Aero Elasticity
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Lecture – 25

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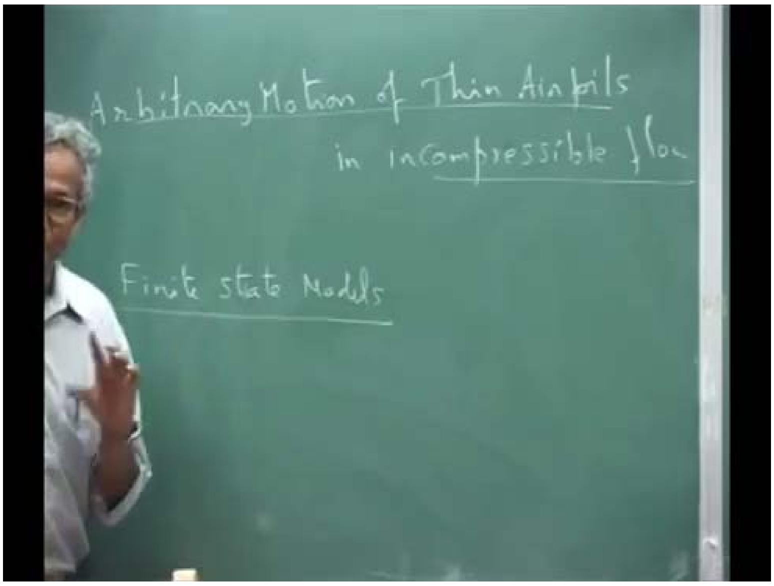

We will take up. If you have arbitrary motion of thin airfoils in incompressible flow and I will also describe something called as a part of this. I will tell you something about finite state models, finite state basically unsteady aerodynamic models.

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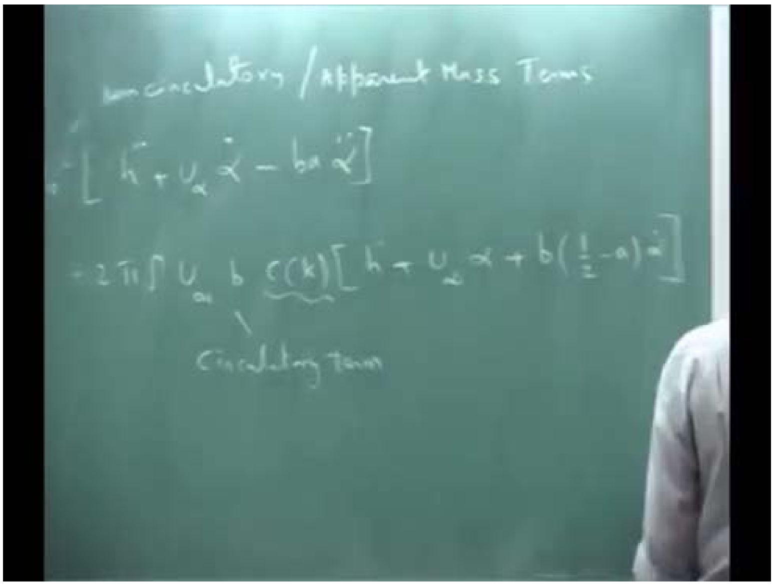

See we knew that for a harmonic oscillation, the lift was given by pi rho b square, just we derived last class, U infinity alpha dot minus b a alpha double dot plus 2 pi rho U infinity b c of k h dot plus U infinity alpha plus b half minus a into alpha dot. Now, if you leave this term, this is the non circulatory or apparent mass terms, this is the circulatory term. Here, the c of k you know that is the Theodorsen's lift deficiency function.

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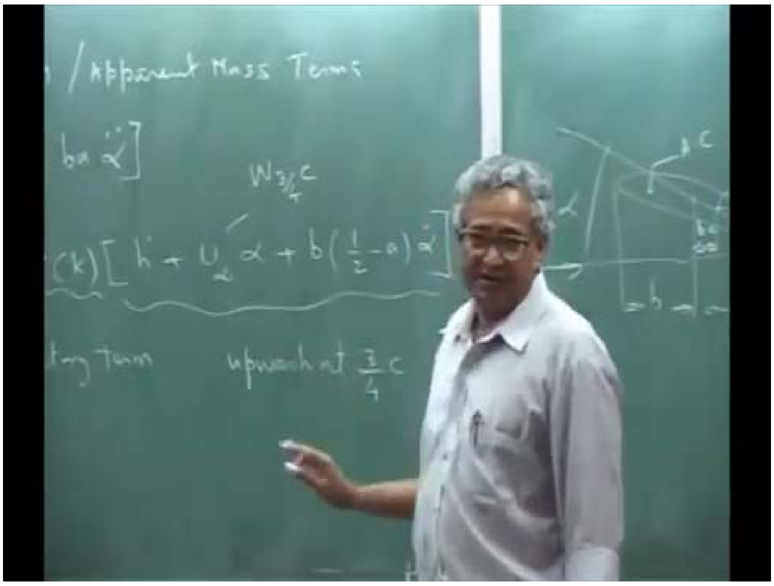

This particular term if you look at it, that is nothing but , because we have the airfoil, we said this is b, b and our elastic axis is given as, this is at b a from the center, this is our aerodynamic center and this is our h and alpha is, this is the alpha. If I want the angle of attack, angle of attack effective at three fourth chord, if I want that then this is nothing but , alpha plus the downward velocity here is h dot then what is that, this is b by 2, three fourth is b by 2, b by 2 minus b a.

So, you will have h dot plus alpha dot b by 2 minus b a over, now if you look at this, if I multiply by U infinity, that is nothing but this term, this is called the up wash at three quarter chord. Sometimes people use a negative sign, W three fourth c is negative of this term people put it, up wash. Technically you say, you take out the U infinity outside then this is effective angle of attack at three quarter chord point, but the lift is at quarter chord, please understand the lift is at quarter chord.

This is only for the circulatory term, because this is the only term which contains a frequency parameter, which is c of k, which is the Theodorsen's lift deficiency function, k is omega b by U. Now, this is fully time dependent no problem, but here there is a frequency sitting in. Now, if I want for arbitrary motion, earlier people thought how do we extend this theory to, arbitrary motion means, suppose you have your airfoil, you give suddenly a step change in the angle of attack and then hold it.

