Aero Elasticity
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Lecture - 6

Last lecture, we got the equations for a beam bending vibration with the mass at the one end.

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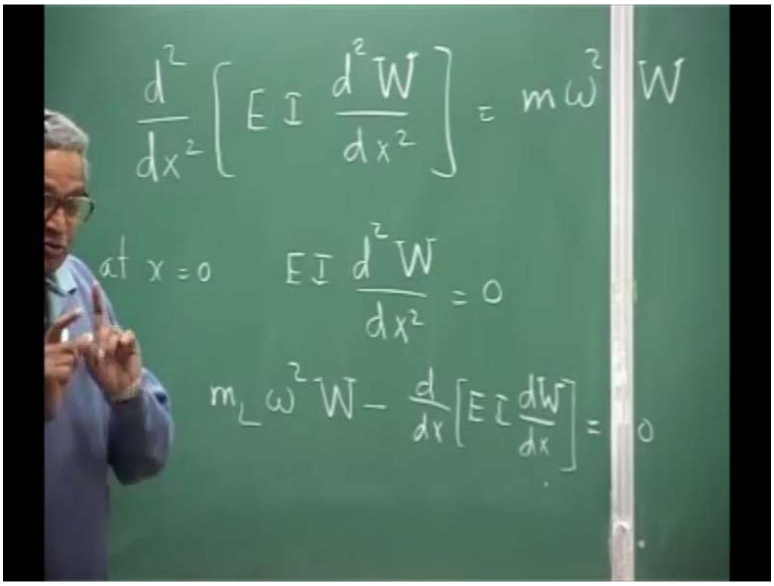

And we got this equation after the separation of variables please note, along with the boundary condition at x equal 0, we had two boundary conditions; which is E I is 0, and then another boundary condition is m L omega square W minus d over dx of E I d equal 0.

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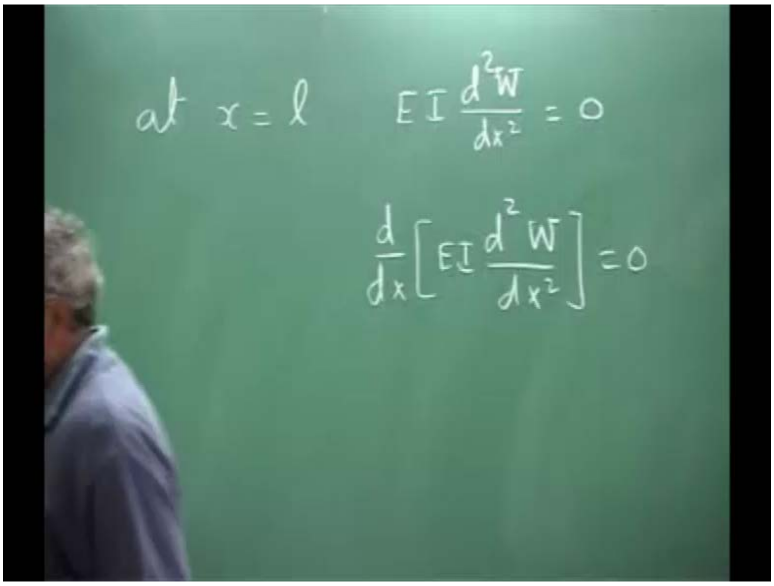

And then at x equal l that is, the other end, we had E I d square W by dx square is equal to 0 and d over the dx of E I, so these are the four boundary conditions, you got that. So, we are continuing from here, but you look at the form of the boundary condition. In three boundary condition, you had the homogeneous, right side is 0, whereas in one boundary condition, there is a omega sitting in, which is again sitting in the field equation. Whereas, that is why, depending on the problem, the omega can be in the boundary condition or need not be in the boundary condition.

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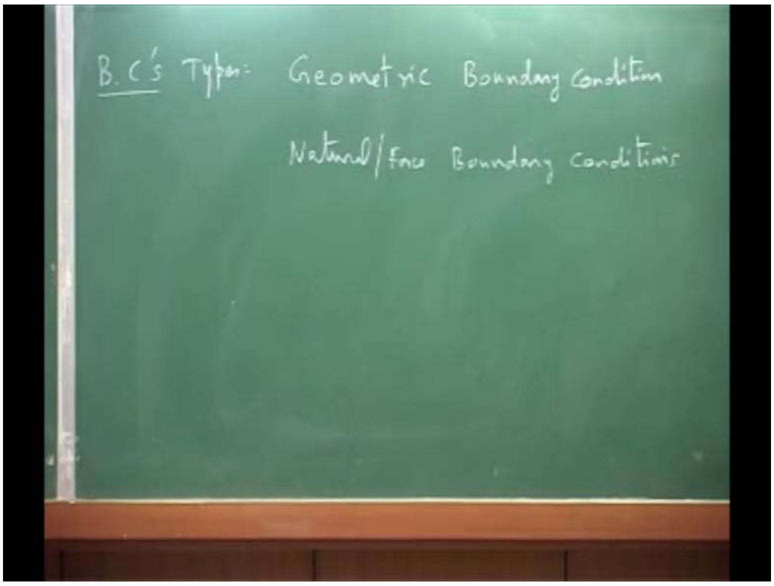

And again the boundary conditions split into two types of boundary condition, one is called the, I will write boundary condition types. That is, geometric boundary condition and another one natural or force boundary conditions, again it depends on the type of problem you are dealing with. In the current case, all the boundary conditions are natural force type, whereas if you have fixed beam like a cantilever beam one and is fixed or if it is simply supported, those may provide geometric boundary condition.

