelength lambda. The height of the crest from the bottom of the bed and kind of length scale delta which represents the negative elevation of the wave and a is of course, trough to crest amplitude, so this what we talk about.

Now, if I take this cnoidal wave and make its wavelength go to infinity then, I would get what is called as solitary wave and this is also known as the soliton. Soliton profile of KDV equation is given by this is second hyperbolic square. As you can see that this really goes like a wave because the phase is x minus c t, it moves at a constant wave speed C. However, speed of this soliton is a function of amplitude.

So, usual surface gravity wave that we have studied so far corresponds to low amplitude phenomena but, here specifically the finite amplitude comes into the play in defining the wave property and you can see that speed of propagation of this crest is a linear function of a.

Now, lots of work has been done since mid 60s in solving the PDE so that KDV equation can be simplified in this particular form, as you can see that this is a PDE in x and t. These first two terms corresponds to your burgers equation kind of form and this is a third derivative which gives it a dispersion effect. Basically, you are seeing a competition between nonlinearity and dispersion here.

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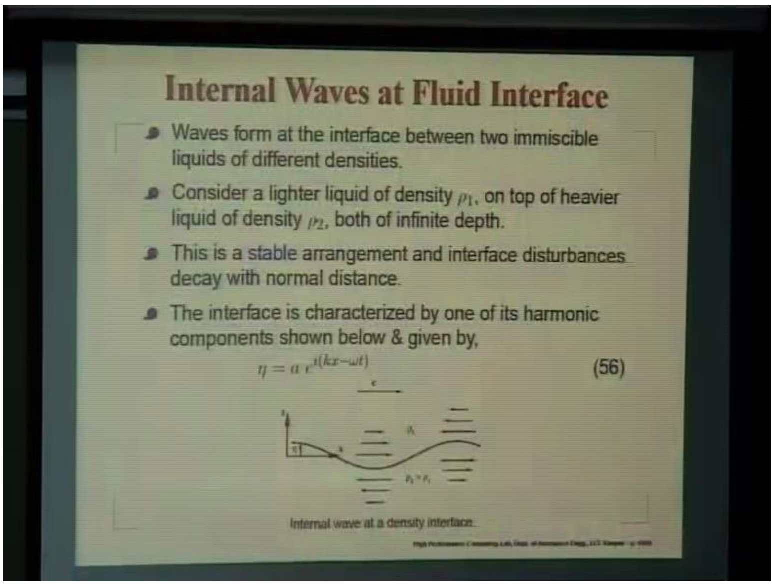

There are lots and lots of papers people still keep publishing days. These are very important areas as I told you in optical communication, soliton pulses are used for signal propagation. So, that is about another example of effect of dispersion, how nonlinearity is balanced by dispersion.

Let us now look at waves that could form in the interior. So far we have been talking about on the surface; we were talking about surface gravity waves. Now, let us look at what happens when waves are created at the fluid interface in the interior. Consider a lighter liquid of density rho 1, which is on top of a heavier fluid of density rho 2 and both these medium are of infinite direction across the normal of the interface. You realize that this is a stable arrangement because, lighter liquid is resting on heavier, if you would have done the other way that would be an unstable configuration that would lead to instability on which we are not going to discuss but, it can happen you can see it near estuaries. You could see the fresh water and the salt water can have kind of layered formation and you can see this kind of scenario occurring here.

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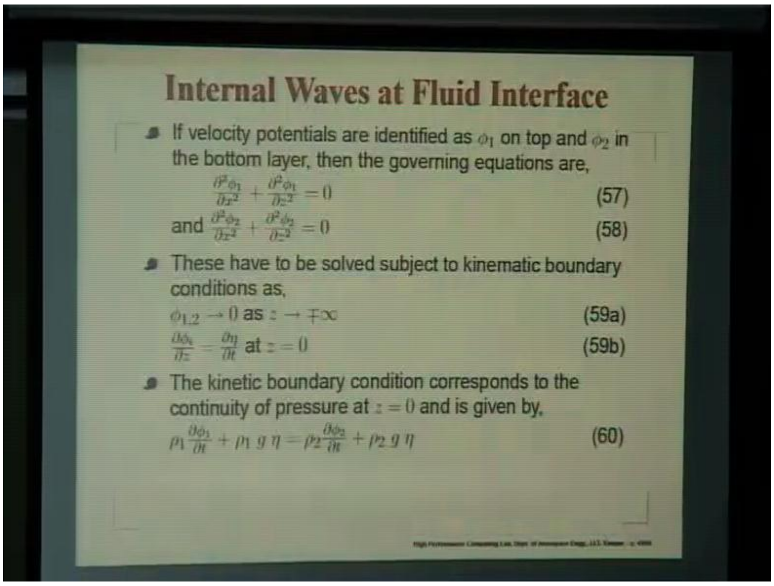

Now if I define the interface in terms of a harmonic component, a times e to the power i k x minus omega t then, we can look at the following development considering the behavior being irrotational. So, we can again define velocity potentials which are phi 1 on top and phi 2 at the bottom layer the governing equations are again the Laplace equation and you need to solve these equations subject to these two kinematic boundary conditions, if you are going far away from the interface. This solution should decay this displacement - interface displacement - should decay.

