Aero Elasticity
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Lecture - 21

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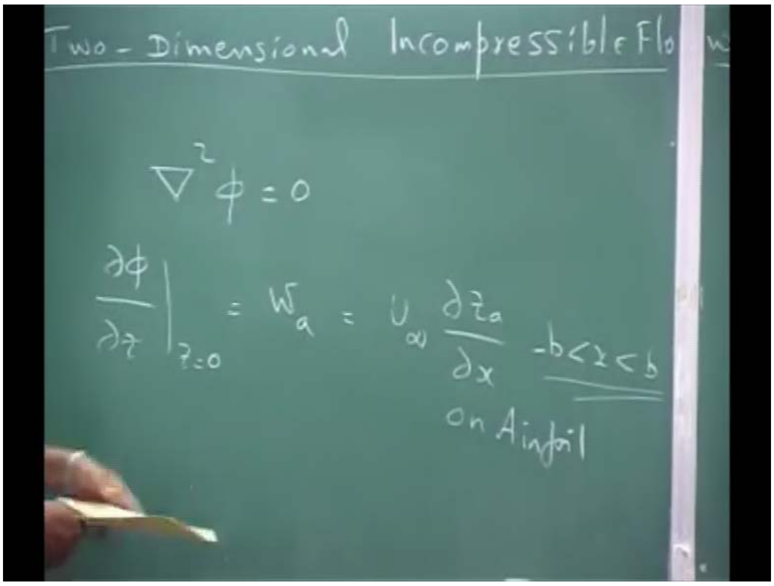

Today we start with Two Dimensional Incompressible Flow. First we take the steady case, and then we will do the unsteady case, in the steady or unsteady, because it is incompressible, your equation is Laplace equation. And your boundary condition on z equal to 0 plane, you call this as w a, and in the zone, where the airfoil is you will have u infinitely delta x, in the I will put minus b x plus b, this is on airfoil. This is the steady flow boundary condition.

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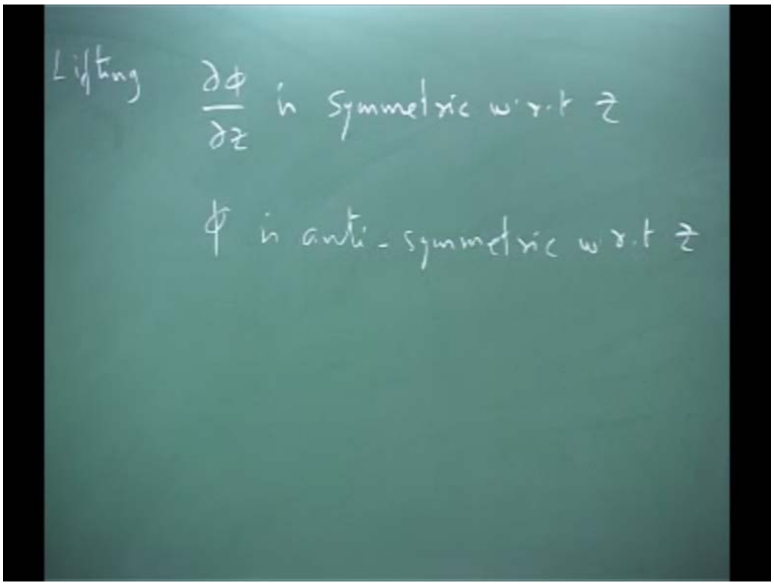

And because we earlier said thickness problem and lifting problem, and we mentioned that lifting problem lifting case it is just a revisiting what we learnt, delta phi over delta zee is symmetric with respect to zee, therefore phi is anti symmetric. That is how we get that the discontinuing in the pressure.

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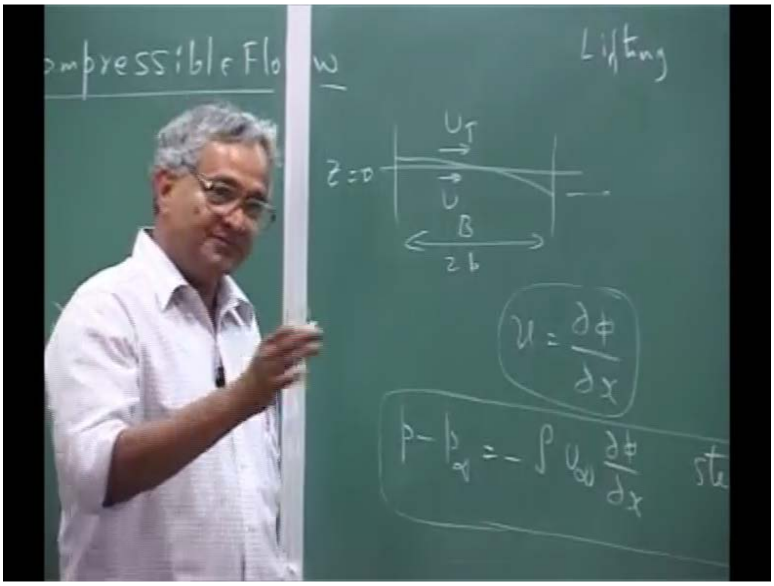