Basically, displacement, slope, that is why the boundary conditions are split and if you look at the highest order of the differentiation in the boundary condition, now here you have to, it is of order 3. And whereas, this is forth order equation, because we have already done the suppression variable and W is only function of x. So, this is the ordinarily differential equation with boundary condition, so our aim is to uptime the solution, which is actually omega as well as W, which is the function of x.

Now, one value is, if you set W is equal to 0 then everything will be satisfied, that is a trivial solution that means, nothing is happening to the beam. But, the aim is to find out, for what values of omega which are non zero, you have solutions, where you have W, a function of x. This is what the whole differentiate Eigen value problem is this, all types of differential Eigen value problem. But, there is a class of problem, which is actually written as a, I will just briefly mention and I will not get into the details of that.

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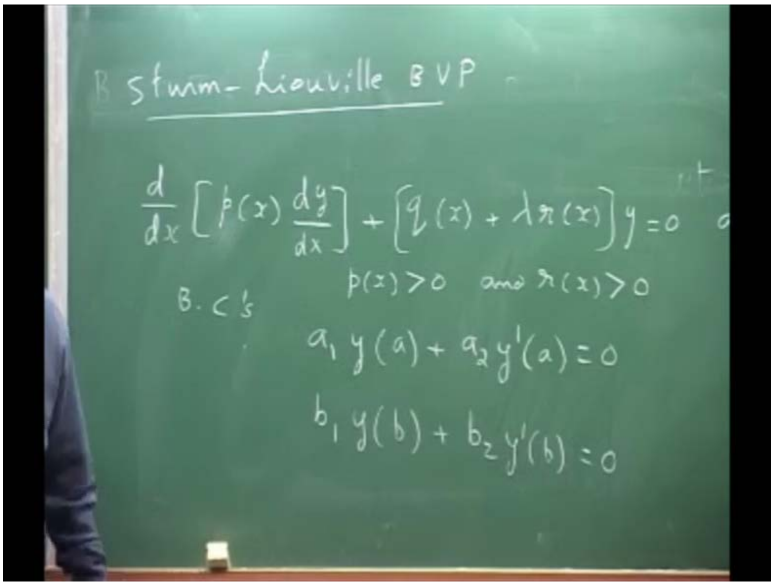

That is, the order 2, this is order 4, this is boundary value problem, the differential equation given like this, d over dx of p of x dy over dx plus q of x plus lambda r of x y equal 0 in the zone x in the domain. Here, this is valued in the domain x l l 0, because this is where it is valued, this is the domain and at the end, you have a boundary condition. Now, with the boundary condition with BC's are given a 1 y of a plus a 2 y prime of a equal 0 and similarly, b 1 y of b plus b 2 y prime of the b equal 0, this is the form of the boundary condition.

But, please understand, this is like one set of boundary condition, this is only the refer to one and a and other end only b, so they called it separate boundary condition. And if you look at, for this type of problems, there are some conditions are all you put in. That is, p of x and write it here, with p of x greater than 0 and r of x and also greater than 0 and they are all constant, a 1 a 2 and b 1 b 2, they are all constants. Essentially, this problem goes towards solving the value of lambda and your y, for what values of lambda you have a solution.

And it says, because it can be proven one later, because that is where the self adjoining is everything, I will try to derived in general form. It says that, we will have this values, Eigen value you may call it, they will take lambda 1, lambda 2 so on, you can have lambda n, corresponding to that, you will have the solution, which you call it.

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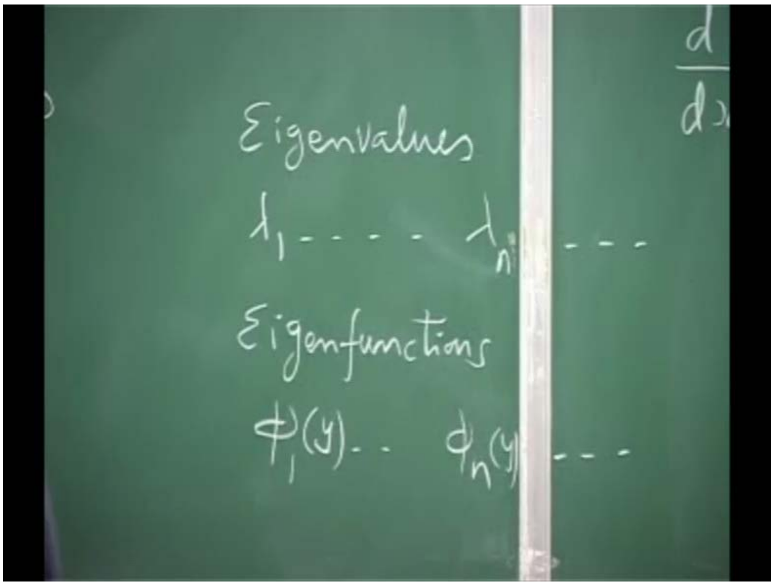

This is Eigen value and there Eigen function, which are p 1, p n, these are functions of y and so on. You will have infinite, but infinite does not mean you can count, that is why we will say, countable infinite Eigen values. Correspondingly, you will have Eigen function and they satisfy and these are all real numbers please understand, real numbers are lambda 1 to lambda n.