At the interface, which we will apply at the mean interface, because of the linearity of the problem that fluid velocity is given by the interface displacement time rate. So, this was what we have already done it and the corresponding kinetic boundary condition or dynamic boundary condition would come about from continuity of pressure. If we exclude any role for surface tension, the pressure must be continuous at the interface and again, we will be applying it at the mean interface z equal to 0 and this is what we get from the unsteady Bernoulli’s equation which we have done it before.

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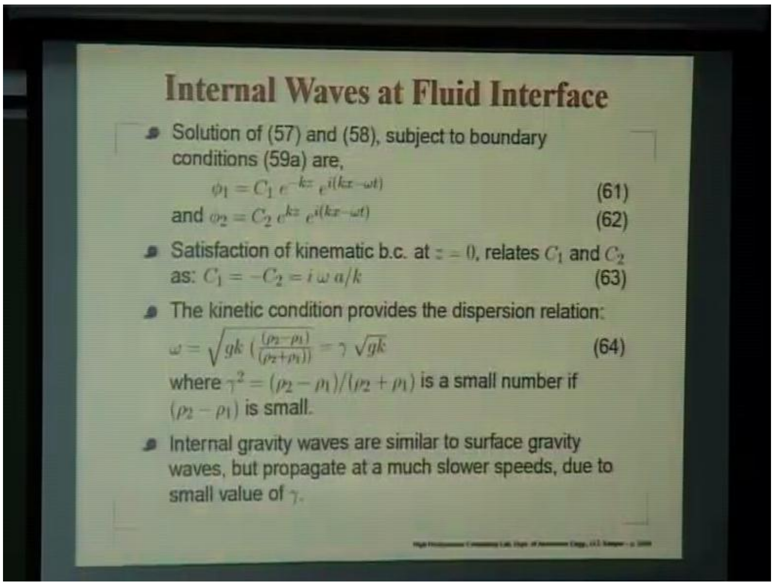

Now, to satisfy these boundary conditions that phi1, phi2 goes to 0 as z goes to plus minus infinity then, we should have this two admissible solutions, only for the top layer we should have e to the power minus kz because that is where z is positive and for the bottom layer z is negative, so we should keep the admissible path is C2 times e to the power kz.

Now, the third kinematic boundary condition which is del phi del z is related to del eta del t at z equal to 0 would help us relating this constant C1 and C2 and that is what we get, so C1 equal to minus C2 that is i omega a by k. Finally what you need to do is, go to this kinetic condition the Bernoulli’s equations substitute these two solutions phi1 and phi2 with this C1 and C2 you will get this dispersion relation, this we have done it before also.

What we had seen before for a single medium surface gravity wave only one of the layers was missing. Of course, this gave us this condition square root of gk that we have obtained for d part of wave but, here what is happening additionally because of this density stratification. You are seeing this factor gamma - gamma square - is this and this is going to be a very small number, if this difference in the density is very small and this may appear to be a trivial issue but it was not so, people who use to get into the river from the sea and then, all of the sudden they will experience that their ship is experiencing very large quantum of drag. Why does it happen? Well you can see that this is because the smallness of this parameter of gamma because, if I put in the same amount of energy because of this quantity being very small rho 2 minus rho 1 divided by rho 2 plus rho.

What will happen? You will create a wave of very large amplitude for the same amount of energy that is put in into the system and internal wave amplitude would be far in excess compare to surface gravity wave and this was a mystifying thing for the seafarers for a long time till ((Bearkans)) came and explained this phenomena.

You can also see because of smallness of the number of gamma, the phase speeds are also going to be much smaller because, we will have the phase PDEs omega by k. So, that will be gamma times square root of g by k. It is a factor of gamma that reduces the phase p.

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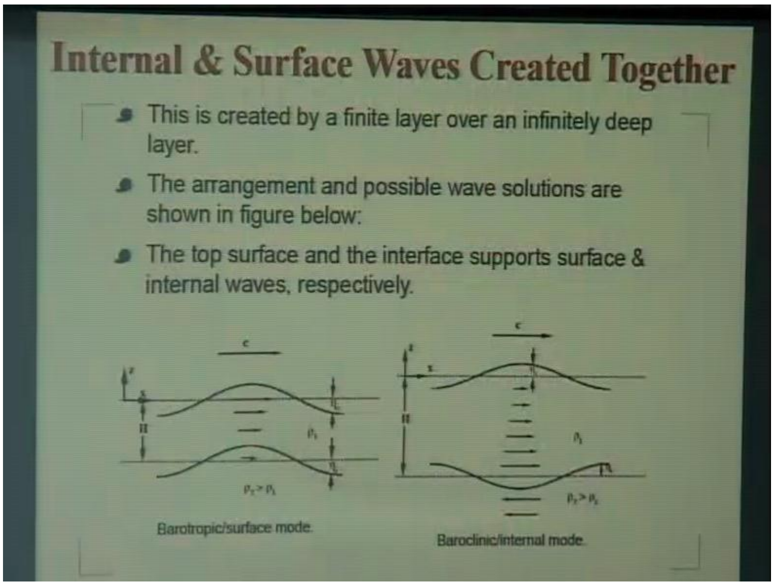

Now, let us look at another scenario where I would have a heavier liquid of infinite depth over which I have a lighter liquid of finite depth. So, the rho1 has a finite depth of H whereas, this phase 2 has infinite depth and what happens is, you can get two types of solutions which we have shown here. One is of course, the surface wave that is given by eta s and there is internal wave that will define as eta i. What happens is, in the first case we see that the surface gravity wave and the internal wave they are in phase and this is what is called as surface mode or Barotropic mode.