That means, you are having 0, you give a step input, how the lift will vary, that is an arbitrary motion. Now, how do you get solution for those type of problems and if you have a vertical gust coming and hitting, how the lift will vary with time. Now, I will first describe what is the classically what people thought, later I will describe how you can get that solution from finite state unsteady aerodynamic model. Now, that you can use it even for your flutter calculation, it is a very powerful, this is very powerful.

So, what we do is, you know that, this is in time please understand, this is a function of time. So, you can call it W three fourth c, that is the velocity. You can say, this is also the fluid velocity, because please understand I am taking h dot is down that is, the body motion, alpha is the instantaneous angle, U infinity alpha because U infinity is coming this way. I can take a component which is basically, which is almost like this way, that is how and h dot alpha dot, this particular term is body motion, this is due to angle you can say.

Now, this in effect represent the fluid flow coming up at that point or else you simply take h dot this, W three fourth c upwards, they will say. It is a general terminology, but they put a minus sign, but here it is not necessary that you have to carry, but so long as you understand what you are doing, that is perfectly fine. Now, this is a function of time, h is a function of time, alpha is the function of time, therefore alpha dot. I am going to call this entire term, W three fourth c t, entire term this is the function of time.

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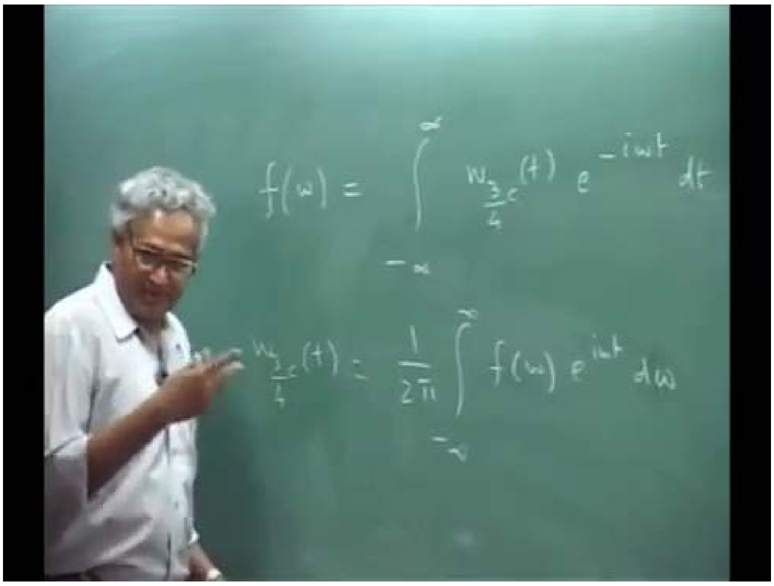

If any function of time can be converted into frequency domain, now I am going to call it, in the frequency domain this is using Fourier transform. Fourier transform W three fourth c, which is a function of t, e power minus i and inverse is, you will get 1 over 2 pi f of omega e power i omega t d t, this is a Fourier transform. Now, what I do is, I say that, I know this, I can get the Fourier transform, if I get the Fourier transform that is, in frequency domain f of omega, now can I get the ((Refer Time: 10:01)) d omega that is, d t.

Now, you can get your circulatory lift please understand, I am not looking at the apparent mass term, the circulatory lift you can write it in frequency domain. Suppose, if you say, this is the frequency content of W c t, for every frequency you will have some f of omega. The f of omega is a function, for every omega you will have a magnitude, that magnitude if you write it in frequency domain, please understand I am writing now, this is frequency domain.

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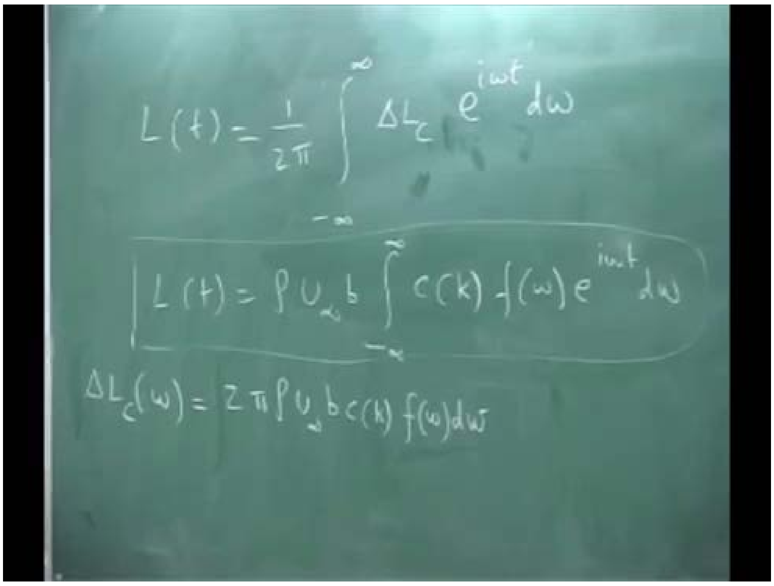

So, I will call this term delta L c that is, circulatory lift in frequency domain due to at a particular omega that means, it will be only f of omega at that small value f of omega, you can put d omega. Now, if you integrate, you will get the full over the entire domain, now you do transform, because you know this quantity, you can do the transform. When you do the transform, so I am erasing this part, if I do the transform, I will have L c of t, because I have to get the transform of this delta L c, full integrated over the whole thing.

I will have 1 over 2 pi minus infinity to plus infinity, I will have basically delta L c e power i omega t. So, I can directly write here, this is I will put it delta L c, if you do not want the integral you can leave out the integral, this is due to only this omega then you will have e power i t. Now, I can write this, I will go back and then put it here, I will have 1 over 2 pi will cancel out, I will have rho U infinity b c of k and f omega will be there, e i omega t d omega, this is my L of t.

Now, I can go back and write my lift expression as, for arbitrary motion please understand, it is left as it is.