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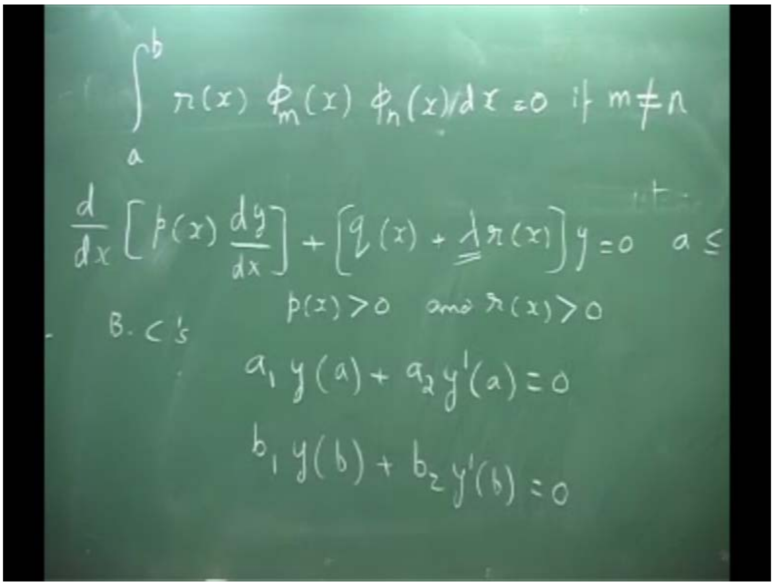

And they satisfy their condition that, the interval a to b r of x phi m (Refer Slide Time: 11:22) not y, these are all x, because our variable is x, this is equal to 0 if m is not equal to n. I am just giving only what is from the math material, I am just writing, this is the Sturm Liouville theorem and what it gives, what is the equation, boundary condition, the conditions on p of x, r of x and the solution will be what type, what they satisfy, this is just for your information.

Now, we leave that part, we will go to our, because you will find that, the torsion problem will be second order, this is the fourth order, torsion problem will be second order. So, we try to generalize all these situation and then write it in simply operator form and then get the full approach, why this is valid, all this proof we will give, various things. After that, we will continue with our vibration problem of our beam, so we take the entire...

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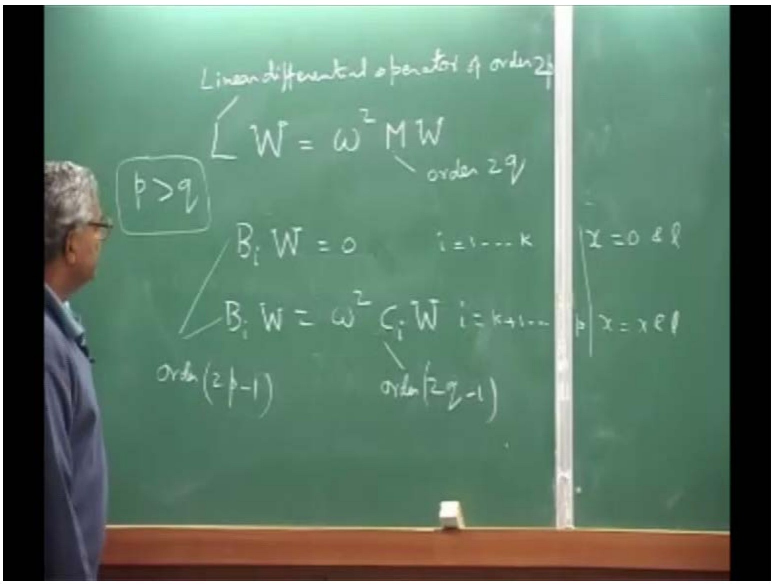

Let us the write the equations L W equal omega square M W and then B i is 0 and then another B i W equal omega square C i. And here, i running from 1 two k and this is i running from k plus 1 to p, but you can say, x 0 and l, 0 and l. Because, you have to satisfy these boundary conditions, l is the linear differential operator, this is linear operator of order 2 p, where p is 1 to etcetera, integer. Here, these are maximum order, B i is order 2 p minus 1 maximum, similarly you can have M also a differential operator of order q, will say order 2 q will write it.

And this will be order 2 q minus 1, but please note that, p is greater than q, but normally per hour if you have a tip mass, we did not have any differential operator, because there was not given. But, where will you get that differential operator, suppose in this situation, the tip mass are within allow, that mass is distributed mass we took. Here also you may get some differential some times, that is why I am saying, tip mass goes here, the distributors mass you put it as a mass.

Suppose, if you add rotation that is, rotary inertia of the elemental mass and take it is kinetic energy that means, we took only half M W dot square, you can also take inertia into W prime, which is dW by dx, that is the rotation, it is time derivative has a dot, one more dot then it is like rotary inertia. If you take then you will get the differential operator here, but most of the problem what deal with, it is like a only math, math, math without taking rotary inertia.

We will not get the differential operator in this case here, same thing to this also, now this is our general problem. We define our idea is what, we need to get the solution omega square and W, which is the function of this. There are different types of functions, we will first define them like admissible functions.

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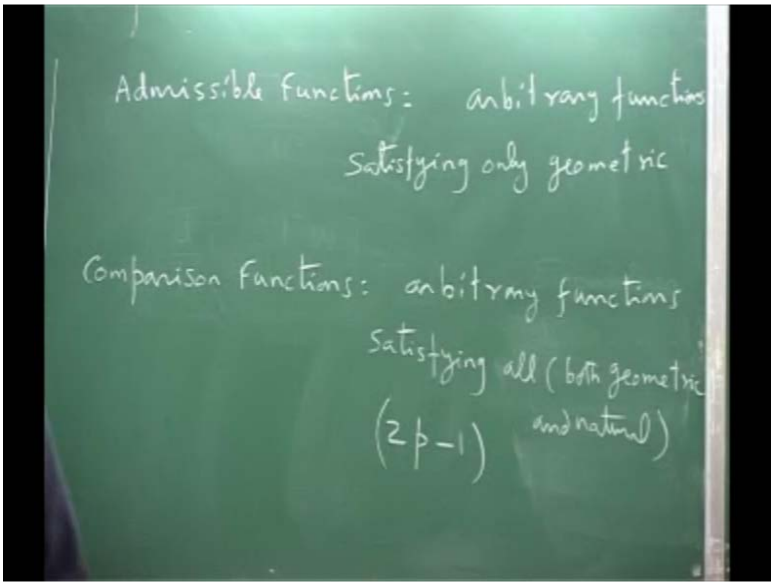

What are the admissible functions, these are function that is, arbitrary function please note, arbitrary functions satisfying only geometric boundary conditions. Then you can have comparison functions, these are again arbitrary functions satisfying all boundary conditions, all that is, both geometric and natural BC's. Now, if I this to have to bring out the distinction, I said here it is satisfies only geometric boundary conditions, here I can say both geometric and natural.