But if phi is anti symmetric if you calculate u which is disturbance velocity, if delta phi by delta x, because what is our pressure, if you look back the steady pressure disturbance pressure that is p minus p infinitely is you will have what expression minus rho u infinitely. You will have this, this is for steady case minus rho u infinitely for steady case, otherwise you will have a delta phi by delta t minus rho into delta p by delta t etcetera, the delta p by t delta theta we are neglecting it here. Now, if you see my disturbance pressure, I will have a anti symmetric here, u is the moment you cross z equal to 0 plane phi is anti symmetric means.

And now, there is a jump in the velocity that is why, you say that there is no pressure, disturbance outside the airfoil, but wherever the airfoil are the thin line is there. You can have, because there is a pressure difference, you can have a jump in the u across, when you cross the airfoil from z equal to 0 minus z equal to 0 plus, you will find that it is like this. If you have a airfoil, this is z equal to 0 line when you come here will have u, here you will have different u, this is u bottom, this is u top.

That can be a jump, because there is a u support of pressure, that is all the lifting happens, but moment you go here, outside the airfoils, this is the airfoil two b. There is not disturbance pressure, therefore delta p by delta x; that means, the disturbance velocity, u is 0 outside, but now u infinity, u infinity is existing. Now, you see I can have a velocity jump in the u direction, this is possible, if you have a vortex.

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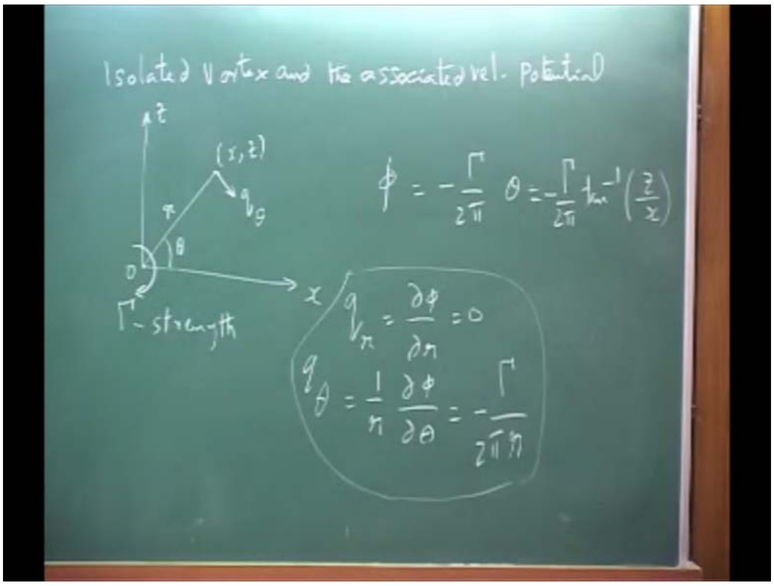

Now, you say this type of flow, there is a jump, if I have a vortex, vortex is what you have a vortex; that means, just the flow above is this way, the below is this side. That means, I can replace my airfoil by a series of vortices, and after that I am satisfying boundary conditions that is, but I can have a series of vortex on my airfoil. So, that is because of this now with this as starting point, we will see what is the potential, how do you write the potential, because that will satisfied the surplus equation for a descript vortex.

First we take isolated vortex, and the associated velocity potential, suppose if you say the vortex is at the origin this is my x, this is my zee, I have a vortex here. The vortex I call it a push strength, I call it as gamma this is strength, then velocity at any point under overtake and the distance is r. This angle is theta, this is at any, now the potential for this is given by for the vortex is minus, but delta is tan inwards zee, this is the origin x, I have kept the vortex at this point.