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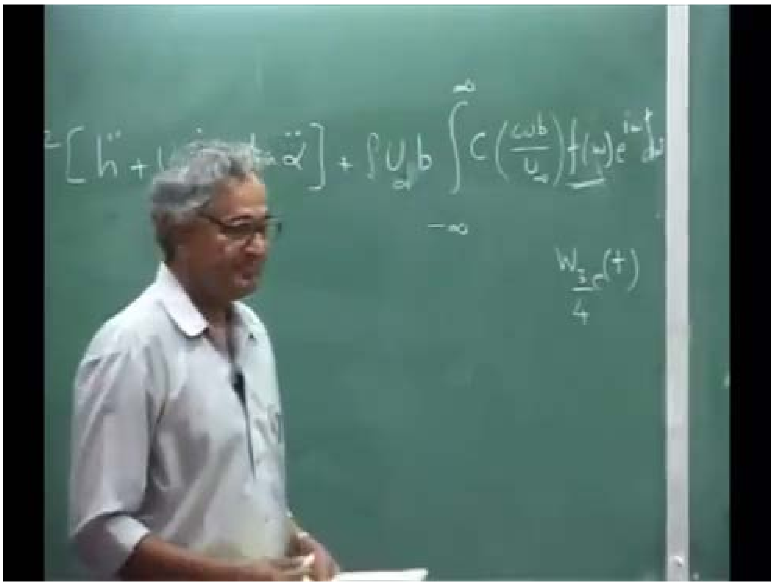

It is pi rho b square h double dot plus U infinity alpha dot minus b a alpha double dot plus rho U b minus infinity plus infinity, c k is nothing but , omega b by U, omega d omega. Similarly, you can write for moment expression also, now I have lift for any arbitrary motion of the airfoil. But, only problem is, how do I get this, because I have to get the Fourier transform of that W three fourth c. And then I should substitute that transform and then do the inverse transform.

Now, this is what initially they started, now you see the two points, which are important is, this is the Fourier transform of this. But, lift is at quarter chord, that is why three quarter chord point and quarter chord point, they are actually called forward and rearward neutral points, it was a term introduced by Kushner. Now, I will come to something, this was a challenging problem initially, first they thought that, I will give you one particular problem.

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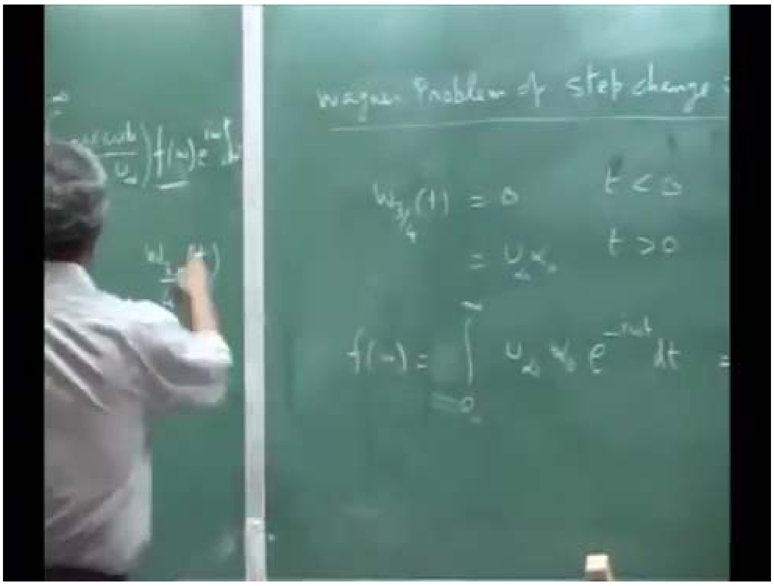

There are two types of problems which are solved, one is Wagner problem of step change in angle of attack that means, W three fourth c t is 0 for t less than 0, this becomes U infinity alpha 0 for t greater than 0. This is a step input, now you know my h double dot, alpha dot, all those are 0, just this check. Now, for this, you can get f of omega because f of omega is minus infinity plus infinity W three fourth c t, which is U infinity alpha naught.

But, anyway minus infinity is not there, so this will become, because the function does not exist before e power d t. This if you integrate, this will be actually U infinity alpha naught over i omega, because you have to set. Because, e i omega, because this is a constant, e to the power minus i omega t divided by i omega minus, you put a minus sign then set the limits then you will get that value.

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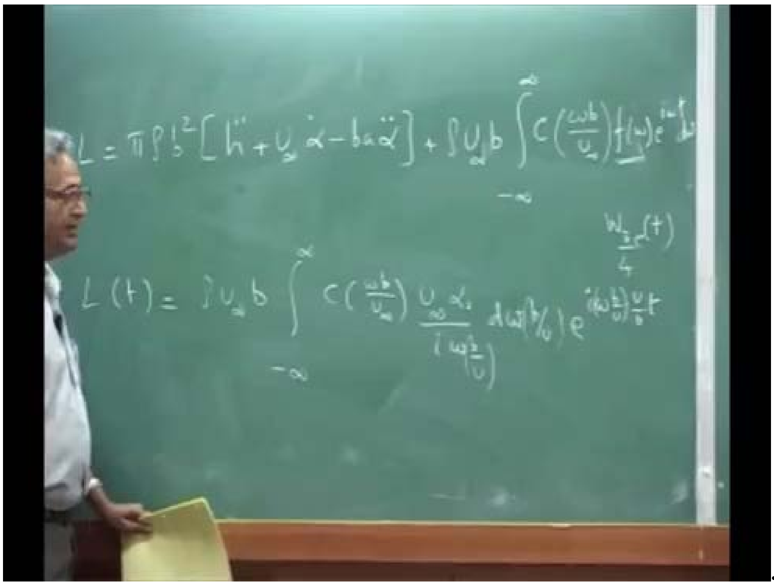

Now, you can take this and substitute here, you will know that L of t becomes, because they always these terms are 0. So, you will have rho U infinity b integral minus infinity to plus infinity c of, f of omega is nothing but , U infinity alpha naught over i omega d omega. Now, what you do is, you convert this, you put a b by U you multiply, e to the power i omega t, that omega also you put b by U into U by b into t.