We saw that, the natural boundary conditions, the order of the derivative will be 2 p minus 1 that means, they must be 2 p not minus 1. I should say, they should, because 2 p times differentiable, let it put it that way, 2 p times differentiable.

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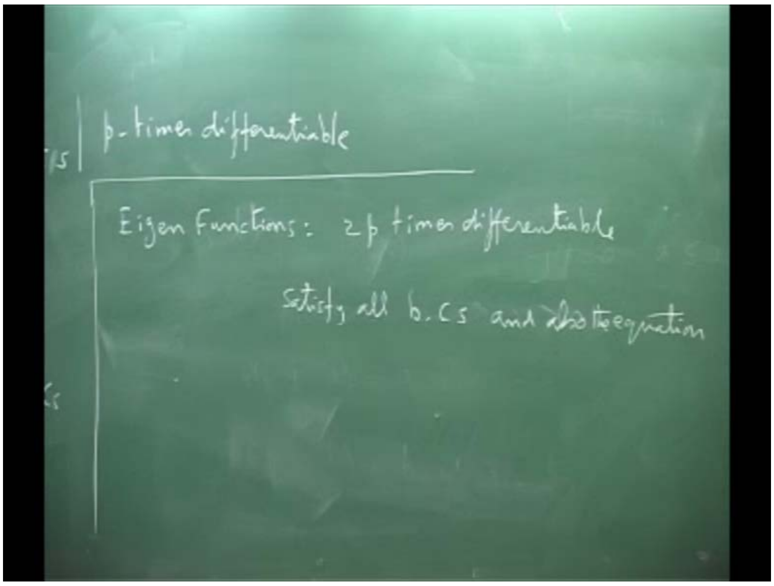

This one if they satisfy only geometric, displacement and slope, the condition on this is, you put it only p times differentiable. Differentiable means, differentiable over the domain, because at the end, geometric boundary conditions, whether displacement is 0, slop is 0. And why that 2 p times differentiable, that will becomes obvious when we solve problem, because we said that, this is to 2 p times differentiable operator.

If you want to substitute that function, it will be possible to go for 2 p times differentiable, otherwise if it is less differentiable, it will be 0, there is no solution. But, admissible function, you will not be put into here, because it is only p times differentiable, you cannot put them here. Then what for we are depending you will understand, if there is a way of, if I modify this into another energy form if I write it then I find the function has to be only p times differentiable, I did not go for 2 p times.

But, they are all approximations then there is a last one which is actually, these are admissible function, comparison function and then Eigen function. These are 2 p times differentiable and then satisfy all boundary conditions and also the equation that is important, because they satisfy the equation also. That means, that is a exact solution, because they satisfy all boundary condition and also the equation, everything. Whereas, here, I do not insist that comparison function, they should be 2 p times differentiable, they satisfy both geometrical and natural boundary condition.

Here, I say, they satisfy only geometric boundary condition, whereas here I say, all boundary conditions and also the equation. Now, the problem is, it is not easy to get Eigen functions for all problems, it is not easy to get. Therefore, you have to, the whole analysis of the solution techniques is, what is the best way, the approximate procedure so that, you can get the good solution, that is all. So, the entire study goes towards obtaining approximate solutions, exact solution is possible only for highly specialized cases, very few cases you can have.

Now, we will define, if will not get into this, you have understood these three types of function, now you will use, wherever I say comparison function that means, you immediately know that means, it satisfy boundary conditions 2 p times differentiable. If I say Eigen function, it satisfy the equation, because Eigen function have certain properties, those properties have to be mentioned like orthogonality or something will come, that we have proved. And are these operators are, what type of operators they are, because if they follow certain class then you say my solution should be of this type, that is what you will learn.

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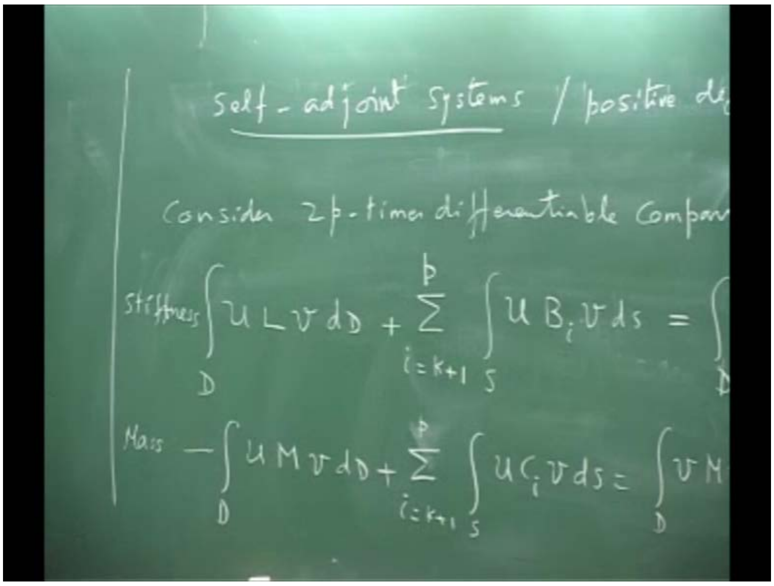

Now, let us take, we call self adjoined system and there is a conservative please understand, there is no damn thing, dissipation, nothing, whatever want that goes into the strain energy. And this is the free vibration problem, please understand this is the free vibration problem, because we do not have any forcing function. So, this is the conservative system and self adjoined systems, what is the meaning of this and they are also will say, positive definite.