Now, because this is in collar coordinates, in collar coordinates if you want because like delta p by delta x is u, delta p by delta z is w, what you have in the collar coordinate, you will get q r is delta phi over delta r. This is 0 because phi is not a function of r, where as q theta is 1 over r delta p over delta theta, which is given by minus gamma over 2 pi r. This is the velocity q theta is along the data, but the minus psi come, because this is the vortex, I am putting it in clock wise direction that is why it came to minus.

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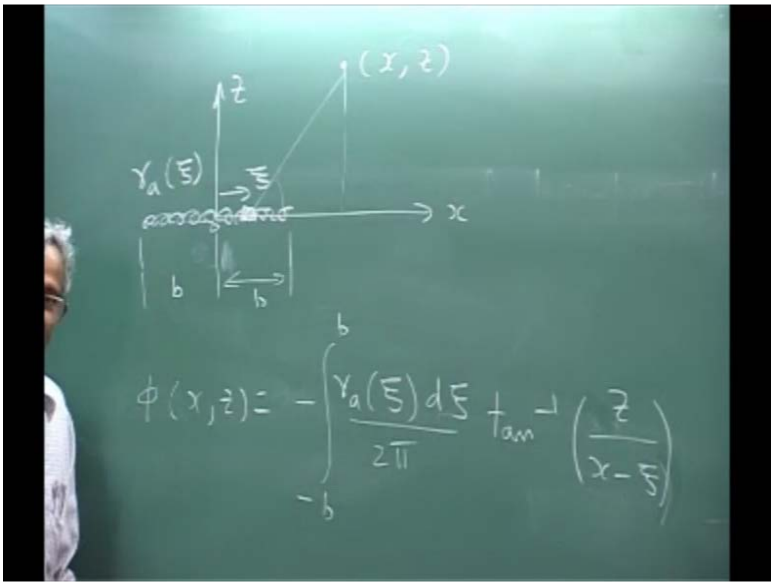

Now, you see if we have a distribution of vortices like this, on say we say this is my x this is my z this is 2 b, this is 0 I can call it b and; that means, this is my airfoil. I am representing that by, because I can have a jump in the velocity which is possible by a vortex. And the strength of the vortex, I am calling it as gamma a psi, where this is my running variable, this is per unit length.

Now, what is the velocity expression, at any point x comma zee, first velocity, and then the you can write the, first potential, then you can get the velocity, because the potential give to a vortex, which is at origin is given by this expression, tan inwards z over x. Now, if you take a small element of vortex here, I want to know the potential at this point, I will have c phi x comma zee is, I will use the same thing minus the strength in the vortex is gamma a which is into d.

Then the strength divided by 2 pi, then I will have tan inwards of this with respect to this, this is x, this is my vortex strength differential, but if I want for due to entire thing I will have minus b to plus b. Now, I have the potential at any point due to a sheet of vortex, which is placed b minus b plus, now you got the velocity potential, now you need to get the boundary condition. That is all have an expression for the potential, then boundary condition, then pressure, that is it you will have you are full solution, but before we go and get it let us look at few things let us same thing you will used even for unsteady aero dynamics.

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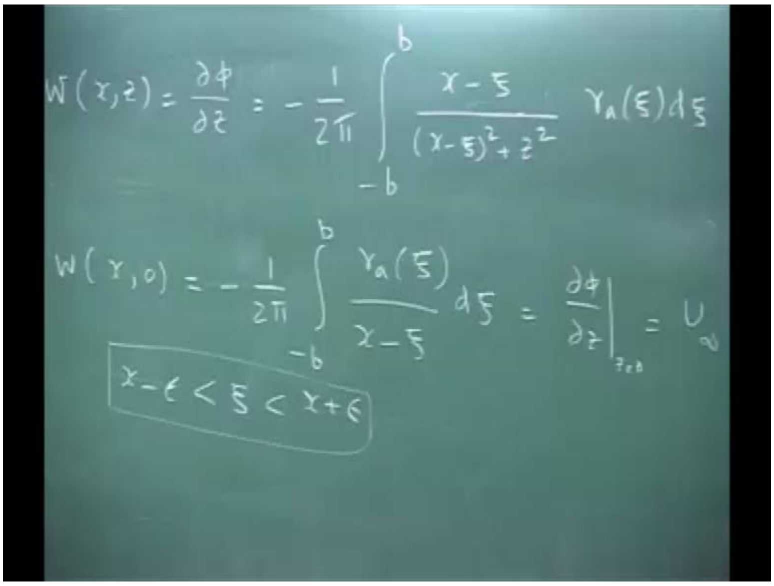

Now, what is my w, w x comma zee is delta phi over delta zee, this is at any point, I have to pick up this integration is only over psi, I can differentiate it with respect to z. I will get minus 1 over 2 pi minus b to plus b x minus psi over x minus psi square plus zee square gamma a, because tan inwards of that if you differentiate a with respect to that. You will get this, because this is a simple differentiated tan inwards, now this is the velocity at any point x z, but when I want to apply my boundary condition I am applying on z equal to 0 plane.

