Aero Elasticity
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Lecture – 8

Today what we will do is, we will classify the aero elastic problem as in a general fashion and solutions.

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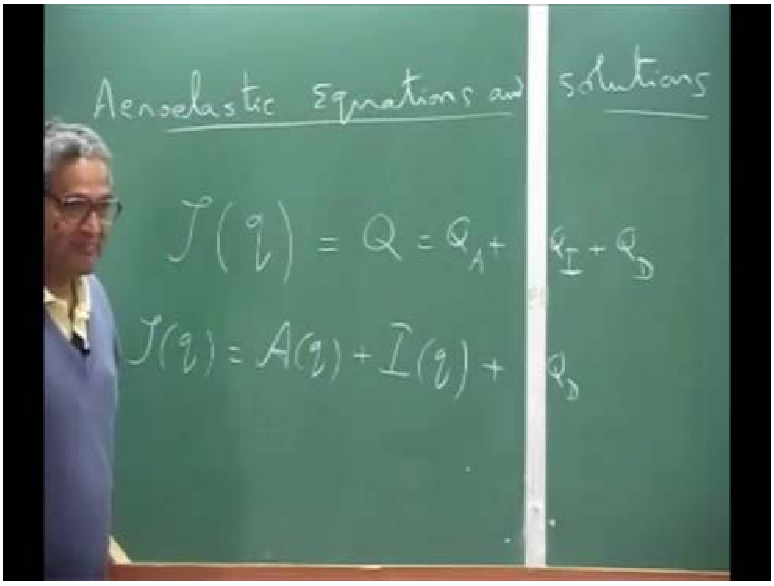

The various types, how do we solve, we will use operator form, because he said that, when you apply a load, there is a deformation. That means, you have a structural operator, which acts on their deformation to give you the load. In instance, we will use the structural operator acting on some deformation, which I am going to call it in a general fashion q, which is the generalize coordinate. This is going to be, see external load which is a generalized force, this is the form of the aero elastic equation. But, now what are the various types of loading, which we will have in aero elastic equation.

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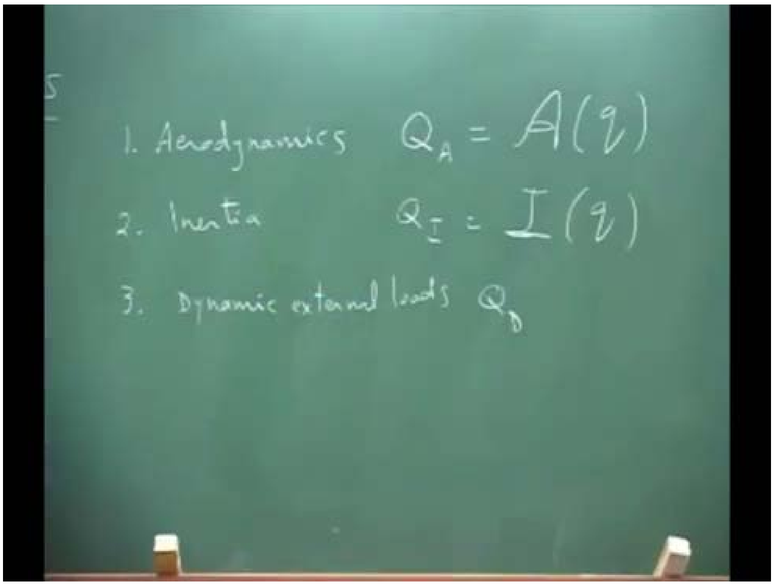

If you look at it, we will have loading, one from aero dynamic and another from inertia and the third some other external, which can be dynamic or external loads you can call it. Now, each one of them we are going to put it in, this I will call it as Q aero dynamic, this I will call it as Q I, this is some Q D, this is the symbol what was I think. Now, if you look at this, the aero dynamic loads are functions of the, so I will use a q. Aero dynamic load is the function of the generalized coordinate or you can say the deformation of the structure see.

Similarly, inertia I will use I, which is also function of q, now I substitute back here for q is usually Q A plus Q I plus Q D, these are the... Now, I put all of them in operator form, I will have structural operator q equal aero dynamic A load inertia plus Q D, this is the... That means, all aero elastic problems are brought under this type of, you can say symbolic representation in operator form. Now, the question is, these operator can be differential operator or algebraic, you have the differential to be in integral also possible.

But then what are the characteristics ((Refer Time: 04:23)), these are the self adjoint operator earlier and for that, we got the solution, etcetera. Now, just we will put it in the classification of this, whether we can take inwards like structural operator operating on the inverse structural operator.

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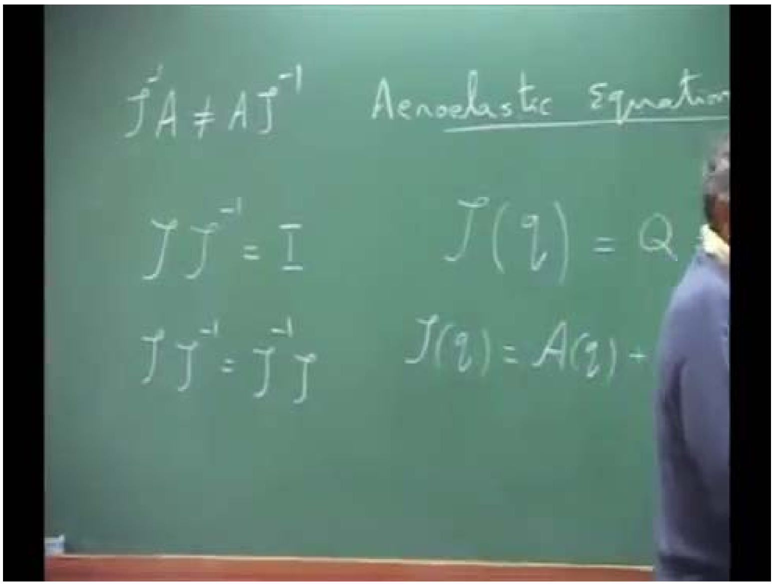

But, please this condition will be valid only under certain special cases that means, they are commutating, commute that is, A B is equal to B A. It is possible that, only when you satisfy certain boundary condition, I will give you that home work to you to show that, what should be the condition, at which this is valid. For such problem, we shall test the valid, because structural operator operating one this thing, this you put it. Now, what condition that this will be valid, but in general, if I take inverse of this, because I need q, q will be structural operator into A, structural operator inverse into I, etcetera. Now, I cannot, this is not valid, they not commutate, they do not commute, therefore each problem you have to solve respectively and you will learn the methods of solution. Now, we will classify the problem, this is the general under different category.