I would not go into it but, this has got something to the weather prediction terminology where the pressure and density goes in phase, that is why it is called Barotropic mode, whereas if you look at the other thing where the displacements are opposite to each other, when you have that, this is what we call as the internal mode or Baroclinic mode. We also called this Barotropic mode as sinus mode because this goes like a sinusoid, whereas this Baroclinic mode is called varicose mode, so it is like a tube of liquid being conducted and you get a local dilation of the radius and that causes that varicose nature of the geometry.

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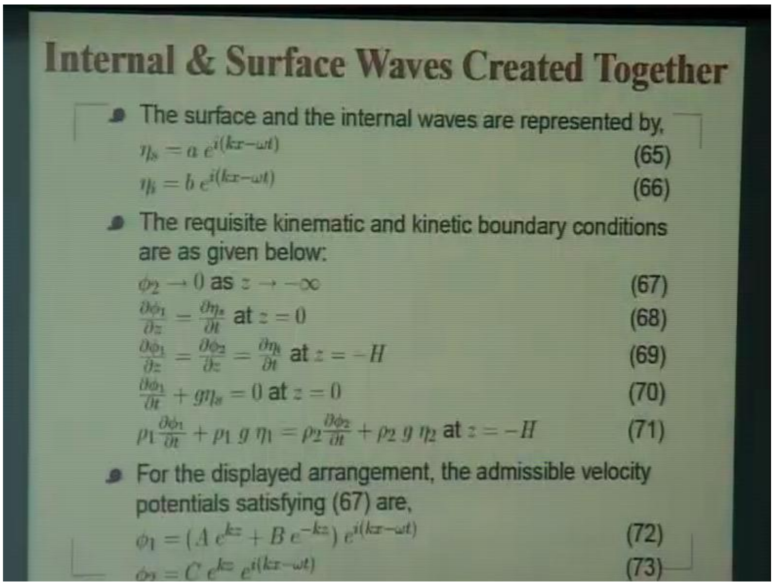

You can actually see when you turn on the tap sometimes you see that the water falls like a sinusoidal and sometimes, you will see that the width of the water column keeps changing with x. So, it is fat then it thins down again it becomes fat. You see those modes almost every day probably, if you are careful to look at. Now in this kind of a scenario, if we define the surface under internal displacement in terms of the amplitude a, let us put in the same kind of solution, we will find out how omega and k related.

Now once again, you would have to be now satisfying boundary conditions for the lower liquid, you will have to say that disturbance goes to 0 as you go far, far down. So, z going to minus infinity phi 2 should go to 0 at the interface where? At the interface of the top that is, where you should have this kinematic condition del phi 1 del z should be equal to del eta s del t. At the internal wave - where you are getting the internal wave -

that is where you should have del phi 1 del z equal to del phi 2 del z equal to del eta i del t.

So, this is what we are going to see at z equal to minus H - the interface of the internal wave. Finally, we will have the pressure prescribed in the surface that is this Bernoulli's equation. If we put the day term is equal to 0, this is what we get and this is the continuity of pressure at the interface of the internal wave, the last one is 71.

For the arrangement that we have seen, the velocity potential must have this form phi1 is unbounded sorry phi1 is bounded between 0 to minus H I suppose, let us get it, yeah Right.

So phi1 corresponds to this phase, so it goes from z equal to 0 to z equal to minus H. So, that is why the phi1 solution should have both the exponentials. So this is finite z case that is what you have both the component being present there.

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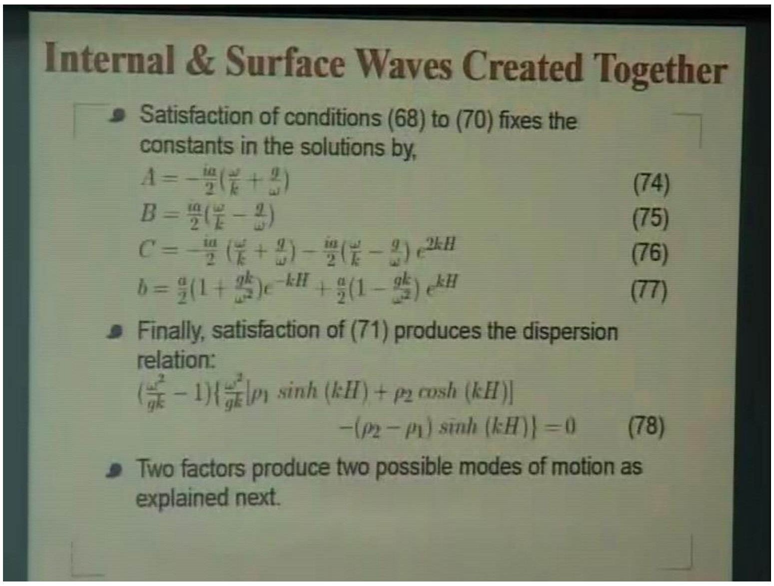

Whereas, phi2 has infinite depth going down, so you just only keep e to the power plus kz, because z here is negative. We have these two admissible solutions and we can satisfy all those conditions that we have lead down; those kinematic conditions would give you four equations A, B, C and this b; b is the amplitude of the internal displacement - wave displacement. Whereas A is, lower case a is the surface wave amplitude. Once we plug that in, we get this four equations this gives you a relation between B and A the last equation. While satisfaction of the Bernoulli's equation, the kinetic boundary condition provides us with a dispersion relation.