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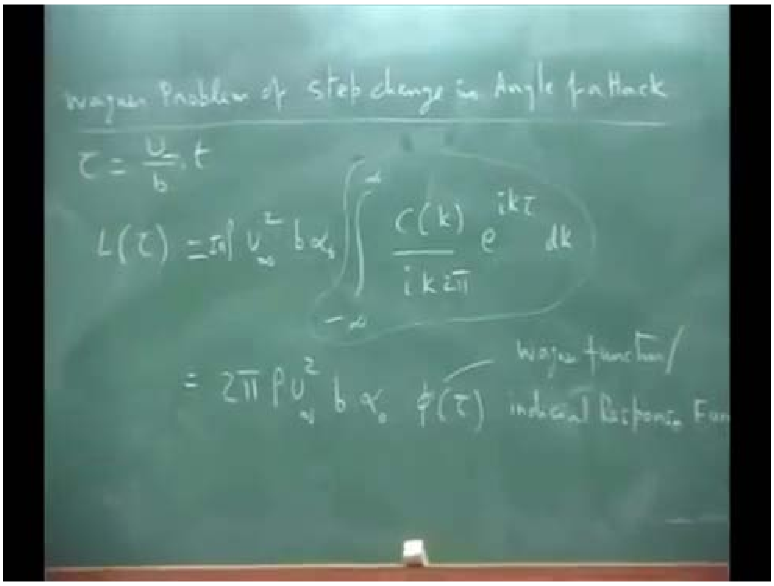

Now, this will be converted into L of tau, please understand I am putting tau is non dimensional time, U over b into t and omega b by U infinity, actually you have to take U infinity ((Refer Time: 20:13)). This is nothing but k, this is also k, so you will have, you can take out the U infinity alpha naught outside, so you will have rho U infinity square b alpha naught integral minus infinity plus infinity c of k over i k e to the power i k tau d k. Now, you can multiply denominator by 2 pi and numerator by 2 pi, this particular expression is written in this fashion.

This is given as 2 pi rho U infinity square b alpha naught phi tau, sometimes it is called Wagner function or indicial response function that is, this function is you multiply divide by 2 pi multiply by 2 pi. So, you can do that, this is 2 pi, here you will have 2 pi and this particular term is this. Now, I will give you, people have obtained, you know that this function c of k you know it, it is given in terms of Hankel functions, which are in terms of Bessel functions of first and second kind.

They have given approximations of this function to this p of tau, that approximation is I will just write the approximation. Several people did, even I have also done that, that is why I introduce, how we calculated that I will tell you by the next approach.

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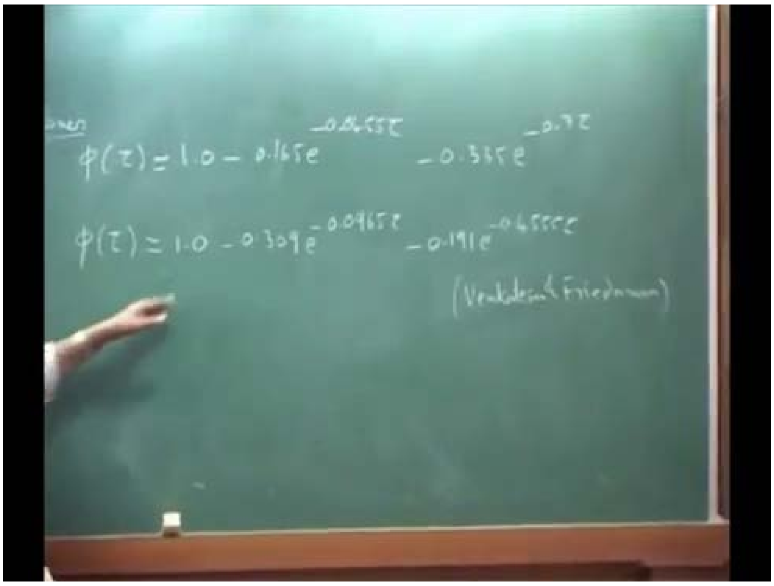

Jones, he gave this approximation, one that is, these all approximate 0.165 e to the power minus 0.0455 tau minus 0.335 e to the power minus 0.3 tau. See, there are several, this is by Jones then there is a Dowell’s approximation. Now, I will write our approximation what we got, this is actually I will put it, we have several orders of approximation, we developed, 1 minus 0.309 e to the power minus 0.0965 tau minus 0.191 minus 0.4555 tau, this is by me and my professor.

I will give the reference and we have several approximations, we also have third order approximation that means, three terms are there. How we obtained I will tell you, because I know about this particular thing, this is they have done some approximation, something they assumed and then try to calculate, but how this function looks, that is the most important thing.

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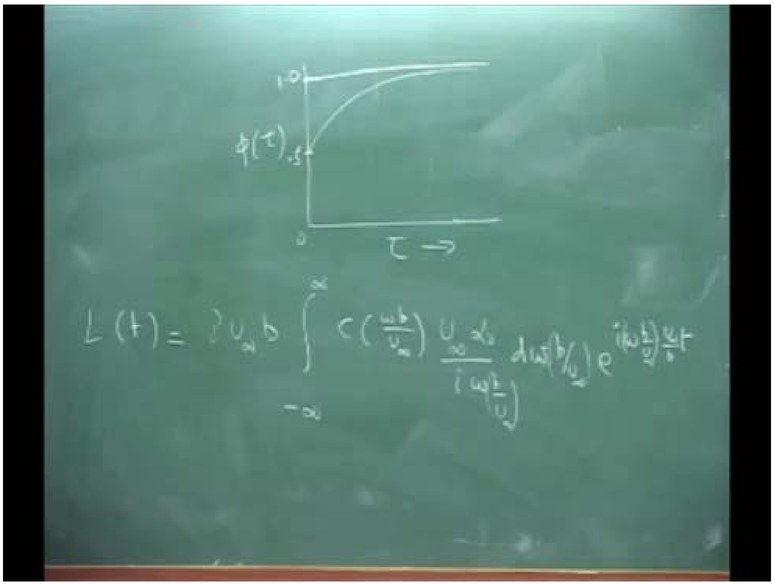

That is, this is tau, this is phi of tau, when tau is 0, it is 0.5, so the function starts with 0.5 and it ends at 1.0 and it will go like this, asymptotically it will reach 1. That means, when I give a step triangle of attack, but please understand these functions will give very good result, but I will tell you the procedure, how we got it. Because, ours is much more elegant and highly, I would say physically meaningful things, there is some mathematical basis for how we obtained.

So, this is called the, now you know in a fixed wing case, whenever you have a change in angle of attack, the step change in angle of attack, the lift will not suddenly become this value, because alpha naught is the angle of attack. It will not suddenly become lift is equal to alpha naught, it will take some time and then it will reach asymptotically like this, that is because of the vague structure. On the other hand, see that is a much earlier, our time we published in 1986, 1984, 1986 period.