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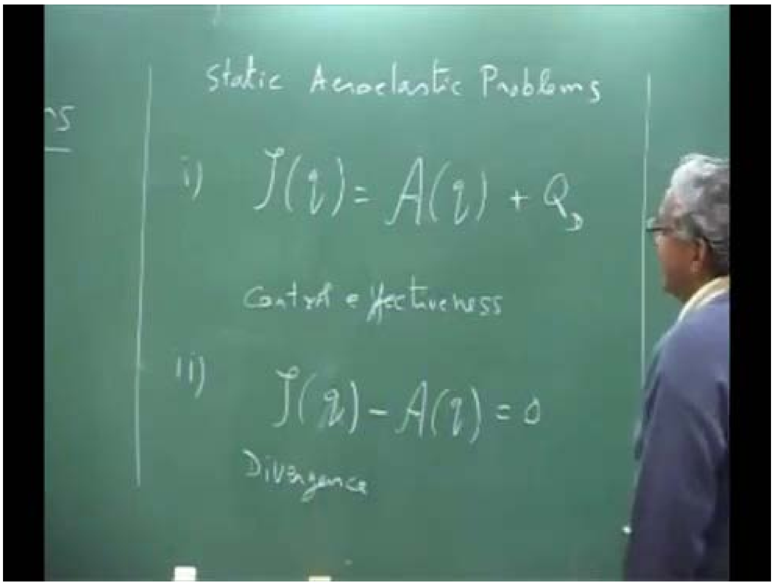

If we classify, suppose if I neglect inertia, this static aero elastic problem, they can be of the form, take dynamic load I put it, but then Q D, essentially it can be some ((Refer Time: 07:41)) part, that is why I put D here. Even though, I put it as the dynamic here, here I put it in dynamic loads in this situation, here it can be due to surface motion, which is control surface motion is independent of the generalized coordinate of the wind deformation.

That is why I put in, this type of problem is basically control effectiveness and control reversal, etc they will come, you understand. If I will put this 0 that is this is one, the other one if I said that Q D 0, I will put like this equal 0. Problem look like a algebraic problem like ((Refer Time: 08:53)), but this is all operator I am just saying that. Now, this is the divergence problem, this is similar to, this is like a Eigen value evaluation, so this is a divergence problem, wing divergence, diversion of full effect of this.

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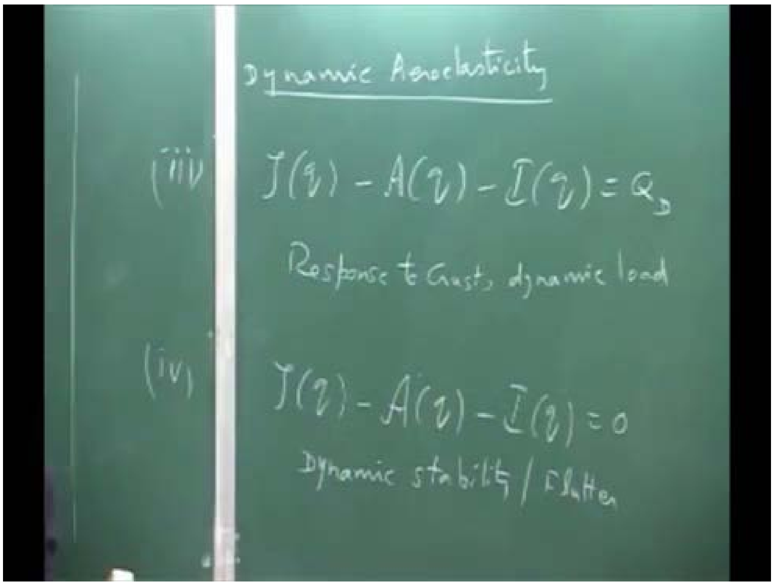

Now, if I take a inertia into account, this is dynamic, under this you will have the full equation. When q minus A q minus I q equal is Q D, this is response to this or any dynamic load, Q D is, you can be landing then when you land, there is an external load complete, impulse type of load. But, if I neglect Q D that is, then this problem becomes our 0, this is again like a similar to another Eigen value type of the problem. But, then this is the stability you can ((Refer Time: 10:40)) dynamic stability or flatter, these are flatter type of flatter.

Now, you see our general equation in this fashion, only thing is the sub part of the equation represent a particular problem, that is all. So, in the whole idea is, formulate the equation of motion or equation for representing the aerospace problem then classifying what type of problem you are solving. If you have static aero elastic issues, dynamic aero elastic, this is where we studied about the various type of operator, adjoined operator. If it is there, that is not self adjoined, we have self adjoined operator is what we saw, today is non self adjoined how do we saw, that part we will do it along the course.

Now, the question is, we saw in the vibration problem that, beam vibration, only for a very special case, we have close form solution, very special case that is, constant uniform property everywhere. But, if in the real life, the properties are vary and how do we solve, we cannot exact solution, you have to resolve to only numerical or approximate solution.

Now, how do we get this approximate solution, that is this part, first is this equation is this part, you are formulate the equation.

Now, how will you get the solution, basically I would say, there are two stages involved in the solution procedure. Two steps, first step is you say that, you have a infinite of degree of freedom system, there is a continuous system, you have in finite degrees of freedom, you are not going to solve infinite degrees. So, you have to first convert the infinite degree of freedom system into a finite system that is, the first step.