There are two words I am introducing, self adjoined system and then positive definite, this is a linear conservative system that is all, you can take conservative system, because there is no dissipation and whatever work you do like inertia and potential strain energy. So, this is an interplay like your simple vibration problem, only thing is, this is continuous system. Now, if let us consider 2 p times differentiable comparison function, what do we mean by adjoined, we have to define what is adjoined.

If comparison function satisfied, we are taking some function, which of course, they are comparison functions. If they satisfy this, this D is the domain that means, domain in the field, in the case of beam 0 to l length of the beam and this is U, this is the operator l, this the operator operates on V. U and V are comparison function, plus you put the summation i running from k plus 1 to p, this is the over a surface, surface means only end point, that is a boundary.

You will have U B i V D surface, this is equal to domain V L U dD plus summation i running from k plus 1 to p, over the surface V B i U. So, you see what is happening is, U L V plus this i k plus 1 to p that is, only this, this is not considered, only this condition, you follow, where there is a those boundary conditions, which are having omega square in their relation. If this is satisfied that means, U and V are any two arbitrary comparison functions, if this is satisfied then l is self adjoined operator.

But now, you see most of the, that is why I wrote this particular thing was, if my boundary condition are of this type, I will simply get. That means, omega does not appear in the boundary condition then self adjoining becomes simply U L V dD equals, this is what V L U over the domain. This is what will happen, because these two are not there and then many times most of the book, because you have do not consider that type of boundary condition at all.

You will tend to think that, this is what the self adjoining is, that is why I brought I just put one mass extra on the beam and show that, how that comes as the boundary condition and then how this is also has to be considered. Similarly, for the mass, mass means that capital M, you also have the self adjoined this is you call it, because this goes to stiffness term, this is omega square, this is the mass term. So, you can say, this is stiffness operator, this you can call it mass operator, but anyway that equation also will look like this, that is D U capital M V.

But, here you add only the other condition, k plus 1 to p ds equal V M U dD plus summation i running from k plus 1 to p, V C i U ds. Suppose if you say, this stiffness operator and this is the mass operator you may call it, if these two are self adjoined then you say my problem is self adjoined, followed. And ((Refer Time: 34:19)), this can be written, if you actually what you have to do, I am only describing in words, I am not going to derive that.

If you take only this part or that part, you essentially that differential operator is operating on V, what we do is, you integrate by parts, one, twice, something like that depending on the your order this. Then what will happen is, this will become quadratic in form, if it becomes quadratic in form then you can change U or V, wherever they interchange, it does not matter will the expression will remain the same expression. That is why, sometimes this itself is written, but you should have the boundary condition.

So that, when you do then you integrate by parts what happens, you are going to get the boundary conditions, that is why that the boundary conditions are 0, you will not need this. Whereas, if they are not 0, like 0 means not homogenous like, you have a omega square sitting there then you need to have the term in that. And if you have that form then this is the self adjoined, wherever you cannot get into the that form, if it is not valid then it is not self adjoined operator.

Now, the question is, if it is self adjoined, you have a well set procedure for obtaining the solution. But, if it is a non self adjoined, what you do that another case, because that we talk about later, first we will say only self adjoined part, you follow. So, the self adjoined, our problem is self adjoined and I just leave it like this, because I am not going put into in the form. Because, when we get the example what I can do is, I can take the example which we derived, whatever that beam with one end mass, we do same thing, we apply that and then show them, maybe that you can take it that exercise for yourself.

For that problem with those boundary conditions, show that that is self adjoined, this is like a exercise for you, I leave it to you. Now, another thing is, I can erase this, for any comparison function U, please understand for any comparison function U, I am putting it like this, U I am putting same U, any comparison function please understand, that is a p, that is why I took it some differentiable comparison function is essential. If this is always greater than 0 then the system is positive definite, only when U is 0, it is equal to 0, you follow.

On the other hand, if for some U it is equal to 0, so greater than 0 is positive definite, why I put equality, here also greater than or equal means, positive semi definite system, positive that we call semi definite system. In this case, even for non zero U, it may be equal to 0, so positive definite, semi definite, this is in differential operator form. Actually, if you see what is the analysis in matrices, because you have learnt spring mass damper, spring mass system, damper you leave it.

If you want to find out the stiffness math matrix, there the stiffness matrix, if it is symmetric you say and mass matrix symmetric then symmetric goes towards self adjoined, where is the adjoint is A transpose is A. So, if it is a symmetric math matrix, you say self adjoined then what is the positive definite term then normally you take any lecture, multiply by the stiffness matrix then transpose of that then you will get 1. If the quantity is positive, you always say the system is positive.

So, this is identical, only thing is here it is all differential, I am not getting into no spring mass damper type of system and then tried to derive the equation and show, both are same it is similar. Now, what you do is, if they are self adjoined and positive adjoined system, all Eigen values are positive, real positive, that is the key, all Eigen values are real and positive. That is why, now I am just describing, if the system is non self adjoined, you cannot guarantee.

They may be real, they may be complex, they may be anything, that is why you first check all your operators, whether our problem falls into this category. If they fall into this category that means, I have a solution of this type, I have these quantities are all positive. Only thing now the question is, how do I go and get them, that is the different part. But, it is guarantee that, they must be positive, that is why self adjoined and positive definite, both the important.