So, z equal to 0 plane I go I said z is used, but then I know that this is integral is a, because x minus psi will cancel out you will have x minus psi, but psi is in between the integral point. So, there is a singular; that means, this is a singular integral you needed, but then that is by the casting principle value you get the integral. Now, this is how it is integrated I will briefly say because this is very simple, you do not integrate right up to x equal to psi that is whenever psi become x is going to be 0, then zee is 0.

You come up to some epsilon minus epsilon, plus epsilon on either side integrated after that set epsilon 0, that is how you get that is casting principle value of the integral. Now w at x comma 0 is nothing but what minus 1 over 2 pi minus b plus b gamma a over x minus d, but w is nothing but we know delta p by delta zee at z equal to 0. On the airfoil it is delta zee, this is nothing but u infinity, this is only on the airfoil otherwise this is any x. Now, what we will do is let us go back, and then that is why this is a singular integral, what you do is you bring up to x.

You vary this till x minus epsilon, and then x plus epsilon, you do like this integral, epsilon you keep a small number, you will get the integration. Then once you get the after integration z epsilon goes to there, and you will get that limit, otherwise this will become singular numerically evaluation also will be a problem. So, that is why you always take a epsilon go around just like a and then send that 0, later now let us go and get the pressure expression.

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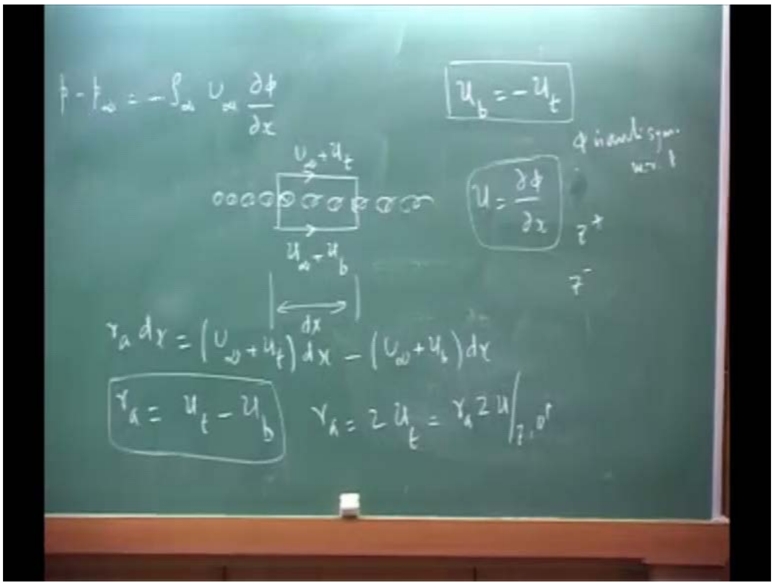

The disturbance pressure is minus rho infinitely u infinitely delta phi over delta x at any point, but this is I can get it as delta phi over delta x I want, I go here delta p over delta x I can get tan for that. So, let us write that u is delta p over delta x, which is minus 1 over 2 pi minus b to plus b gamma a phi, then you have minus zee over x minus psi square plus zee square d, but here if you send z equal to 0 you say. Then what everything becomes if because you set this what happens, now this is why you say that at, because we are put here, the vortex is here.

This is giving as a jump in the velocity above and below, what we do is not by this we express the relation between u and gamma not through that. We will do like this, you take a maybe I erase that part, because you know that u is delta phi by delta x, that is all that you know it, so let us I erase this part and you take this is my vortex. Now, I am taking a close contour, u the top I call it u infinity plus u top and this is the velocity in this direction, and this is the velocity u infinitely plus u bottom, this is over a distance d x.

What is my strength of the vortex, strength within that is gamma a d x, and then vortex, if you want what is gamma u, you close integrals, but velocity d l. So, you will get gamma a d x is nothing but u infinitely plus u t d x, and this side this is a you will take a very thin thing w, this d x is small this and this will cancel out, and here you will get minus u infinitely plus u b d x. Then you cancel out d x, you will get gamma a is u t minus u b, that is u top u bottom, but u is delta d phi by delta x, this is the disturbance potential and d phi is, but we started it is anti symmetric.