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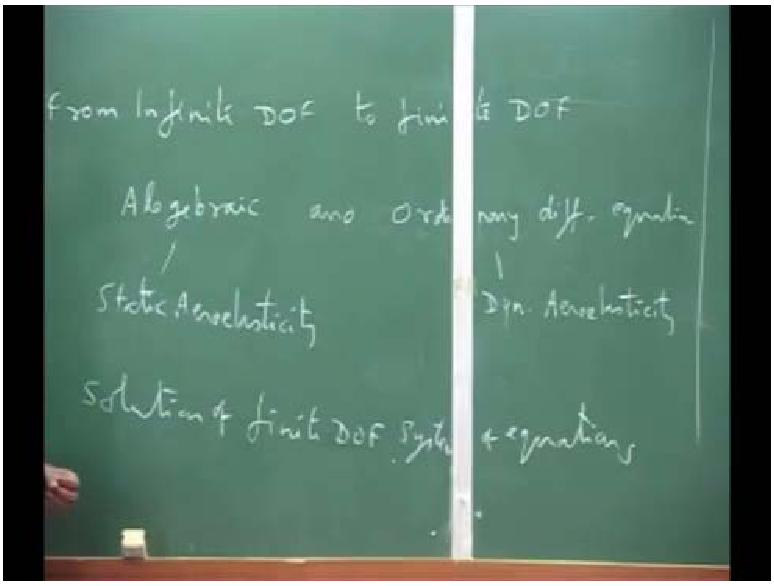

So, I will write here, the two steps are, one is production of the, production means you are basically making approximation, infinite degrees of freedom system, you first put it into from infinite DOF to finite degree of freedom system. This is the first step, you reduce it, now this is where how many finite when we say, how many you should treat it. This question will arise, that is why I said, in the case of rotor blade, we take about 8 modes, three in one side bending, two in the other bending and torsion and axial like that.

Whereas, I guess an aircraft wing, actually some interesting ((Refer Time: 14:20)) that will take about 40, 50 or 100 modes, is it really necessary to have highest modes, that depend upon how accurately you want your results predicted. So, this is the reduction of an infinite to finite, once you reduce it, this stage this equation will become, because in the infinite degree, it will be partial differential equation. Because, we have distributed loading, we also in the vibration, it was the partial differential equation.

Later, we did the separation of variable, we convert it into Eigen value problem and we have orthogonality principle to convert to ODE Ordinary Differential Equation, you saw last class. So, when I convert to finite degree of freedom system, the equations can be algebraic if it is of static aero elastic problem. But, and it will be ODE Ordinary Differential Equation in the case of dynamic aero elastic. But, you know how to solve algebraic equation, but they will all be coupled, you can say coupled algebraic equation or coupled ordinary differential equation.

And then comes the second stage is the solution of this finite degree of, finite DOF system of equation, this is the next. So, I first reduce it then I go and solve it, this is another technique, there is nothing special about this. When this algebraic equation is coupled, what you do, you take a matrix form, you take inverse of something and then get the solution. If it is an ODE, you know the system of ordinary differential equation, you can get the solution for that.

Three is, how do we reduce from here to here, in the step 1, there are what type of approximate technique, I would say what type of techniques are used in approximating infinite to finite. So, that I will describe now, so that is, the technique that is adopted from reducing here to here, is what I am going to say, is what it is, it is known. Once you get the algebraic equation, you know how to get the solution.

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Now, there are four types of approaches, one is collocation method or collocation technique, it is the first technique. I may put it, this is I take this part, you may call this direct collocation. And number 2, you have collocation with generalized coordinate and assumed modes and number 3 is Galerkin's method and the last one is Rayleigh Ritz approach. These are the four approaches, which we use, but I will just briefly describe these methods.

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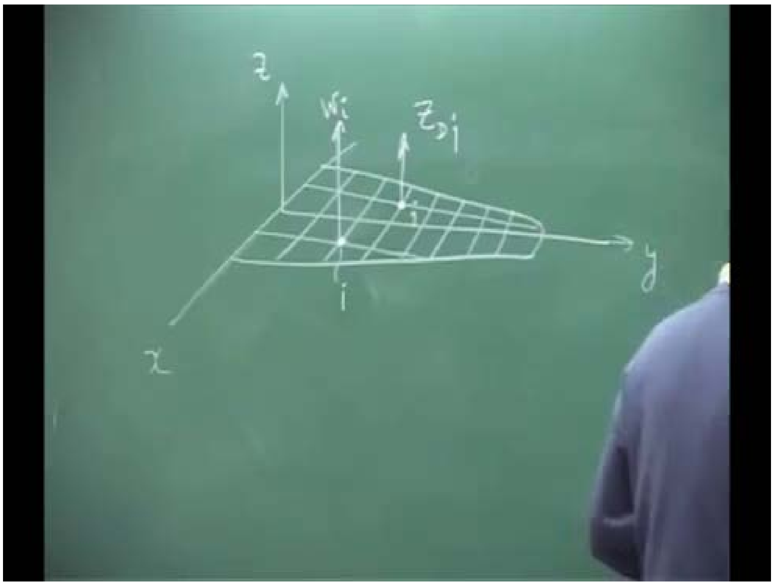

The first one is direct collocation, suppose you have a wing which is, this is your x axis, this is your y axis, this is your z axis. You have a wing, you make the wing into a grid, several grid, you called these as any point here, these are nodes, you may call them nodes or anything. Now, I apply a load there which I call it, this is the j th node, this is the i th node, the load is, that is the external load I am calling it, applied at this location and this is the deformation, which I call it as W at i.

That means, I have my equation continuous system equation, but I am not going to solve a general solution, I will only solve if I take some 100 points, you try to solve at those 100 points. That means, I am dividing my structure into 100 or 1000, depending on the numbers into the node. Then, your equation will get converted into in this fashion, this is direct collocation.