In the case of a helicopters, it is a little different, if you change the angle of attack rotating, this will go up and then down, that is the difference between the helicopter and fixed wing. So, for helicopters we gave a first, it is part of that paper, that is why I given, this also we got, we got for a helicopter also. Now, that is the different characteristic of your system, the aerodynamics associated with that for a step and there is a experimental evidence for helicopters, that is how this became very fundamental work. But now, I will briefly describe, how we obtained this, I will give the two references then you will know and it is a very cute, I would call it a very elegant thing, I leave this part, let us throw it out.

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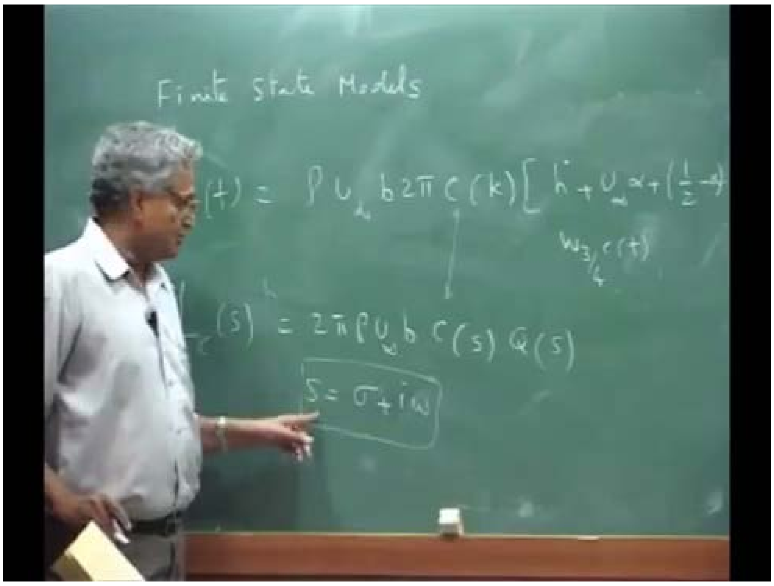

But this I will take it as finite state models of unsteady aerodynamics, again compostable theory nothing else, because there is not much time left, I thought I will directly go there. What we have is, we got what, L c of t is, this is only circulatory, leave out the apparent mass term, that will always be taken, rho U infinity b, you have 2 pi c of k h dot plus U infinity alpha plus half minus a b alpha dot, this is as usual W. Now, I am going to call this, if I write in, this is q of t some function, this is W three fourth c, I use the symbol q.

Q is the velocity at three quarter chord point and lift is happening, in between there is a c of k, I said that control theory, there is an input, there is an output. That means, in between this is the transfer function and this is given in time domain. Now, if I want to convert into Laplace domain, I will simply have what, L c of s Laplace, that will be 2 pi rho U infinity b c of s, C of s is please understand Theodorsen lift deficiency function applicable for entire Laplace domain. This is c of k is given when the Laplace variable is only imaginary part, the real part is 0 and then I am putting it as Q of s, where s is sigma plus i omega, because this is what Laplace variable is. The moment sigma is 0, this is nothing but , the Laplace becomes ((Refer Time: 31:19)).

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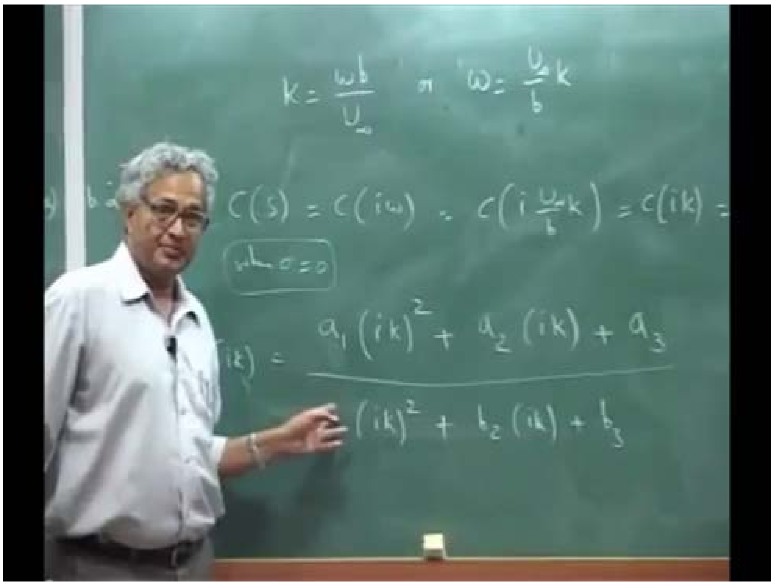

Now, we know c of k is omega b over U infinity or your omega is U over b k, now C of s becomes c of i omega when sigma is 0. Because, sigma is 0, this is i omega, i omega is nothing but what, omega is c of i U by b k U infinity ((Refer Time: 32:09)), which I can call it as c of k, that is what I am normally call this. Because, I say this is c of k, k is i k that is all, because Theodorsen lift deficiency function is a complex. Because, you know that, this is nothing but , h 1 2 k over h 1 2 k plus i h 0 2 k, this is the Theodorsen's lift deficiency function, which is given in terms of Hankel functions.

Now, I am going to say, I can represent them by rational approximation, rational approximation means, I am going to write c of i k as some, I have used a different step, may be I will use the same thing a 1 I can say i. Because, this is omega, omega I can put omega, omega I know U infinity by b over k. So, I will put this particular thing i, omega what is that, I can change it, let me do one small change, c of i k I will put it a 2 i k whole square a 1 plus a 2 i k v naught. I will say a 1 b 2 i k plus b 3 that means, this is a rational approximation, this is like any Laplace, any transfer function.