Now, the dispersion relation is noted to have two products. One is this, which is familiar to us that we have seen for surface gravity wave omega equal to square root of g k, whereas this one is new, because we have an internal wave that gives you the second factor.

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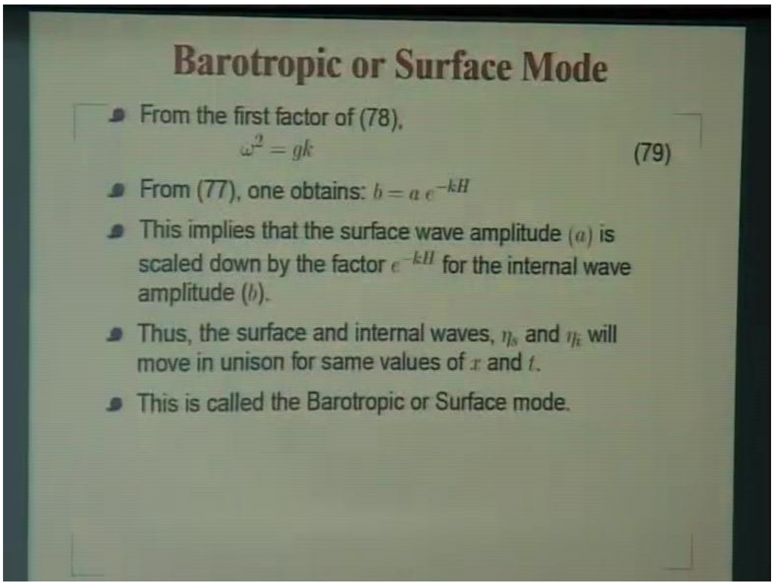

From the first factor as I told you that we have omega square g k and then, we have the relationship between B and A. If I put omega square equal to g k, the second part drops out and this becomes a into e to the power minus kH.

So, this tells you a very interesting thing that the internal wave amplitude is actually the surface wave amplitude times e to the power minus kH, so it scales down. You may have larger wave amplitude on the surface, as you look at the interface at the internal wave that amplitude actually comes down by this factor e to the power minus kH. You also note that they are of the same sign, so eta s and eta i will move together and this is what we call as the Barotropic or the surface mode.

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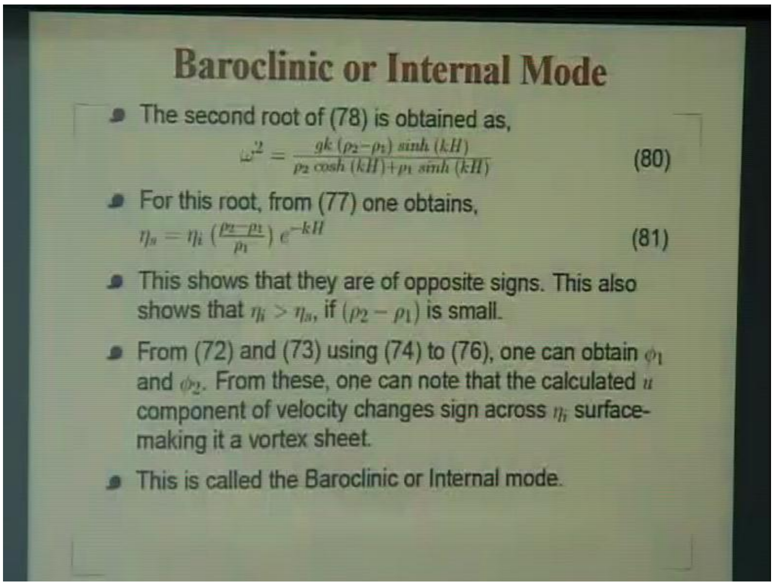

Now, look at the second factor in the dispersion relation that gives you omega square equal to this and substitute this in that kinetic boundary condition that we had 77 and that would give us this equation 81 which relates the surface elevation with the internal wave elevation or the function of this factor rho2 minus rho1 by rho.

They are of course, of opposite sign why because rho2 is - I think, I have lost a sign somewhere there should be a minus sign over here. I think it is a mistake. What we will find that, there is an opposite sign; please correct it in your notes, you can check it for yourself, how I have missed it up.

So, if the density differential is small then, we can see that this eta i is going to be much larger compared to eta s. That is, what we also talked about that internal wave amplitudes are always going to be greater than that side. I mean, As and Bs are the amplitude but eta n and eta s are including that factor that multiply the face part also. So having obtained this, we can also calculate phi1 and phi2 and from there, we can calculate the u velocity and what we notice that the velocity changes sign across internal wave interface.

So it basically tells you it is the internal wave are something like vortex sheet because, on top you have velocity going in one direction, bottom in other direction. So, that is an attribute of a vortex sheet and whenever you have Baroclinic mode you do see that happening.

Now, I come to the last part of this discussion on waves, because what we have seen so far that try to develop tools for scientific computing purposes. We need to have model equations. One of the model equations was of course, that D'Alembert's solution of the wave equation but, there you have the problem of the condition being that you have non-attenuating, non-dispersing wave solution. We want to come out with an alternative model where we can actually see the effect of dispersion that comes about in what is called as a shallow water equation?