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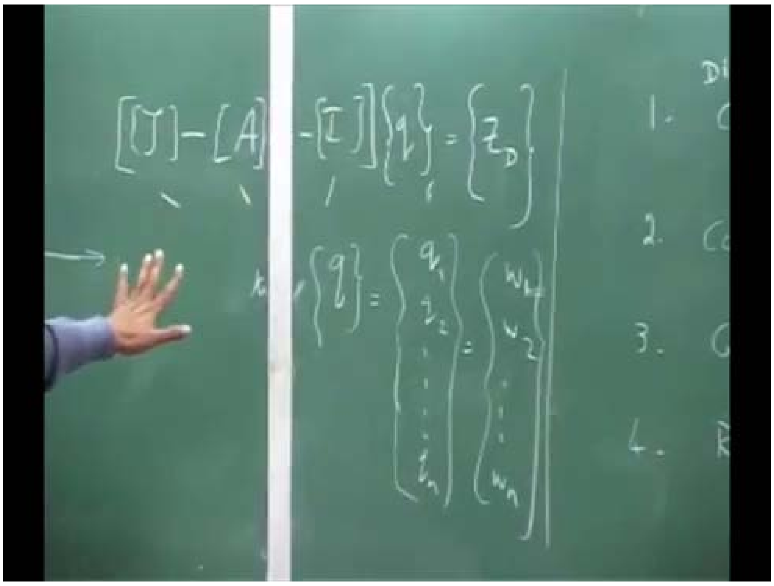

What will happen your equation, your equation will be simply your structural operator minus, this is one operator, aerodynamic operator, inertia operator. I am assuming the system is linear, so I am taking it out, this is acting on my q, the q s are W that means i, these are all n by n system. This is equal to my generalized force, which here I put it as z D, they will act that z D, I would not put an i here, I am putting it, because this is what I am having.

This q is a vector, which contain q 1, q 2, etc, q n, please understand this q are nothing but my W 1, W 2, W n and this z D, again it will have external force that D or Q D you can call it. This is basically the external or whatever dynamic load, if it is 0, that will be 0, you can take it. Now, this is pure equation in this form, n unknowns, n equation and you just solve this equation. But, I am taking it as though they are linear, I have taken it out and then fixed like that, this is called the collocation.

That means, this is like I told, you have the influence coefficient, you formulated, you put a load in one place, you followed the deflection at couple of location, that influence coefficient matrix, which you formulated. And those are these matrices that influence coefficient, you put a load, deflection, load deflection, here this will become like a stiffness matrices, it has stiffness into deflection is force. So, it is stiffness influence coefficient, that flexible influence coefficient or stiffness influence coefficient.

Now, you formulated that, but this is one type of formulation that means, I am trying to satisfy my equation only at n locations on the structure, this is one form of that is, the direct collocation method. Now, if you take a next method, collocation with generalized coordinate and assumed modes, please understand this is a little different from the first direct collocation. Direct collocation is you simply take your equation, put ((Refer Time: 25:43)), you formulate the all this influence coefficient matrix. And then combine them, any external loads you will have this, but you may ask how do you get these things, that is another question.

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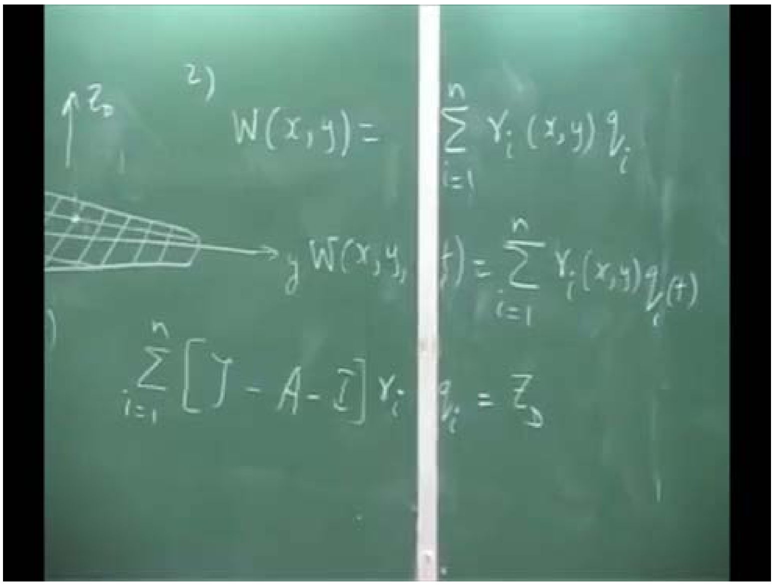

Next, we take the, now this again I am using this same thing only, but now I will not put in this fashion. I am going to represent W is in this direction, which is the function of x and y, because this is a planar. So, I will write, this is the technique two, collocation with analyze coordinate and assumed modes, here I am writing it as i running from 1 to n, gamma i y into maybe I will use a q. But, please note, if it is the function of time, this will become like this, this will be in this fashion i running from 1 to n gamma i (x, y) q i t.

So, if it is the static problem, it will be like this, if it is the dynamic problem, this will be in this fashion. That means, what I am trying to represent is, I am assuming the deformation to be a summation of some modes, these are assumed modes gamma A assumed. But, what they should satisfied this is the question, they were satisfy your boundary condition, because we are going to substitute in the equations, that is the different later.

So, you must choose the function of the assumed modes, they must satisfy your boundary condition, all the boundary conditions, both geometric as well as natural boundary conditions, that is very important. And number 2, you are going to substitute if it is a differential equation, you are going to substitute this back in the equation, so they should be differentiable also. So, these are important, now you see this is a little different, but my unknowns are q i, this is essentially like a separation of variables and the q i's are my generalize coordinates now, these are assumed, q i's are generalized.

Now, what I will do, I will go, substitute this in my aero elastic equation and then I try to satisfy this only at n locations. That means, n point I must satisfy, I will not be satisfying everywhere, so I can make grid like this and then I will write that equation here. So, you will have when I substitute this in the aero elastic equation, it will be i running from 1 to n, you will have a structural operator, aerodynamic operator, inertia operator operating on, because i you will have gamma i q i.

Summation is there, this is equal to external loads, your external load you may call it z D, which you is what this load is, any dynamic load or control success load, whatever you may call it. Now, I need to form, I do not know this q i, how many q i's I have, I have n q i's that means, I go and satisfy this equation only at n points on the wing like in the earlier case, we fix grid then way you did the grid. Then, you can find if you satisfy, what will happen, I am choosing this point, this is our n, now here this will be x i y i, that is the location.