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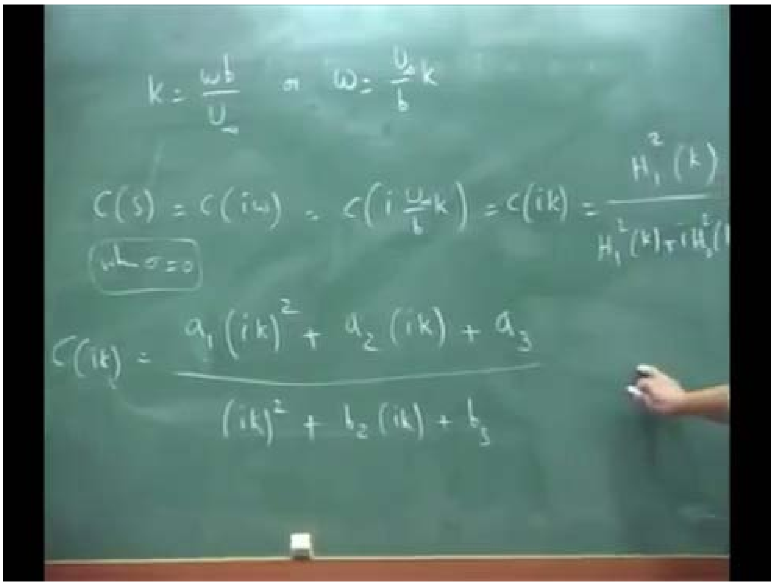

Because, you know that, if you are given a transfer function C of s what will you do, you will write it as a ratio of polynomials, say a n S power n plus then b n S power n plus a 0 plus b 0, this is called the rational approximation of c of i k. I can take that into that, because I can take as the constant, I can take it out, it does not matter, I can always, because this b 1 I can it out and then put. Now, you know I can find out what are the values of a 1, a 2, a 3, b 2, b 3, because I know this function, I know this.

So, I can always find by curve fitting, now I can write to any approximation, but there are certain characteristics features. Characteristic feature is, you know that, c of k which is given as f of k plus i g of k, because this is known real part, imaginary part.

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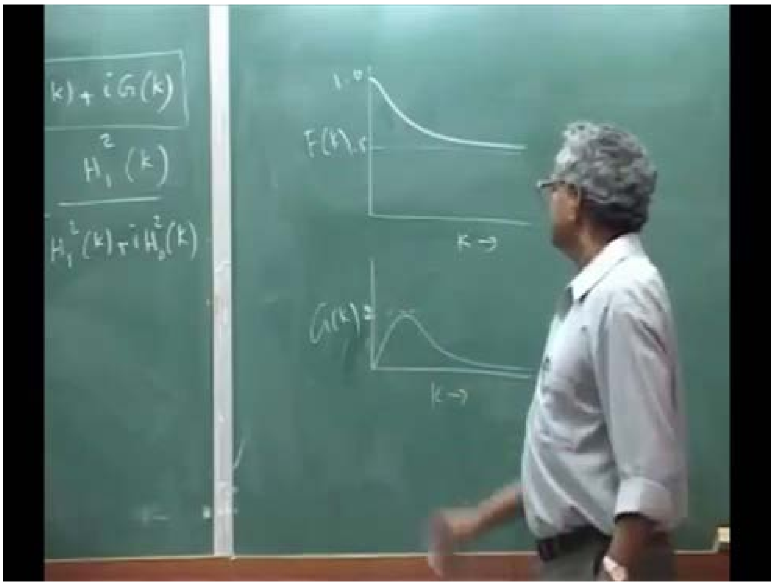

F of k goes from 1 to 0.5 that is, if you plot f of k versus k, it starts from 1, it comes to 0.5, this is 1.0, this is 0.5 and g of k it goes up and then it comes down like this, this is around 0.2 or something like that. Now, I know these values, how they vary that means, at high frequency, my value becomes 0.5. At low frequency, low means, when k is 0 a 3 by b 3 must be 1, when I go to k infinity that means, this is done, a 1 becomes 0.5, because I know that, this is 0.5.

So, knowing the transfer function, I can fit a ratio of polynomials, this is called the rational approximation of your Theodorsen lift deficiency function. So, I used, because you can get all these things from bode plot knowing the nature, bode plot is real part imaginary part as a function of omega, that is what you plot, you can take it either 20, log 10, something like that in dB's you can put it. But, that gives how your transfer function form is and it will also tell you, where the poles of the transfer function must be, where the zeroes of the transfer function should be.

You can immediately figure out all the poles and zeros from the magnitude and phase diagram and then once you know, you can easily fit these values. Once I fitted then the function will give you final result so that, that way what we fitted was c of k.

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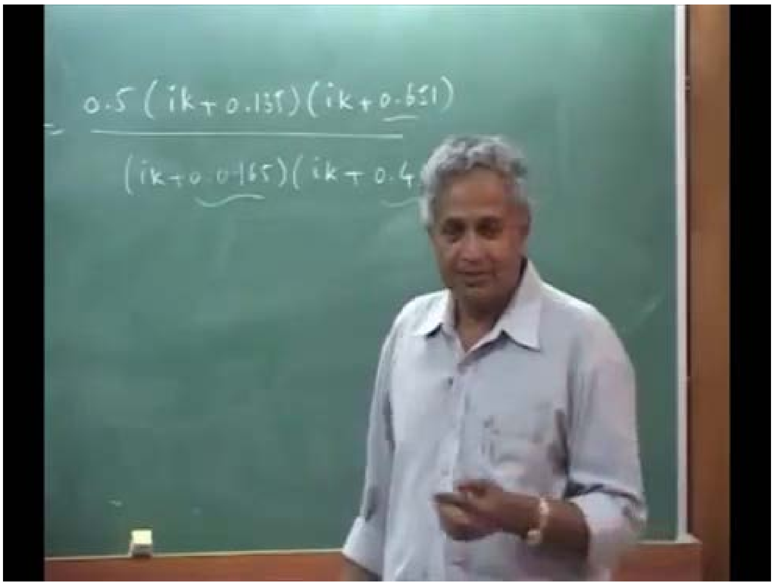

C of k became, for me 0.,5 i k plus 0.135 and i k plus 0.651 and divided by i k plus 0.0965 i k plus 0.4555, I can have third order approximation also, I have a third order also, there are you can have any order now. So, I actually fitted second order, third order, but third order it is almost exactly matching with the c of k, Theodorsen’s function exact, even second order is reasonably close, but first order it will gives a lot of error. So, you can have any order approximation, now you see the zeros, these are like a transfer function, i k you replace to s, this is any transfer function between input output.

And these are the poles and these are the zeros of that transfer function and it will fit exactly at that location zero, you do not have to put arbitrarily. What Jones and other people did was, they kept arbitrarily the poles, so our research was, it is not necessary, you do not know how to do that. Because, they did not use the control theory approach, they were just trying to fit the curve, whereas we brought the control theory and then finally it fitted very nicely.