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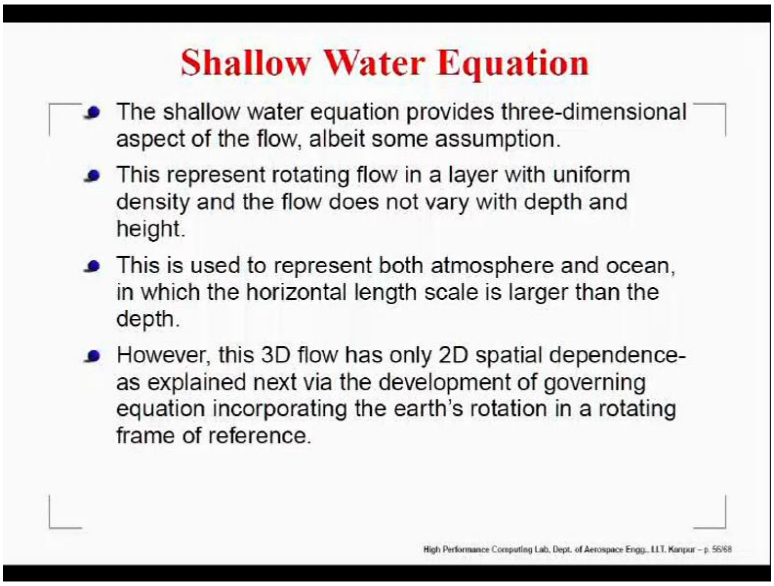

This is very important equation because this allows you to investigate the three-dimensionality of the flow problem. Of course, we make some assumptions but despite that its pretty good model equation. This actually represents rotating, flows in a layer with uniform density and the flow actually does not vary with depth and height. Where does it apply? It applies to both the atmosphere as well as the ocean phase of atmosphere.

What we are talking about has practical utility and we are talking about flows where horizontal link scales are much larger compared to the depth. You all know the height of atmosphere is very limited, ocean is also very limited and depth is always small, whereas, if you look at circum navigating the earth that will go into 1000 of kilometer, whereas this depth could be less than 100 kilometer or so far even for atmosphere for ocean it is of course, much lower.

Despite the three dimensionality of the flow problem, the development in the theory is such that you end up with a two dimensional variation and that is of great interest for us to really look at this shallow water wave equation. Here, unlike what we have done for surface gravity wave, we have neglected earth's rotation, because we are looking at local effects because we did not talk about large horizontal scales like what we are talking about here. Here what happens, because of taking the large horizontal link scales, we will be talking about the effect of earth's rotation by Coriolis term.

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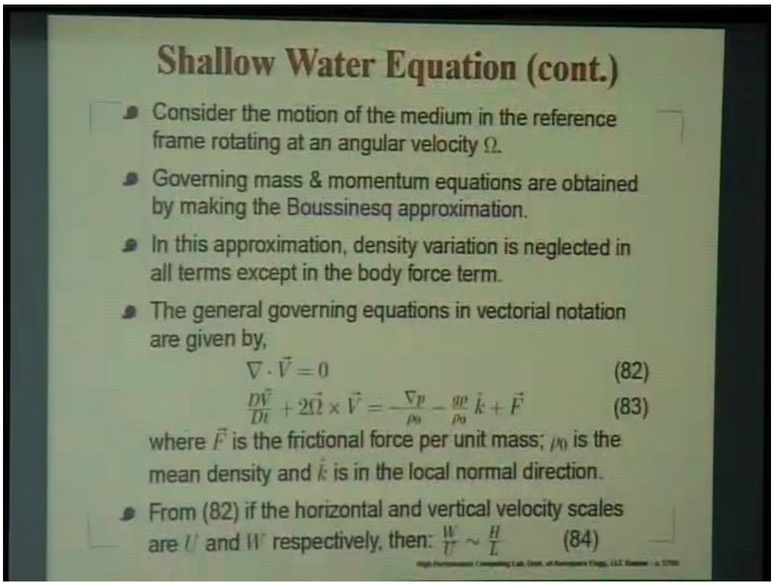

So, come back to your conditions of mechanics once again and we look at the governing equations stating the mass and momentum conservation. This could be obtained via Boussinesq approximation. What is Boussinesq approximation? In this approximation you neglect density variation everywhere except the body force term.

What happens is as a consequence, if I neglect the density variation mass conservation gives us this. So, this is like what we have for incompressible flow del dot v equal to 0. Whereas, the momentum conservation equation, we are writing it in a rotating frame of reference that is why you have this Coriolis force term to omega across v. So this omega is of course, the rotation above the North Pole for the earth. That is a kind of a constant, we will talk about its value and density is taken at some kind of phenomenal value that we defined it by rho naught.

However, body force would take care of this variation of density, this is what the essence of Boussinesq approximation, that in the body force term we allow the variation of density with height so thereby, will be height or latitude and thereby we get this additional term, if rho is equal to rho naught. This we can factor it out and it does not come into the picture.

In addition in 83, you look at the last term. We did not write the discussed terms but, instead, we write something like a vector F which represents frictional force per unit mass and this k H of course, is the local normal direction that is the direction of gravity. If I now look at this equation, so if I am talking about a globe then, I have a horizontal plane that is wrapping around the globe, so that is your horizontal plane, whereas vertical is the normal to the surface of the earth.