If it is simply self adjoined, that is only real symmetric matrix, the real symmetric matrix you will get real Eigen value, they may be a positive, they may be a negative also. Whereas, if it is a positive definite, you can show that, this will be greater than the 0 for our problem, that is why I would say, I left it to you, you show that for our given operator, show that this is the greater than 0. Then you know that, this thing must have real positive Eigen values and some corresponding Eigen functions. But, these are all shown with comparison function, now let us prove one more condition, which is the other orthogonality principle. That is look little bit messy, but it all I thought, I will show that.

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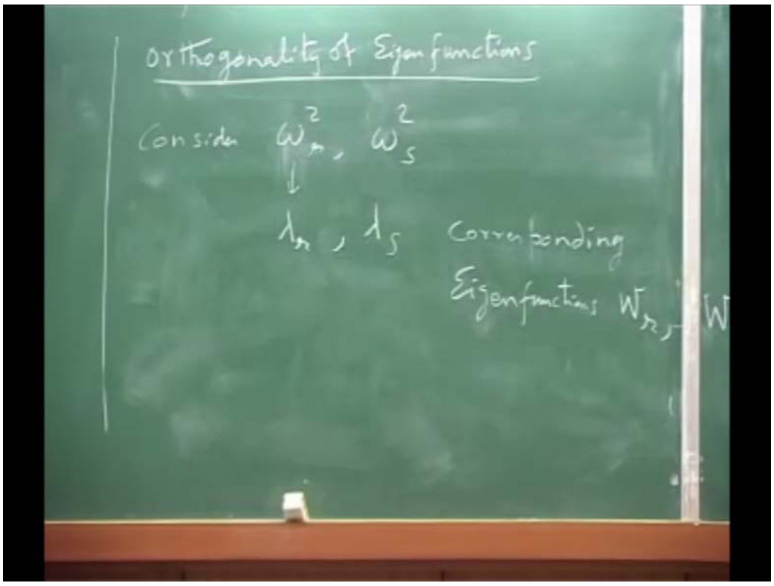

Orthogonality of Eigen functions, consider lambda r and not lambda, you take it as a omega square itself, I think that is better. Omega square r, omega square s, I may for simplicity, I am writing it has lambda r, lambda s just for, because I am going to use only not omega, I will use lambda r, lambda s, I am just denoting them. These are two Eigen values, corresponding Eigen functions, please understand Eigen function now, you will note that W r and W s, these are the two functions. Now, you know that, Eigen function must satisfy my equation, so I will write that L W r is lambda r M W r, similarly L W s lambda s.

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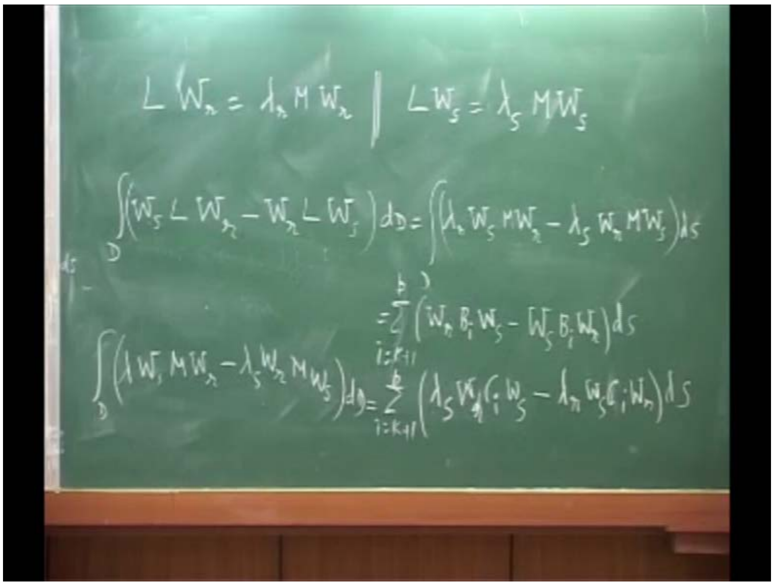

Now, let us multiply this by W s, this by W r, please note that this is the operator, so I am multiplying before, premultiplying and then integrating. So, I will do integral over the domain W s L W r and then I am subtracting this, so I will put minus W r L W s d domain. This is equal to, again we have to do the same thing here, lambda r W s M W r minus lambda s W r M W s ds, this is the condition. Only thing is, I have to do some algebra, that algebra is, you know that, there are L M, they are all self adjoined, etcetera. You need to go and what you can do is, self adjointness, we have defined a long expression.

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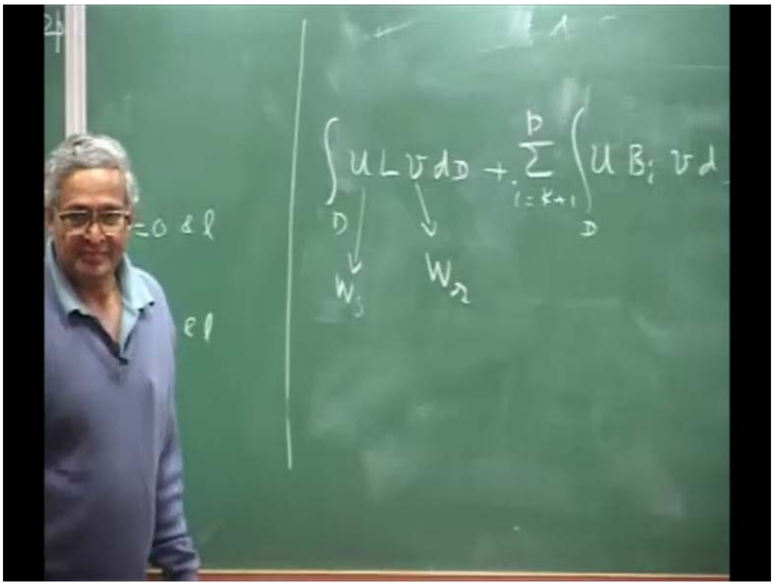

what was the expression we defined, we wrote the self adjointness, this one U L V dD plus summation i equal to k plus 1 p U B i V d s, this is over the domain, this is over the, we wrote this is equal to D V L U dD plus summation k plus 1 to p V B i U ds, we wrote the value of this. Because, now they are self adjoined, they are this, now what I can do is, instead of U and V, you have W s W r which means, I am taking and replacing here this by W s W r, U is W r, this is W s.