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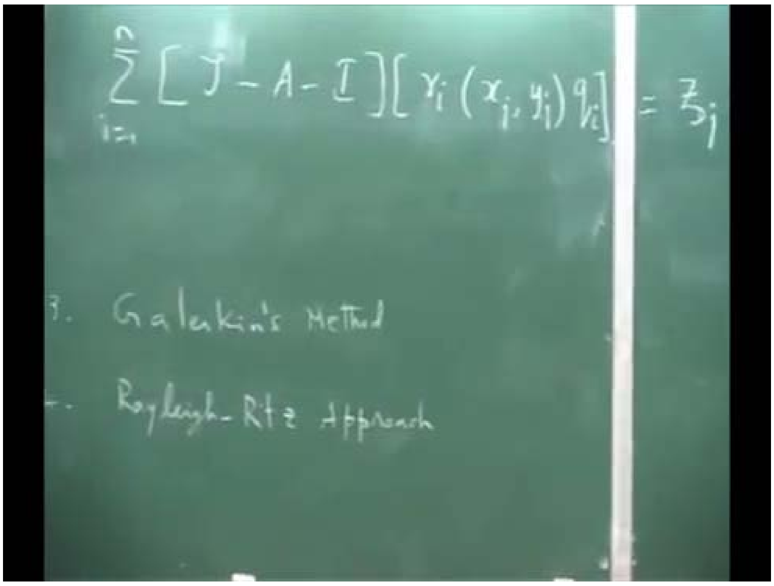

So, I go and then write this in a very general fashion, maybe I erase it, my equations will be written as summation i running from 1 to n, structural, aerodynamic, inertia, you put it here, gamma i at various locations. So, I am going to change the location with the subscript x j y j and q i equal z D, which location I am looking at, at the location j, now j running from 1 to n, you follow. Now, I am actually taking the load at j th location, finding essentially the deformation at the j th location.

But, this is due to various q i's, every q i will come, now you form n equations and solve for this q i. Once you get the q i, you can substitute back here and get your W, if it is the that is why such static problem, you will get algebraic equation, static aero elastic problem. If you have a dynamic, this will become a ordinary differential equation, now this is the direct, I would say what is that assumed mode. This is also collocation with generalized coordinate and assumed mode, there is a little different between the direct collocations and this. The next technique is the Galerkin’s method, I erase this part, but please understand, all of them, they will have some link, how the problem is approximated.

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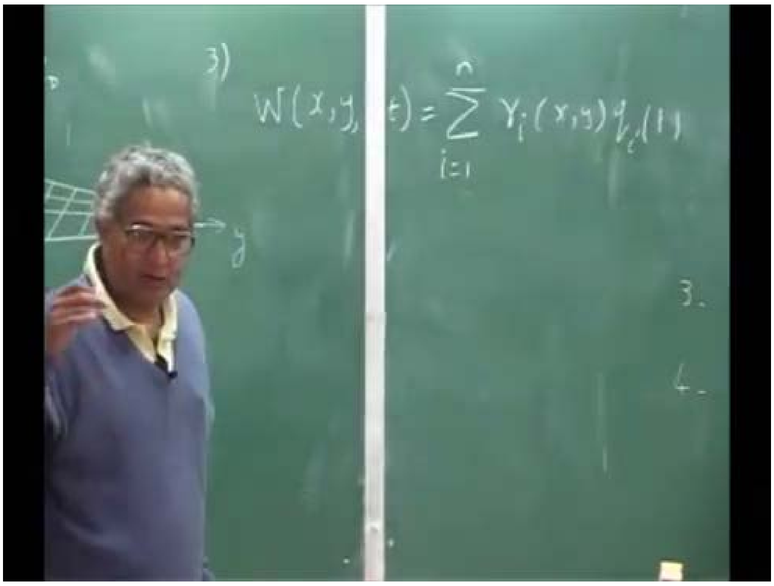

Because, next I showed the Galerkin's method, which will look at, again here also and starting with W (x, y), this is three. You can put t also, if you have t, you will have t, if you do not have time, you will have only that gamma i q i, i running from 1 to n, this will a function of time, if it is a static, t will come. Now, here again I am choosing, so please understand, this is a little different from the method, which I mention earlier, you take this, you go and substitute in your equation.

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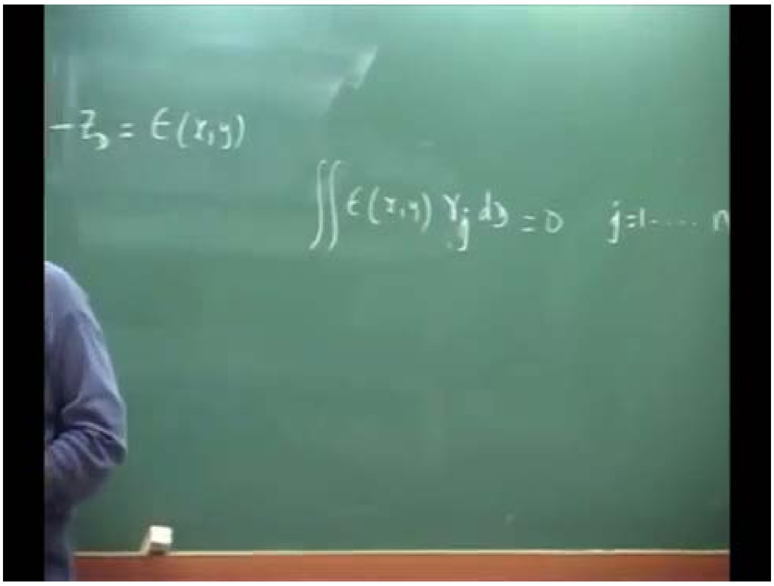

That means, when you substitute in your equation, equation will be summation i equal 1 to n, you will have structural operator minus aerodynamic operator, inertia operator into gamma i q i. I am just substituting for the deformation directly, now this is equal to the loading, whatever the generalized force, so I am subtracting the right hand side to the left.

Please understand, this is now over the entire domain, because these are all fully is the function of x and y, this may be differential operator, etc, you will get the differential ((Refer Time: 36:26)). At every point, if you ((Refer Time: 36:29)) because z D is also a function of x y, so this is generalized force you can say. The generalized force when you get it, what will happen is, this will give you an error, because it is not going to satisfy, this is assumed.

When I substitute in the equation of motion or equation of the system, it will not satisfy exactly, you will have a error. Now, the question is, you want to this technique, this in the 1913 it was proposed. He said that, I want to find out gamma i such a way that, the error. Should be that is all, you can say the error should be minimum, but there is no, you can choose.