So, if I look at that kind of horizontal and vertical velocity scale, from this equation I can see that the vertical velocity divided by the horizontal velocity would be like this. Since, H is much smaller compare to L; L is the longitudinal scale and H is the vertical scale. So, you can see that W is much smaller compared to U.

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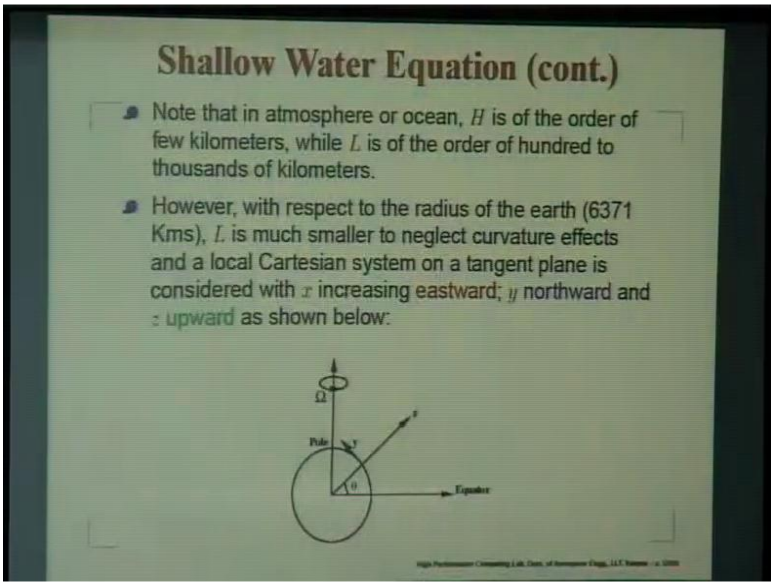

This is the schematic of what you see; you have the earth rotating about the pole and as I told you that H is of the order of few kilometers for oceans, for atmosphere you do not need to really go about even what about 11 kilometers or 20 kilometers where you can account for about 80 percent to 90 percent of the mass of air, you do not really need to go very far. Although for aerospace applications, we do consider atmosphere to be a little deeper than that because of re-entry problems. This is your earth radius and you know that it is slightly oblate, the equatorial radius is about I think some 40 odd kilometers more than the polar radius of earth.

What we can do? For the purpose of analysis, we can neglect curvature effect because of the fact that we are saying L is much larger compared to H. We can actually fix a local Cartesian coordinate system; local Cartesian coordinate system, how do we define? We define z which is perpendicular to the surface y is directed towards a pole and x is perpendicular to the plane of the figure.

In this context the way the earth is rotating, so x should point east wards, y is north wards in the latitude direction and z is of course, normal to the surface of earth. This is a Cartesian co-ordinate system that we can adopt for this shallow water equation which we have purposely chosen because, we can neglect curvature effects.

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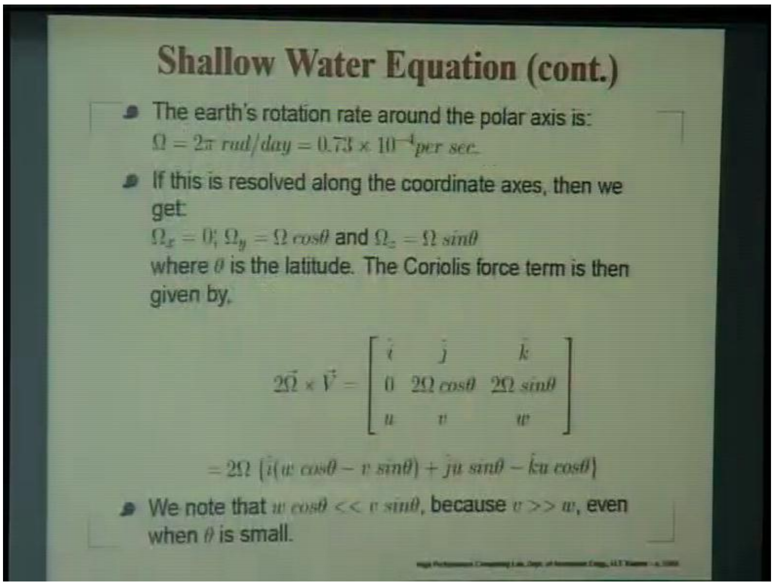

So If I do that I know what this omega is earth rotation rate is 2 phi per day - radian per day. That works out to a very small value 10 to the power minus 4 but, what we could do is, if I go back to this; this is the direction of omega vector so, we can decompose it in the x, y, z direction and that is what we have done here, what we find? That we get two components omega y and omega z given by this, whereas of course, omega x equal to 0; theta is the latitude and the Coriolis force is 2 omega cross v and we obtain this term.

Since, we have already shown that w scale is much smaller than the horizontal scale, so what I could do is w cos theta term we could consider it to be negligible compared to v sign theta term, so in the X part we can omit this term. Well, excepting the equator this observation should remain valid everywhere.