Therefore, this becomes anti symmetric, if it is anti symmetric means u t, then you will jump here it is minus here u t, so you will get delta phi by delta zee is anti symmetric, phi is anti symmetric with respect to z. That means, the moment I go z plus then z plus or z minus just above 0 plus 0 minus phi is changing; that means, phi changes delta p by delta x changes signs, so you have to change psi.

Therefore, I will have because of the anti symmetric nature of phi, I will have u b must be equal to minus u t, because of the anti symmetric nature. Now, if u b is minus u t or u b you put that gamma a becomes what 2 u t, which you may call it as just gamma a 2 t, because this subscript and not given out, because u is delta phi by delta x at z equal to 0 plus, this is the z equal to 0 plus.

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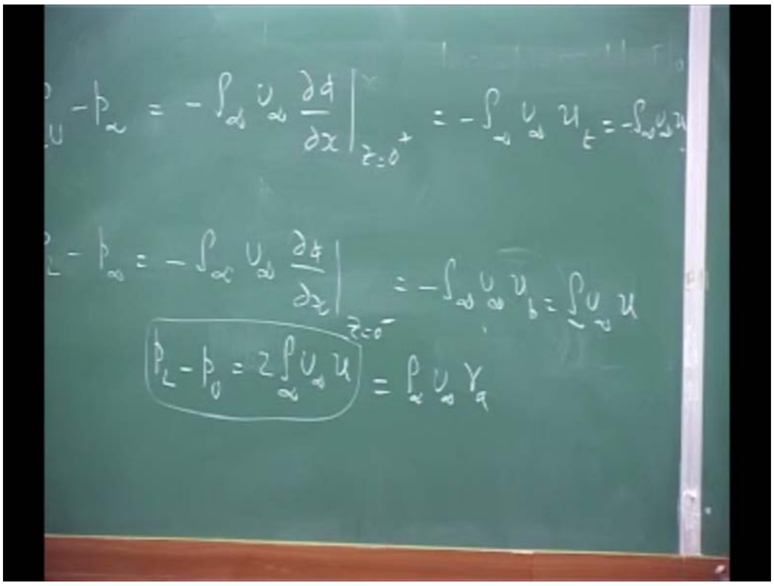

Now, let us go back and then write the pressure expression, now the my pressure expression is on the tops of surface, uppers surface minus p infinitely, this is upper surface, upper or top surface. This is equal to minus rho infinitely u infinitely delta phi by delta x at z equal to 0 plus 0 plus means my velocity is u t, so I will have minus rho infinitely u infinitely u t which you may call it u t is u.

So, I am just calling it minus rho infinitely u infinitely u, whereas that the bottom surface phi lower minus phi infinity delta phi over delta x at z is equal to 0 minus, which is minus rho infinity u infinity u d. Because, delta phi by delta x are at the bottom of u b, but u b is minus u t, so this should become plus and you will have rho u infinitely u, because u b is u t u b is u t.

Now, if you want the pressure p l minus p u, differential pressure this is this minus this, so you will get two rho u infinitely u, and what is 2 u, 2 u is gamma a, gamma a this is not gamma a this is 2 u gamma a is 2 u. So, I can write pressure is rho or rho or rho infinity, you can take it, I put the it is rho infinity u infinitely gamma a.

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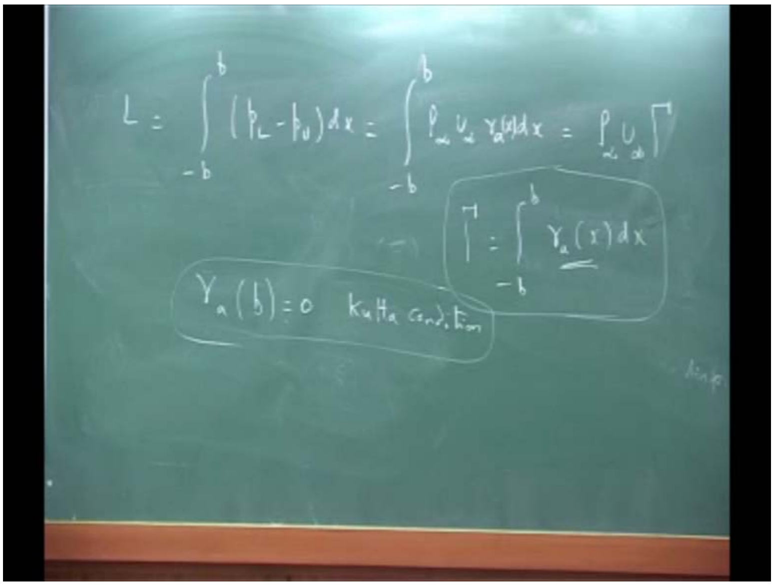