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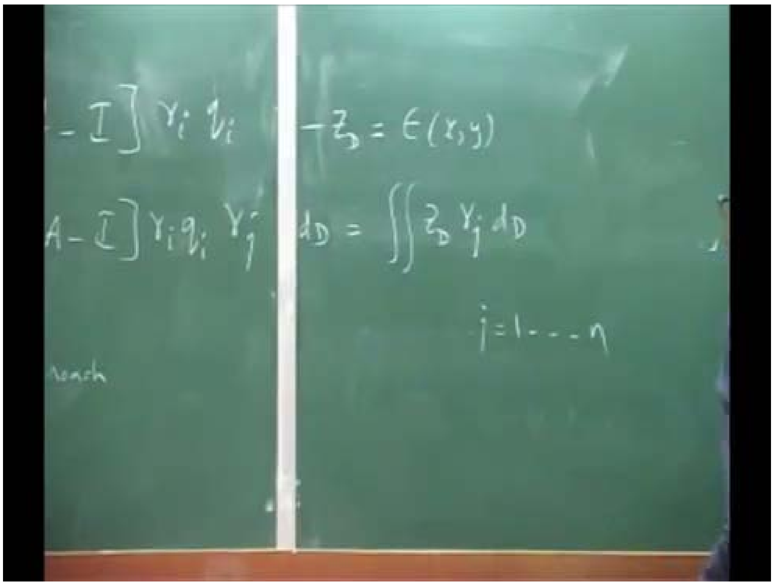

What he did was that, Galerkin's technique itself it is like he said that, this is you know that, this is the function of x comma y, because every point you will have an error. He said that, error is orthogonal to that means, error (x, y) gamma i d domain or you can say d domain, this is 0. Now, you can say gamma i or j, you can choose any one of them does not matter, i running from 1 to n. You can use gamma j also, you want put it maybe when I am doing that, you can j itself, that will be better, where j running from 1 to n.

Now, what I have done is, this is the technique, this is orthogonal to the assumed shape function this one. Now, you understand, these functions must satisfy my boundary condition, all boundary conditions, geometric and natural. That means, the function satisfy my boundary condition is good, when I substitute in my equation, they are not satisfying it.

You get an error, which is everywhere there is an error, I want the error to be minimized and this simple technique is, the error minimum is easier that, the error is orthogonal to this. Please understand, it is not the minimization he is showing, he shown that it is orthogonal to this.

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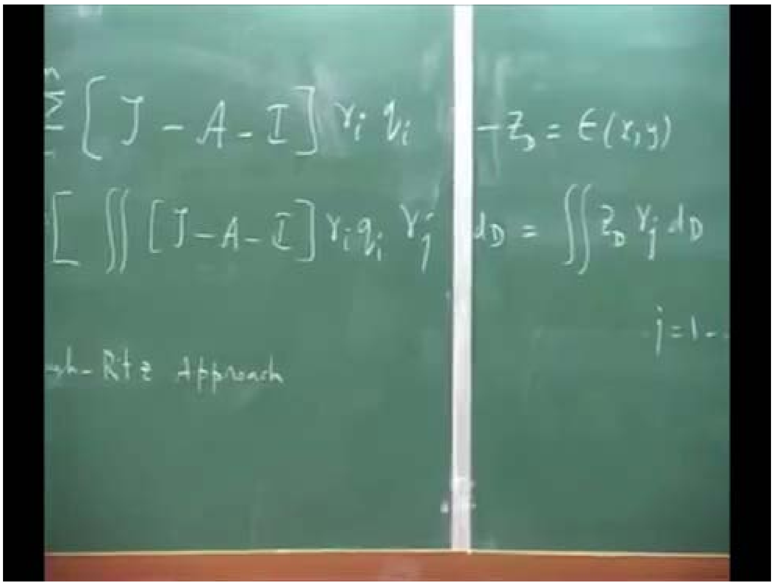

And if you do that, your equation will become you will get essentially n equations, because you will put it as summation i running from 1 to n, you will put, this is an integral over the domain, that is all, I gamma i q i gamma j d domain equal integral. This is over the domain z D gamma j d domain, now j running from 1 to n. Now, you see you will get n equations and you just multiply these n equations, you solve them. So, this is how you, like I said infinite degree of freedom system, however representing by infinite ((Refer Time: 40:48)).

Now, the question is, how many modes you are assumed, that is your choice, your convergence, you can take 1 2 3 4 if and solving every problem and then see how many results are ((Refer Time: 40:59)). You can stop at the point, where it is converge, this is Galerkin's approach. Now, the next most widely used is Rayleigh Ritz, but Rayleigh Ritz you start with, again I will briefly describe that part. Rayleigh Ritz, you do not go and use your, this is the fourth approach.

See, till now what we have used is, we use the equation of the system that is, structural operator, aerodynamic, inertia, all those are obtained and those equation we have used. Now, in Rayleigh Ritz, you do not go and start using their, what you do is, you actually use Hamilton principle or you can use Lagrange.

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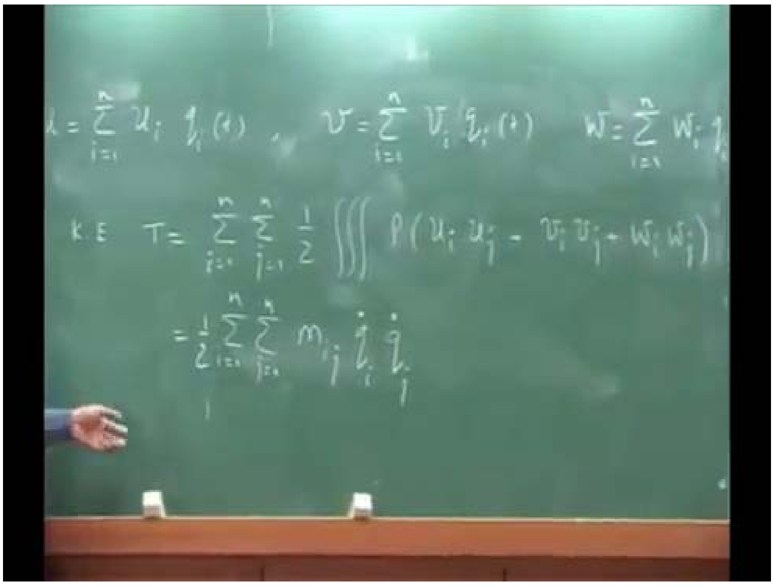

But, starting point, like if you say, very general case I am giving, every point has a u, which is the function of u i q i t, i running from 1 to n, because these are u v w, I am writing in the very general fashion. So that, please understand, this is not the, this is a very general Rayleigh Ritz approach v i q i t and w is again, please understand you see all them have similar expression, where u is a function of x y z t. Because, these are displacement, u v w are displacement, if I do not have this, only w.