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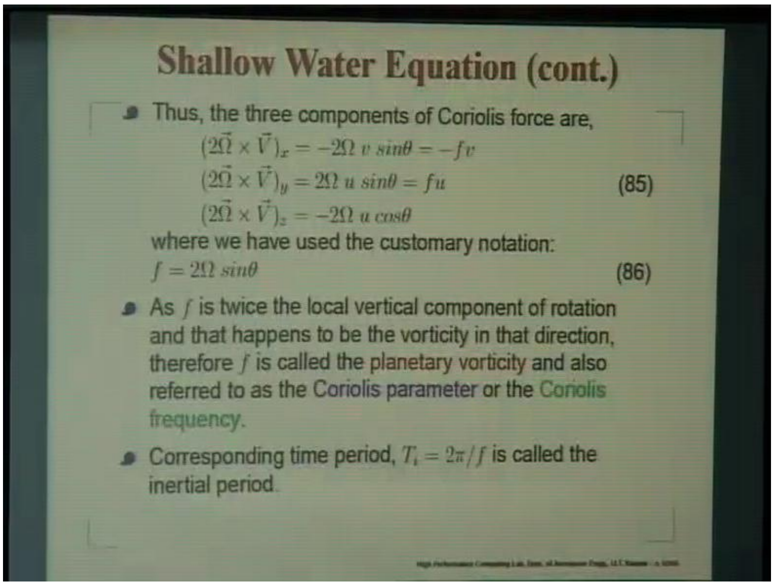

We can write out these three components of Coriolis force and X component is 2 omega sin theta v; we write 2omega sin theta as F; F is a customary notation that is what we have shown here and you can also see F is kind of a rotation rate about the local vertical about the z direction that is what we have done.

Twice the rotation rate is also the vorticity that is why people do refer F also as planetary vorticity, which is also called as Coriolis parameter or the Coriolis frequency, because the dimension of this is one over time, so you can call it a Coriolis frequency and the corresponding time period is called the inertial period, because that refers to the motion of the earth about its axis and it is a large scale motion working on the inertial scale.

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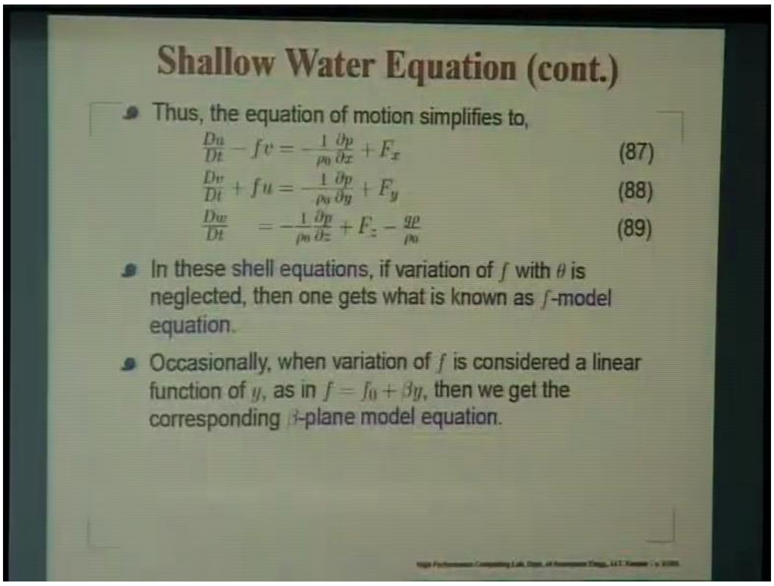

What happens is, we write down the governing equation now in this moving frame of reference. So, this is our substantial derivative U, V and W. We notice that the Coriolis term in the x component is minus fv, fu in the y component and nothing in the z component. These are the pressure variant term this fx, fy, fz are those bottom friction that we have chosen and the last term in 89 corresponds to that body force term that comes from the Boussinesq approximation that is your g rho by rho naught.

Now, what you have noticed that f is 2 omega sine theta, so it is a function of latitude whereas, latitude increases it changes. There are ways of analysis, so these are equations 87 to 89 equations apply to the shell of liquid that we are talking about, be it the atmosphere or the ocean it is a very thin shell.

If we neglect variation of F with theta then, we get what is called as an F model equation. When we do include the variation of F with theta; we could write it in terms of F as a function of y as we have written the mean value of F as F naught plus beta y. So, if you are trying to study, let us say, the atmospheric motion around a mean location where F is equal to F naught then, we actually get y dependent term on the left hand side and those equations are called beta plane model equation of shallow water equation.

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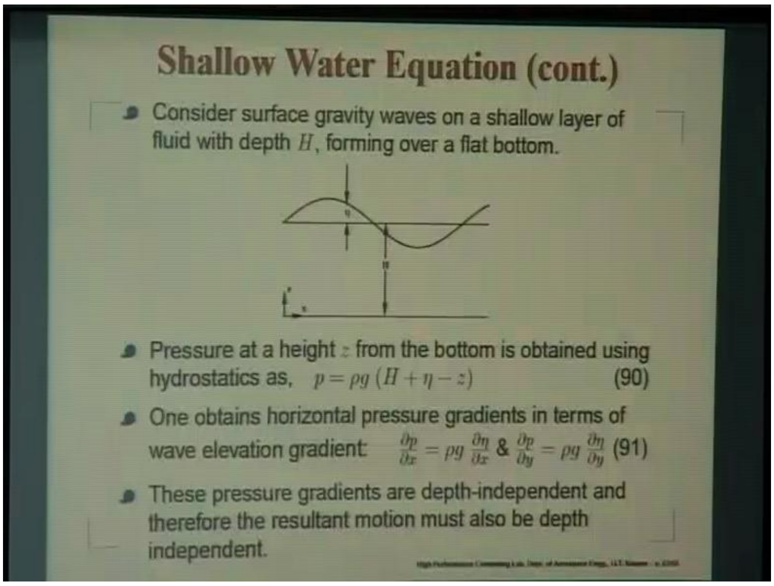

Now, look at this that we are talking about surface gravity wave on very shallow layer of fluid with a depth of capital H forming over a flat bottom. Now, If I look at the hydrostatic pressure at any location then, we could related to rho g H plus eta minus H, that will tell you about the column of fluid above that height including the atmosphere plus the whatever the phase that we are talking about here.