Now, if I want a lift you integrate the pressure difference over the full card, that is you will have total lift becomes minus b plus b this is gamma a, but please not one thing gamma is a function of x, it is not a constraints, it can be different. Now, this give you rho infinitely u infinitely, which is gamma a is that total minus b plus b gamma a x d x, but then you also have one more condition. You said that away from the airfoil, you cannot avoid discontinuity, therefore d u minus d into phi is 0; that means, d l minus d t is 0 at the straining edge.

So, straining edge p l minus p b is 0 means gamma a is at b is 0, this is the condition, that is all because you do not have any disturbance pressure, therefore delta p by delta x is 0 delta p by delta x is 0 gamma is 0. So, you said straining edge, but you may say why not

I said the leading edge, leading edge is also in the front, but then experiments just purely experiments, but you said this is 0. The result is get is very close to the experiment except near the leading, that is why you can always set leaner point also, but we leave that leaner point this is purely from external law operation.

Now, we need to go back to our, how I get my gamma we know that my lift is rho u infinitely capital gamma, only thing is now I need to get the variation of my gamma with respect to x. How do I get that, we will start here see this is you means done with the confirm matrix approach, but here we will follow a different technique.

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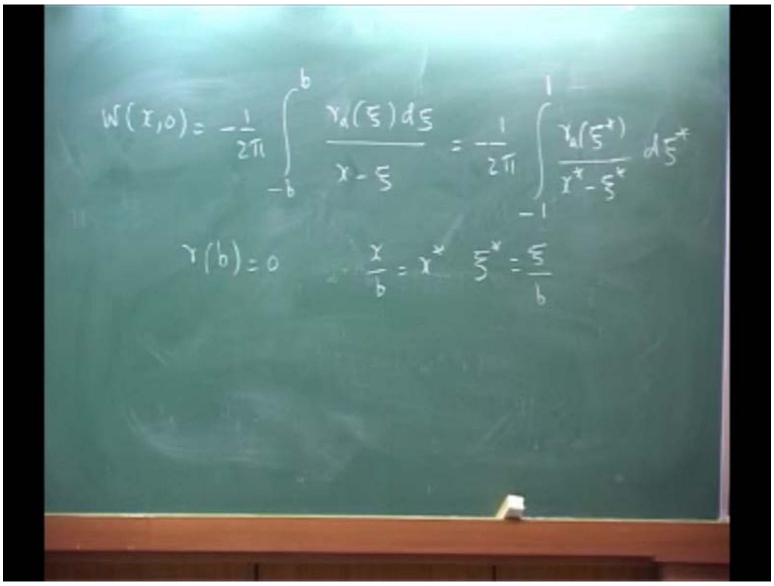

We take w x comma 0 is given by 1 over 2 pi minus b plus b gamma a over x minus phi, you first with the condition gamma b is 0, first you do non dimensionalisation, that is non dimensionalines with respect to length. So, you divide by x by b you call as x star, this is psi by non dimensional's you substitute here, if you non dimensional you will get minus 1 over 2 pi minus 1 to plus 1, because limit is changing. You know gamma a psi star x star minus phi star d phi star, now how do we proceed from there is this is from I will just briefly give you.

There is something called a that equation on the airfoil, I know w, because w is u infinity delta zee by delta x on the airfoil. So, if I somehow do a inwards, this is an unknown, this is a known on the airfoil, if I can relate that to this on the airfoil, then I have the full solution gamma a. Now, this particular thing this what the conformal unit you may have done, but here we are going to use something called this inversion integral.