Comparison functions, but they are self adjoined, they have to satisfy boundary conditions, Eigen function also will satisfy that. And then you take this term, this you bring it to the left hand side that means, this particular expression you can write it in terms of B i, that is what you will do now. So, you will write this is also equal to, please note that I have taken the left hand side, this I have brought in and I am replacing, now I am writing summation i running from k plus 1 to p, this is B i U V and here V is W r, so W r B i W s minus W s B i W r ds.

Similarly, what we had is, we had another relation, which was we had this relationship, so I can also write that as, this is one and before I go for that, you can write from here, W s L W r minus W r L W s. What you can write, you know these are all, from here please remember I am substituting for B i W s and B i W r has correspondingly, because that W s I must put corresponding s, this am going to substitute there. That will become, what will be that, this is essentially summation i running from k plus 1 to p, W r will remain as W r.

Because, that B i W s should be omega s squares C i W s, omega s square I am calling it as lambda s, W r C i W s. Same thing here, minus W s remains same B i W r, that is again lambda r W s C i W r ds. Now, I have a relation this to this, next I erase this part, I think this is I keep this, because everything is important. Next, what I am going to substitute is, I will take that term, I am going to substitute here, L W r is what from here. Then what will happen, this will become domain, please understand that is the domain W s L W r, L W r going to be lambda r M W r.

So, I am going to write here lambda r, this is this type, that is different then these two are equal, that is all you can take this, maybe I can rewrite. So, maybe I will write it again so that, that becomes clear, domain d that is there, lambda r W s and W r minus lambda s W r M W s dD, this is equal to this quantity. Now, what you do is, you have to do some substitution, because from this equation, you have to do some substitution. Let us write this equation, U as W s, so I am going to replace this here w s and this is W r and here also, U is W s, V is W r.

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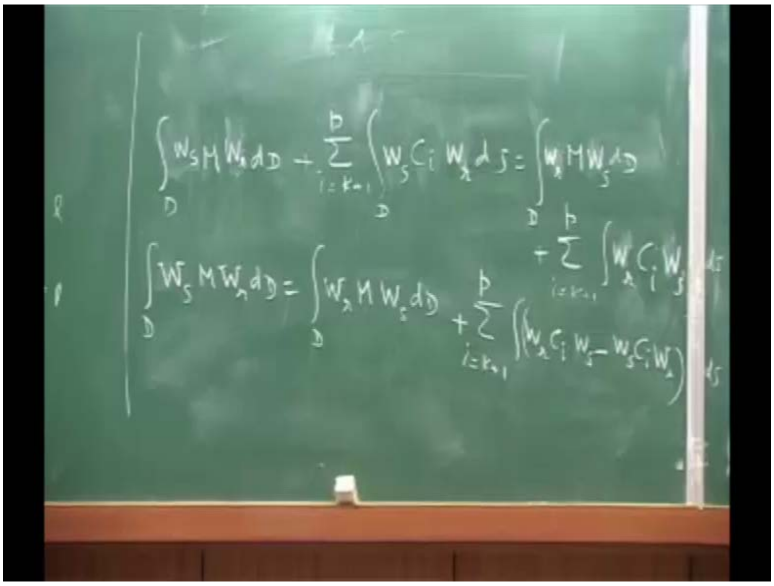

What I am going to do is, I am changing, I keep this term, I transfer all the term to the right hand side then what will happen. I will have integral D W s M W r dD is equal to integral W r M dD over the domain plus summation i k plus 1 to p W r C i W s minus, this term will come W s C i W r ds. Now, what I have to do is, I take this term, this is integral W s M W r over the domain, I will simply take this entire term put it here, in this place. I am replacing, please understand I am replacing this entire term ((Refer Time: 58:25)).

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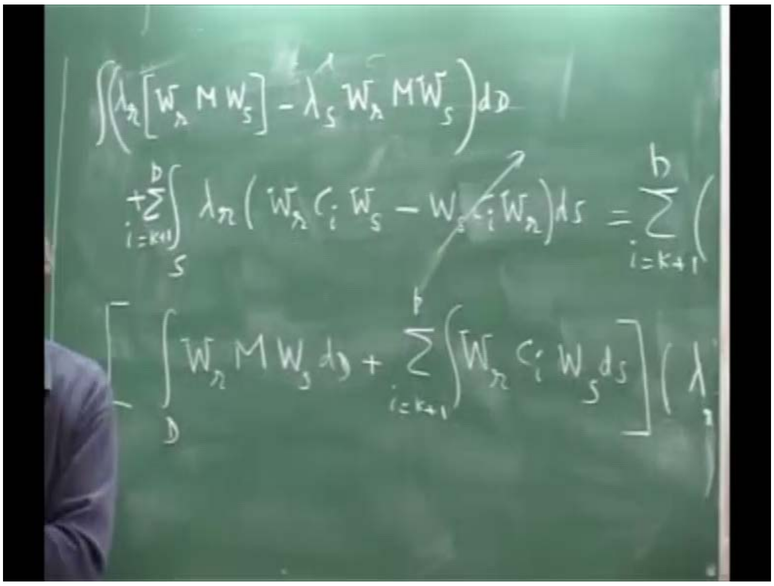

When I do that, I will write my final, because I erase this par, because this is going to be little thing, substitute here this will become, that equation will becoming lambda r W r M W s. And then you already have this term, lambda s W r M W s this whole thing is d domain then you have to add this term also, you will add plus summation i k plus 1 to p, this is over the surface, again you will multiply lambda r, lambda r W r C i W s minus W s C i W r ds.