You see, this looks similar to what we have assumed, everything look like a assumed mode approach, all of them are assumed mode only, but only thing is now, this w i what condition I should have for this. It is enough if they satisfy my geometry boundary condition, please understand that is the key, whereas on the Galerkin's and all the other approaches, they have to satisfy all the boundary condition, both geometric and natural. whereas here, the condition is, it is enough if they satisfy geometry boundary condition and of course, they should be differentiable, that is important.

Now, you write your kinetic energy of the system, kinetic energy that means, you are formulating your equation either by Lagrange's gives or by Hamilton's method. Please understand, you are starting from beginning, writing energy expression, kinetic energy, strain energy, external work done apply directly Lagrange equation or you can apply Hamilton's or anyone of them and you will get the equation of the system. Now, this is summation i summation j, both of them 1 to n, you will have a volume integral.

Because, I am using the whole structure, density into u i u j plus v i v j w i w j into q i dot q j dot d v, this expiration I can write it as m i j q i dot q j dot, because the integral is over the domain, the volume. Because, these are all only functions of x y z, u i v i w i like we used earlier. Whereas, u is a function of x y z t, this is also a x y z t similar to that, now you have the ((Refer Time: 46:10)) expression. Next, you go and write strain energy and then the external what.

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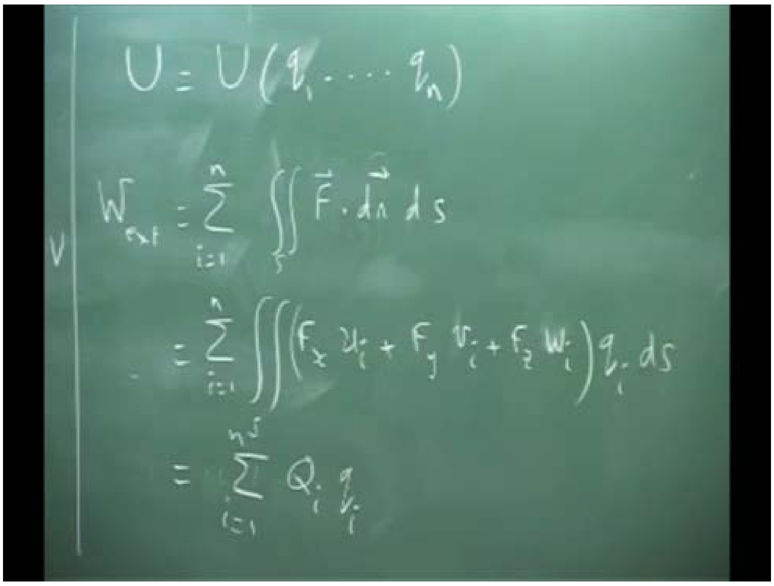

So, we will write the strain energy expression, you will get always U, strain energy will be, U will be a function of your q 1 to q n, that is all, that is the general expression. You have the substitute, this deformation, you know strain energy expression, substitute them and integrate, you will get the strain energy, which will be only a function of the generalized coordinate q i.

Now, external ((Refer Time: 47:00)), this will be if it is a surface load, you are assumed that, F dot d r which is over the surface, F is F x F y F z, you take a tab product with respect to all of them u v w, you will get F x u i. This is summation over the surface F x u i plus F y v i plus F z w i into q i d s, this is i running from 1 to n. This you can write it as summation i 1 to n, capital Q i over q i, this is my generalize force, this is my generalize coordinate.

Now, you can directly go, apply Lagrange’s equation, because you know the expression for kinetic energy, strain energy and external force or you can use Hamilton’s principle. You can use any one of them and if you use Hamilton’s, I will just write that for it. It really does not matter, now you see the application of either Lagrange’s or Hamilton's, that I have the expressions, kinetic energy, strain energy, external.

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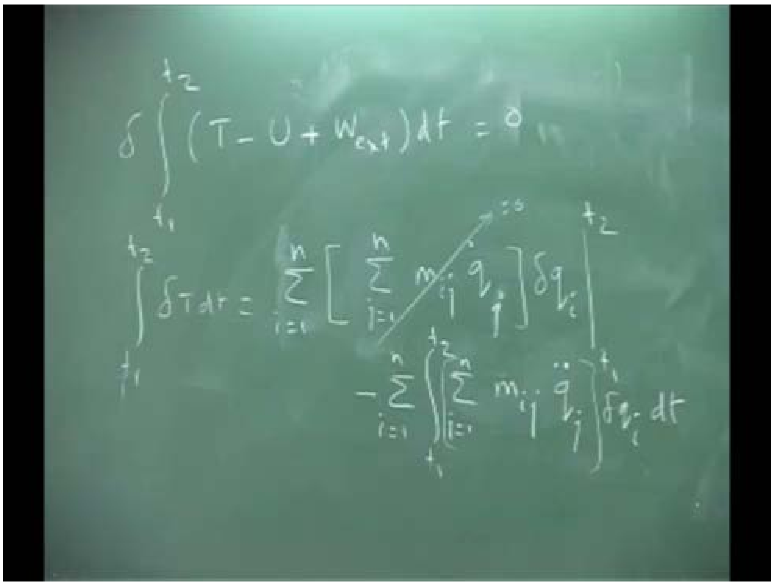

So, my Hamilton's principle if I use, that will be delta t 1 to t 2, T minus U plus W external dt, this is 0. So, you write each term separately and then you will have integral t 1 to t 2 delta T d t, this is actually you integrate by this part, integrate by parts when you do, this actually quadratic expression. So, what will happen is one half will go up and you will have your, I will directly write that expression.