And then this is now useful even for, now you can do time domain analysis, you do not know how to do v g method, you directly use this, go and do vector calculation just by Eigen value problem in time domain, full. You have a time domain unsteady aerodynamic model, but incompressible flow, it is valid, that is why if time permits, I would have asked you to do time domain vector calculation. We did the calculation, we got very good result, v g method, this method everything will be fine, you will get very excellent result. Now, let us see, why do we call it finite state, I will just use, here I erase all this part.

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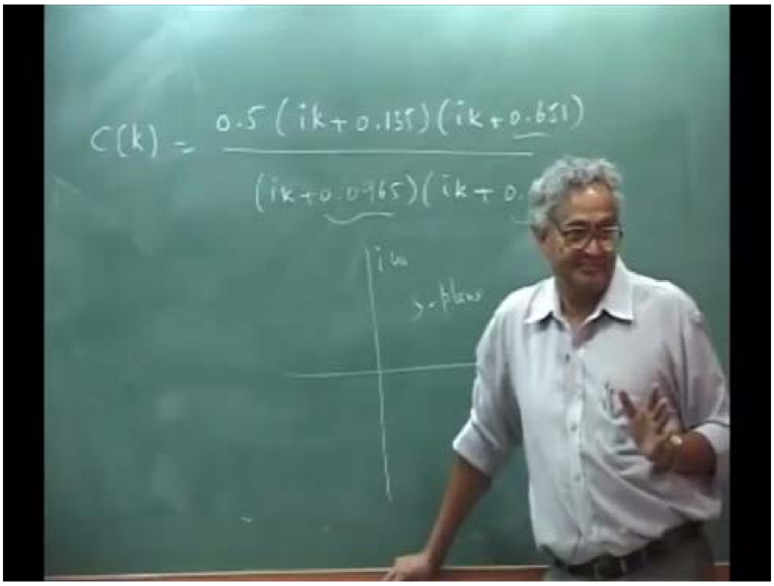

You have L c of s is rho U infinity b 2 pi, now I am there is a Q of s, C of s I have please understand this is i omega that is, in k. So, you know that, between the relation, between k and omega, because you know omega b over U infinity is k or in other words, omega becomes U infinity b k, you can use any one of the forms that does not matter. So, I am using 0.5 S b by U 0.135, because please understand this is i k, by multiplying the i k, I want omega, I want S.

So, you have to multiply by b by U, because if you multiply by b by U what happens, b by U omega is k. So, that is how this becomes, because i omega, S is equal to i omega and this becomes i k, that is what that expression is, into S b over U plus 0.651 over S b over U plus 0.4555 into Q of s. Now, please understand what we have done is, we have fitted the curve, curve means I got this approximation on the, because if I draw the S plane, this is sigma, this is i omega, this is your S plane Laplace plane.

I know the function c of k, the transfer function only along this axis imaginary, because c of k, k is omega b by U. I know c of k only on this axis, I do not know at other places, what is done is, you fit the curve in this line and then you say that, I am assuming that the same function is applicable for the entire Laplace domain, this is an assumption which we made. Because, we do not have C of s please understand, we have only c of k you follow.

But then you fit only c of k, that is why k fitted, but here what I have done, I have actually extrapolated it to say that, it is valid over the entire Laplace domain, this is an assumption in the absence of any transfer function. Because, transfer function what you do, because it is a linear theory please understand, linear theory Laplace domain once you have, because it is perfectly valid, you take the Laplace, whenever you put S is equal to i omega, you get the frequency response.

So, the linear control theory you can directly apply and here that is what we have done, I have fitted the curve on the omega and I expanded to full S domain, which is from the control theory, it is perfectly fine, because it is a linear problem. Now, can you write a differential equation for this, you can write and once you know Q of s, Q of s you take a Laplace inverse, Laplace inverse there are various ways of doing it, you take this, you multiply here, you will get a differential equation in lift, L c dot.

Because, S square will become double dot is equal to q dot q double dot and some q, I have a time domain model, once I know how q varies with time, I can immediately get lift, how it varies with time. So, this is perfectly a valid theory, it is useful for all control studies, if you want to do active control thing, because you need to know the transfer function, use the control theory to come up with a basically closed loop control. So, this is what our study was, I will give two references which we have used.

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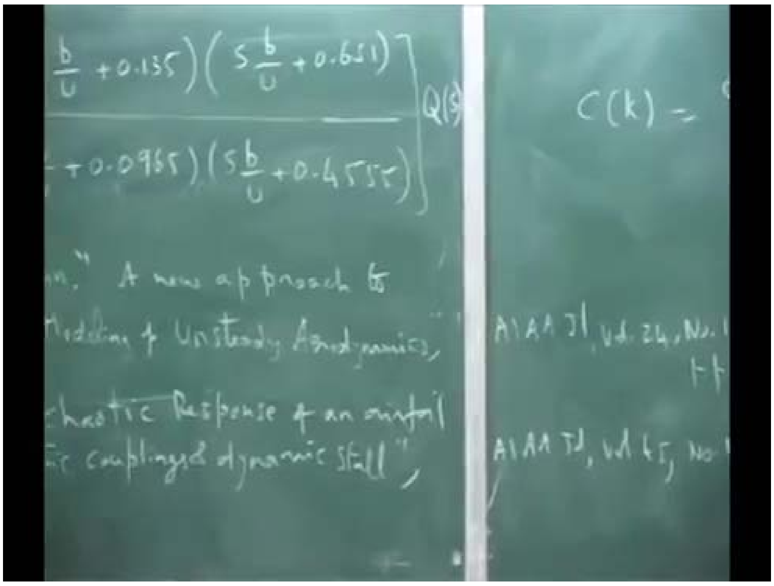

I thought maybe you can have a, this is the, so one is I will just write this is Friedmann and the title is, A new approach to finite state modeling of unsteady aerodynamics, this is AIAA journal, this is volume 24, number 12, 1986 and page number PP 1889 1897. And the second one is actually what we did with that is, with Laxman, because what we did was, this is for attached flow only, we extend to even stall that is, this Laxman and this is chaotic response of an airfoil due to aero elastic coupling and dynamic stall, this is again AIAA, this is volume 45, number 1 page 2007 and this is PP 271 to 280.