Then, this is you have to equate to that term, I erase this part, you are equating to summation i k plus 1 to p lambda s W r C i W s minus lambda r W s C i W r ds, you have here, lambda r W s C i W r. So, those two terms will cancel out, leaving behind please understand leaving behind, now I am the erase everything, put it in the very compact form. The compact form will be, because this is same, W r M W s, W r M W s, you will have integral W r M W s dD, it is over the domain.

Here, it will find W r C i W s, that is common term, this will come left side, so you can write plus summation i k plus 1 to p W r C i W s ds multiplied by lambda r minus lambda s equal to 0, finally that what you are going to get. Now, you say if my Eigen values, which are lambda r and lambda s, if they are difference that means, this minus that is not 0, which implies this term must be 0, that is why you say distinct Eigen values.

But suppose, you have repeated Eigen values then what will happen then for this self adjoined positive definite system, you have distinct Eigen functions. Even though, the values may be repeated, but you still can have this thing Eigen functions, which are orthogonal to each other, that is why now you say, lambda r is not equal to lambda s, this is it.

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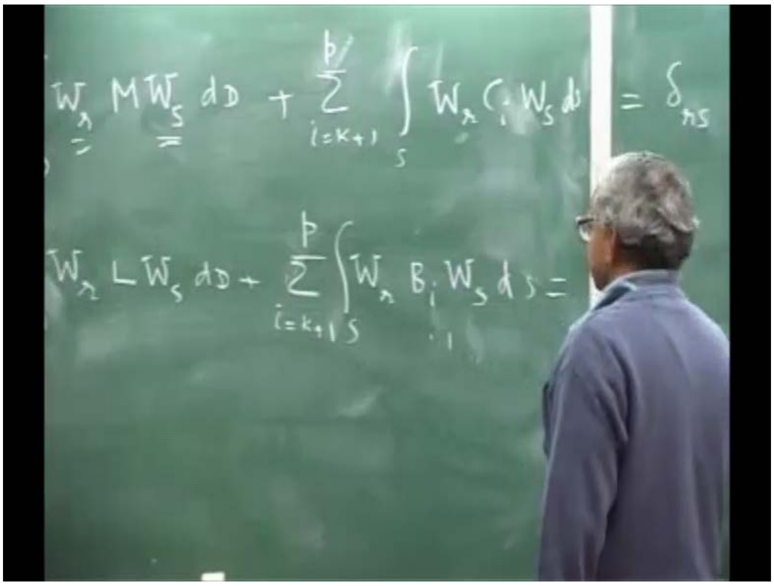

If it is equal then what, then also for this class of problem, you can get the orthogonal Eigen functions, they are distinct. So, the orthogonality condition is defined by this expression that is, integral W r M W s dD plus summation i running from k plus 1 to p integral surface W r C i W s ds. This is you write it as, because it is like a r equal to s, you get 1, how do you get 1, can you guarantee 1, that is where you now talk about a Eigen these functions.

I make them normalized such a way, when W r M W r, W r C i W r this is equal to 1 that means, I am normalizing the Eigen function. Even if you do not normalize nothing will happen, problem will be fine, because this is called orthogonality of Eigen functions, orthogonality condition of Eigen function. Now, we have defined it over M, you have also define it for, but please understand your M is sitting here and your C i is there, that is why this is only those boundary conditions, which have omega square sitting over there.

If all of them 0 0, omega square is not in there, you will not get that term, now what you can do is, you can go and substitute for M W s. Suppose I multiply that equation by lambda s, this because I can put into the lambda s I can multiply then lambda s M W s is

L W s. So, I can replace this by L W s, so that also I can write it D W r L W s dD summation, because here that going to become W r B i W s equal lambda s delta r, these are orthogonality conditions. Now, you have the complete set of, I would say conditions which you have developed.

Student: ((Refer Time: 1:08:12))

No, 0 means, I am putting when W r equal to W s then what is that, why I am a writing delta r s. See when W r equal to W s then you cannot say basically lambda r minus is lambda s is 0, lambda r minus lambda r, the other term need not be 0. But, if you have repeated Eigen values then you can have orthogonal Eigen functions for this, that can be proved. So, I am not getting into those prove, because if you do theory of vibration, normally it is given in the discrete system.

You can give the proof, self adjoined system self adjoined positive, everything you put it positive definite then you can show that yes, they will have orthogonal Eigen functions like this. Now, you have the prove for this, but please understand these are only for Eigen functions, not for comparison functions, only Eigen functions are orthogonal, not comparison functions. So, if you want to get exact solution, only very special case you can have a exact solution.

But, that is like an ideal case, uniform beam, you put a uniform beam, put a cantilever or simply supported, you can get nice solution, but ((Refer Time: 1:10:14)) is not a uniform beam and you can have masses attached that different locations. Then the question is, you cannot get a closed form nice solution, so the whole technique is, how do you get approximate solution.

So, this is where, what we will do is, we will study that part later, because I am not going into details of this part, because this you must have done vibration course. Theory of vibration if have done or this is also come, they call it structural dynamics, if you do the structural dynamics course, there it is taught, because I just want to spend 1 hour on this. I will immediately go for, how do you get the approximation, that is what our aim is, so that we will do in the next class.