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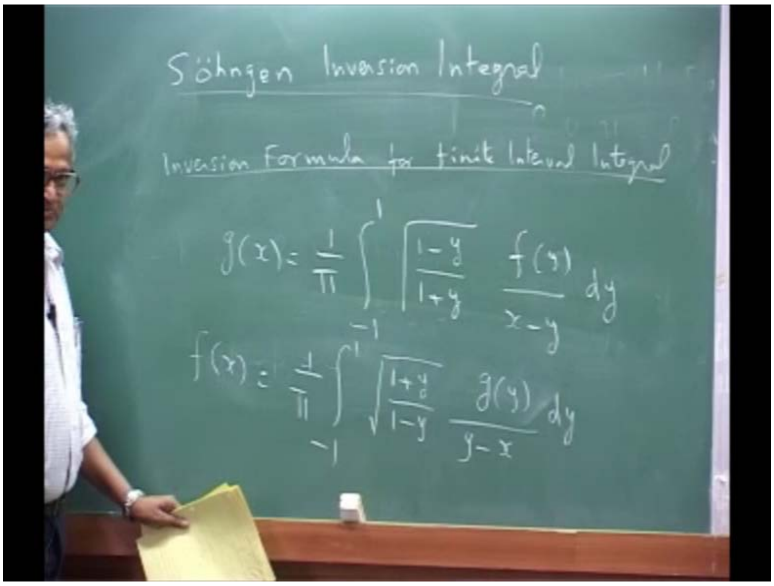

This called Sohngen inversion integral this is from the theory of linear operators etcetera, I will just briefly give you one expression, the two integral conditions this is like this. Inversion formula for finite interval integral, you look at this is I have two function, they can be discontinue, they can have g of x 1 over pi minus 1 to plus 1 root of one y over 1 plus y f of y over x minus y d y. If this is g n f are related like this, then you can write f of x is 1 pi minus 1 to plus 1 square root of 1 plus y over 1 minus y g f y over y minus x d y, and there are some condition, because this is another condition.

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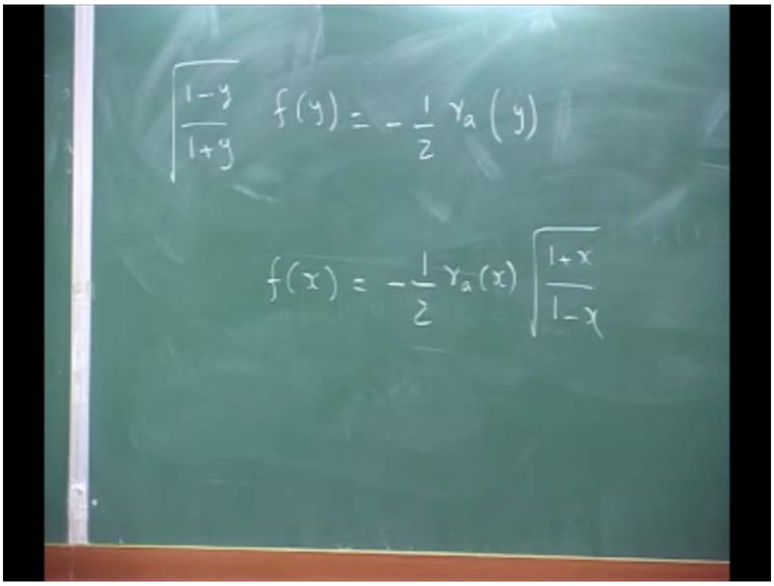

That is integral minus 1 to plus 1 minus x over 1 plus x mod f of x square d x is essentially, minus 1 to plus 1 plus x by 1 minus x mod g of x square d x, we use only this. Now, what you do is you look at that integral minus 1 to plus 1 gamma a, but will not be using this, that is the condition, which is they satisfy, if you have this I can get this; that means, I having w to in terms of gamma a, this is w this entire thing I call it as gamma a.

Then I can get gamma a in terms of w, but in minus 1 to plus 1, because that is the motion of the airfoil, u infinitely delta zee by delta x; that means, I have my gamma expression exactly new. And this is what is very elegant and this particular thing is used, even in the un steady aero dynamics, now let us I am going to write like this, that is root of 1 minus over 1 plus y f of y. This I am writing it as, because to look at this expression and that integral this is minus 1 over 2 gamma a, into because this is y dependent, so I am calling it as instead of psi a I will have y.

Now, you know what is f of y or you know what is f of x, basically because you know this function you have defined, so I can get what f of x is in a sense just to change y to x. This will become minus 1 by 2 gamma a x root of 1 plus x over 1 minus x, because this goes that side, now I have f of x, and I can go and write. Now, let us write what is this is g, g is nothing but my w, so I will put w here, so when I write that expression that is going to become, because what I am having minus 1 to plus 1 I am having gamma a.

So, I am putting here f of x is this expression, now I have used a non dimensionalisation x over b over x star etcetera, here it is only everything is minus 1 to plus 1, there is no star that is way these all are minus 1 to plus 1 only, there also minus 1 to plus 1.