Because, this will be like summation i running from 1 to n, summation j running from 1 to n, m i j q j dot delta q i, this is in the limit t 1 to t 2 then minus summation i running from 1 to n integral t 1 to t 2, summation j running from 1 to n m i j q j double dot delta q i d t, this is what you will have, if I integrate by parts. You know that, time t 1 and t 2, my variation in the generalized ordination that is, 0, so this term is 0. So, this term directly goes to 0, leaving behind only this term.

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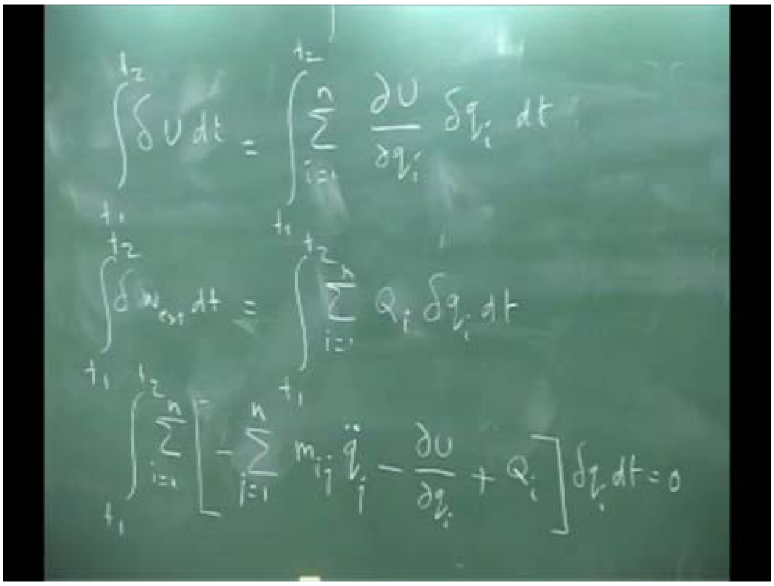

Similarly, your strain energy expression, you will write delta of integral delta U, this is nothing but you will have summation i running from 1 to n, delta U by delta q i and external you will have delta W external dt t 1 to t 2, you will have again t 1 t 2 summation q i delta q i d t, this is i running from 1 to n. Now, you have to combine, put it in this, you say all my delta q i's are independent, therefore if I substitute in this, you will have your equation t 1 to t 2, summation i running from 1 to n, you will have minus, I will put it here j running from 1 to n, m i j q j double dot that is, this expression. Then, I have to use minus this, that will be minus delta U over delta q i then this is the plus, you will have plus q i, delta q i d t is 0. Now, since all delta q i is independent, they will be 0.

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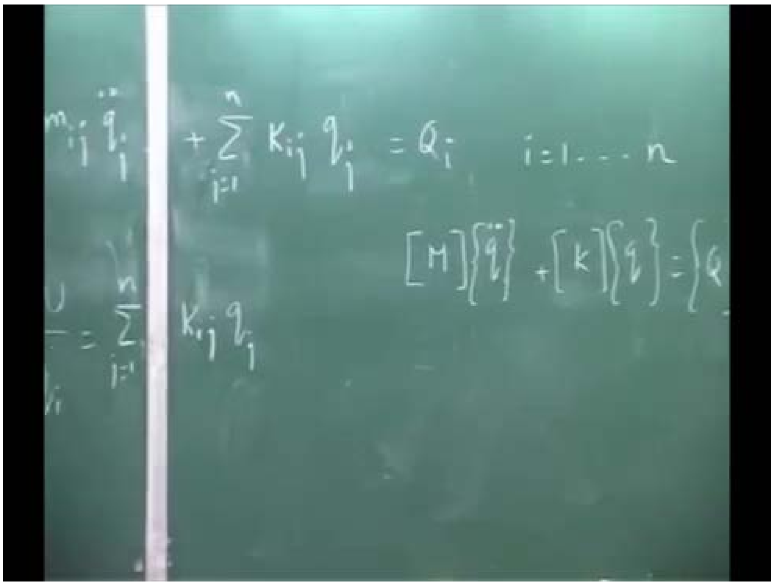

I will write my equation like this, summation j running from 1 to n, m i j q j double dot plus delta U by delta q i, I will write it like this. This is also a quadratic function of the generalized coordinate, so I am going to write it as some stiffness matrix into q, that is how I will write it here, j running from 1 to n, k i j q j. So, please understand, I am taking U is a quadratic function of q i that is, a strain energy is a quadratic function of the generalized coordinate.

When I do that, I will get this form, which will be again summation k i j q j, this is because I have taken minus sign one side, so the right side will become q i. Now, i is running from 1 to n, this is what my equation, I can put it in matrix form, which will be like m q double dot plus k q equals Q, this is a n by n system. Now, you can get the solution, so you know that, m and k represent the mass and stiffness matrix of the system, Q is the generalized force. Once you get the Q, you can go back and then substitute in your deformation, you will get all the deformation of the system. Now, unless you solve one problem by all these four techniques, very simple problem all these four approaches, you can do one thing.

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You can take a cantilever beam, which is the representation of a wing, take a beam, I am just saying, I am not going to solve, I am just saying that. And you have some distributed load, which you may put it like a wing loading kind of a thing and then applying all these four method and get the, just the static problem deformation of the beam at any point using direct collocation, assumed mode, collocation with assumed mode then Galerkin's then Rayleigh Ritz, all the four technique.

Of course, I would like to mention a few points here in the sense, the Galerkin's and Rayleigh Ritz will give same, they are same if the operators are self adjoined, that is the key. Now, you understand what is the self adjoined method, which I was telling self adjoined operator and other things. But, if it is not self adjoined then these two will not be same, in which case you directly go and use Galerkin's method.

Galerkin's method, not Rayleigh Ritz, you go and use the Galerkin's technique for having divergence of a swept configuration, because you derive equation and then apply Galerkin's technique. So, you see, why these operators when we want to say, is it self adjoined, not self adjoined, why this become too mathematical, because of the solution technique.

And you say that, if with the operators are of this type of form then they have this kind of a solution. Now, you have seen, we have reduced the infinite degree of freedom system to finite degree of freedom system, static aero elastic problems will be algebraic equation, whereas dynamic aero elastic problem will be ordinary differential equation in the finite domain.