So, actually this is, if you look at the time gap, this was in 86, this is 2007, almost 21 years, but this we extent to dynamic stall. It is similar, but dynamic stall is non linear please understand, but we use the similar thing, but we also use the French one era dynamic stall model. Actually we updated that model, so we call it as a modified one era dynamic stall model or you can call it improved model, but the concept is same. So, this is what is called the finite state modeling of unsteady aerodynamics and this is very powerful method.

And this has been I heard that, this is being taught in some of the universities, because the same thing, because the idea is, you got the picture. Now, I can do that first and I will get inverse Laplace then I know L c dot, this may be a differential equation, but if I do not want in the differential equation form, what I will do is, I will add the expand this, put it and then I will call this as another name. I have a differential equation for this, this is called the finite state model, because finite state is, these are the finite states, states of the aero dynamic models.

Now, when you do Eigen value analysis, you will start getting Eigen values, which may not have the frequency or whatever you will get it, that will not have a corresponding structural value. Then, you say, this is due to aerodynamic modes, this is in some helicopter we have done that study, we found that, there are some modes that is, through dynamic inflow model. You can get some in the experiment that was measured, but there was no structural mode associated with that. So, you will find that, it is possible to get aero dynamic modes and if you want, I will write that modified form.

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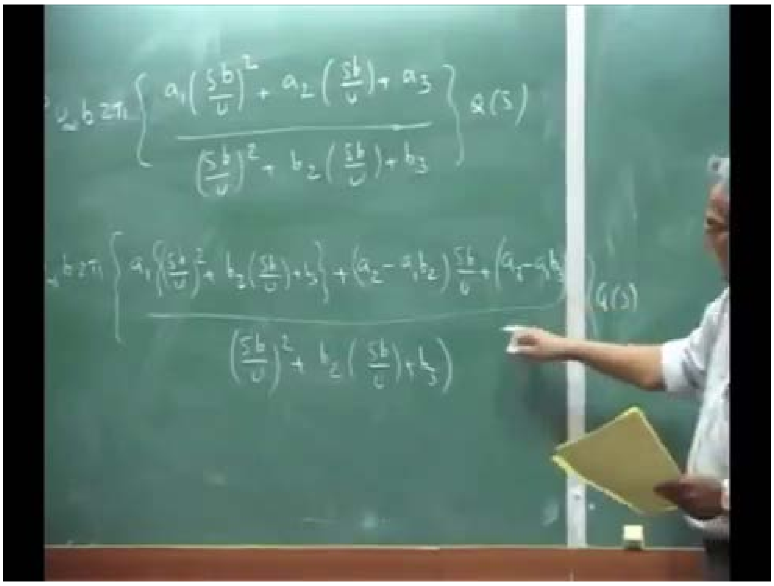

Because, you write it that first expression L c of s rho U infinity b 2 pi, you put it in this fashion a 1, I am writing it directly in a different fashion, a 1 S b by U whole square plus a 2 S b by U plus a 3 over S b over U whole square plus b 2 S b over U plus b 3, this is Q of s. What I will do is, since basically this is product, you multiply you will get this form, you go back and write like this, a 1 S b by U whole square plus b 2 S b by U plus b 3 then you subtract a 2 minus a 1 b 2 S b by U plus a 3 minus a 1 b 3, divided by denominator is same, b 2 S b over U plus b 3 into Q of s, this is 1.

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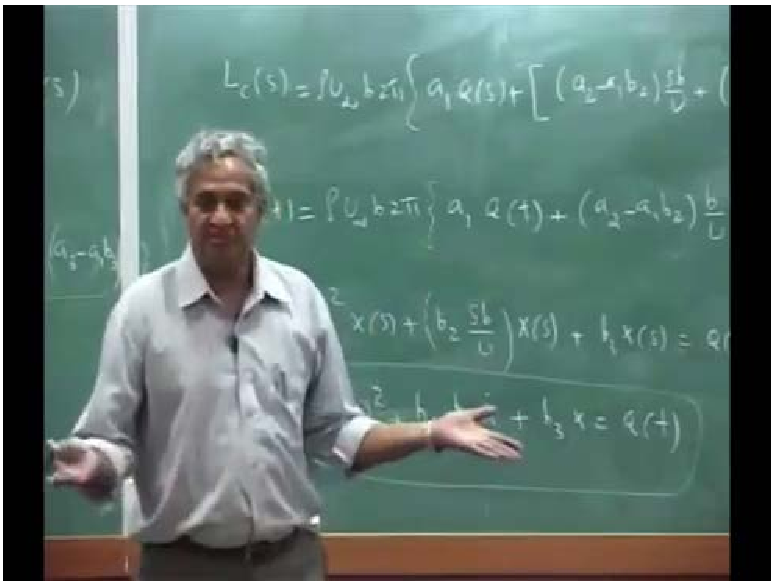

You will get L c of s becomes rho U infinity b 2 pi a 1 Q of s plus, you open a bracket you put a 2 minus a 1 b 2 S b over U plus a 3 minus a 1 b 3, you write Q of s over S b over U whole square plus b 2 S b over U plus b 3. Now, you see this particular term, if you call it as some X of s, this you call it X of s then what will have, I will have my time domain L c of t becomes rho U infinity b 2 pi a 1 q of t plus a 2 minus a 1 b 2 b over U x dot.

Because, S X s Laplace inverse x dot plus a 3 minus a 1 b 3 x, these are functions of time. Now, you have to get that, what is X of t, X of t you will get actually, you are writing what, x of let me write it this way, S b over U whole square X of s plus b 2 S b over U what, X of S plus b 3 X of s is Q of s. Now, inverse, you will get X double dot b over U whole square plus what, this will be b 2 b over U X dot plus b 3 X equals Q of t. So, this is my equation, this can be written again in state space form, if I know Q of t, I can get X of t then you know X of t, these are aerodynamic states.

Additional aerodynamic states, because you see, L c of t is not only a function of Q of t, it is also a function of this X 1, which are driven by again Q of t and these are the additional states of the aerodynamic model, that is why it is called finite state. You can have, now you see I made a second order approximation, if I make a third order approximation, this equation will become third order and I can have any order you can go on.

So, the step input you can put Q of t and then put it in Laplace domain, do the inverse Laplace transform, you will get the same result, as what was obtained by Jones other people. But, only thing is, the coefficients are different, because we obtained from applying the control theory between input output, there is a transfer function. So, the aerodynamic act like a transfer function, that c of k is a transfer function.