So, having obtain p like this I can calculate its horizontal gradients by differentiating the quantity with respect to x and y, so this is what we get. Now, these pressure gradients are of course, depth independent they do not depend on z.

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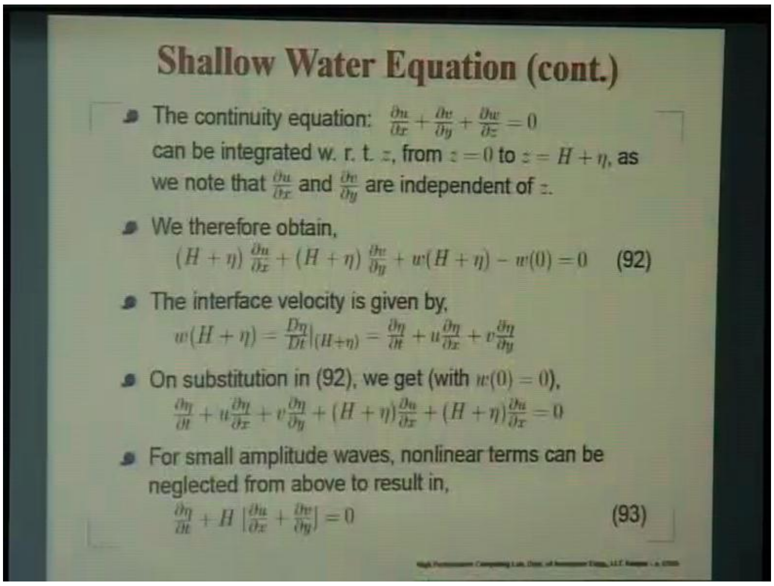

What we say is that if the motion is created, triggered by such pressure gradients such motions also will be independent. So, this is the cardinal assumption of shallow water wave equation and then what happens is we could integrate this equation - the continuity equation - with respect to z from z equal to 0 to z equal to the displaced portion.

That we can do because, we have made the assumption that the pressure gradient does not create a variation of u and v terms. So, this del u, del x and del v, del y are independent of z. We can integrate that equation on top and we get this equation, please note this is the argument this is not multiplied by H plus eta.

So, we have integrated the last term, del omega del z, sorry, del w del z to give us w at the top minus w at the bottom. Of course, at the bottom we have w is 0, so we do not need to worry about it. Whereas, the interface velocity w at H plus eta, we can obtain it from the substantial derivative of the interface description eta, that would be given in terms of this.

Now, you can see that being three dimensional motion, you have to have both U del eta del x plus b times del eta del y. Substitute this in the equation 92 with the bottom vertical velocity 0. We get this (Refer Slide Time: 58:00).

Now, we are again restricting our self to small amplitude waves, then we can knock of the nonlinear terms like this - U del eta del x, V del eta del y plus this kind of terms eta time del U del x, eta times - I think that should be del U del y, No, that should be I will have to check if this term - last term - is correct or not - I think that should be del v del y - last term should be del v del y; no doubt about it so there is a mistake there.

What happens is, this equation 92 simplifies to this. What we have done here basically, we had a 3D description of the flow and that we are rendering it to a 2D description in x and y in the horizontal plane coupling it with the time variation.

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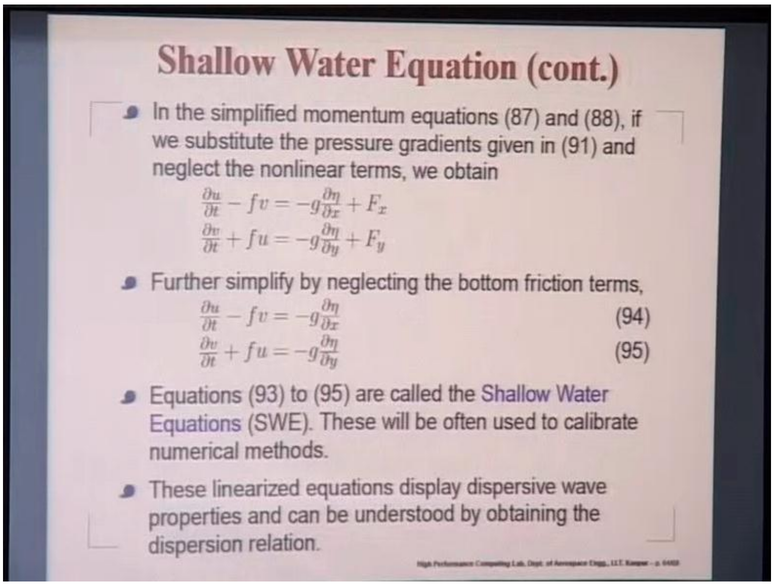

So, this is one of the consequences of shallow water assumption that is what we get. Now, let us see what happens to the momentum equation. I think, we will pick it up from here tomorrow and we will conclude and move over to the next topic.