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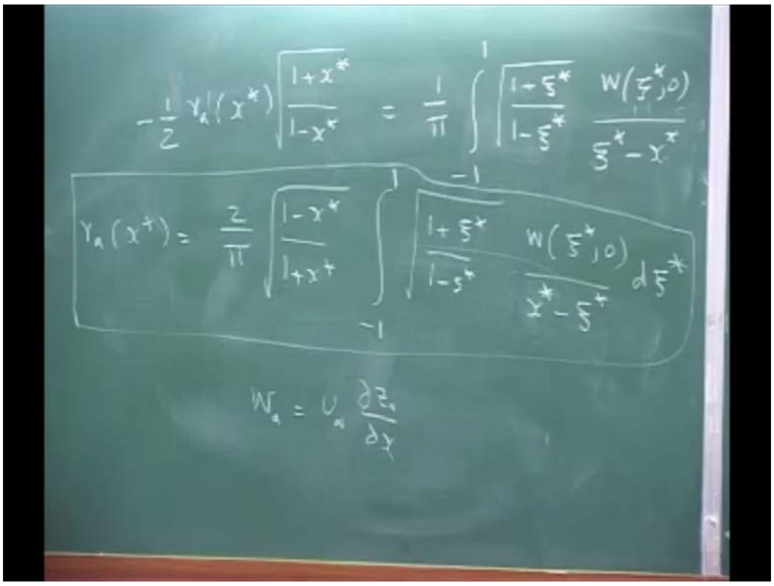

I will write this, you will have minus half gamma a x star square root of 1 plus x 1 minus x star, that is my f of x, this is equal to I will have 1 over pi, so I will put 1 over pi integral minus 1 to plus 1. I have to have this, but y I am calling it as the running wing integrals 1 plus y 1 minus y and my variable is this into w at comma 0 divided by y minus x, y is this minus x star d.

Now, I can rewrite the entire thing as I want only gamma a, so I will write gamma a x star, this will become 2 over pi square root of 1 minus x star over 1 plus x star integral minus 1 to plus 1, because this is minus psi what I will go is I will change in x and sign. So, this will become square root of 1 plus psi star 1 minus psi star w x star 0 divided by this is x star minus y star d psi star, now this is my…

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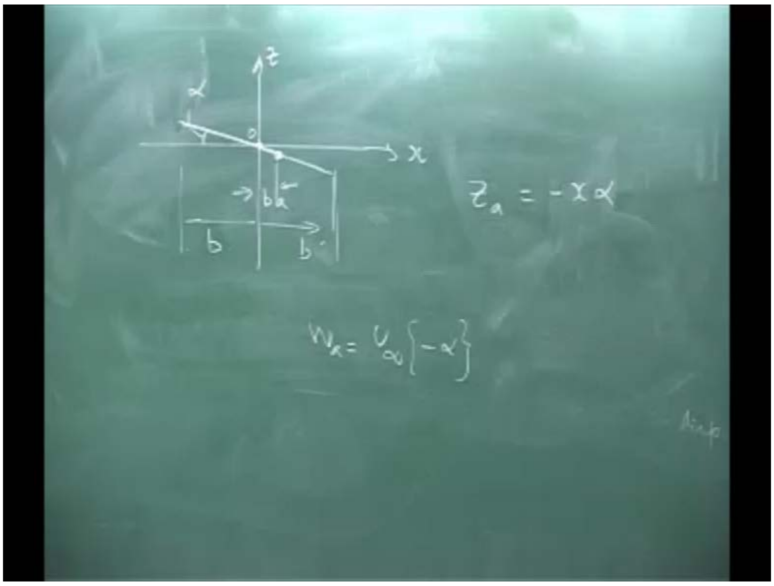

Now, for an airfoil what was our boundary condition w, airfoil is at u infinity delta zee a over delta x, but if my airfoil, this is I am measuring this is b, and this is b, and this is my axis about, which my this I call it b a this is my 0. This my x axis this my zee axis, now there by zee airfoil is x this angle if we say this is alpha x minus what is that, if we take this is nothing but z a is if you take what. If this is the center point you will say simply minus alpha x, but if you say it is rotating about some other point, see what is the equation for z a.

Suppose, if this is passing through that point z is what z that is x become minus psi this is minus x alpha z a, now your w z a, becomes what u infinitely minus alpha and that is in the entire minus b to plus b, which is basically minus 1 to plus 1. This quantity is nothing but minus u infinitely alpha, so your gamma a start you can write it as gamma a.

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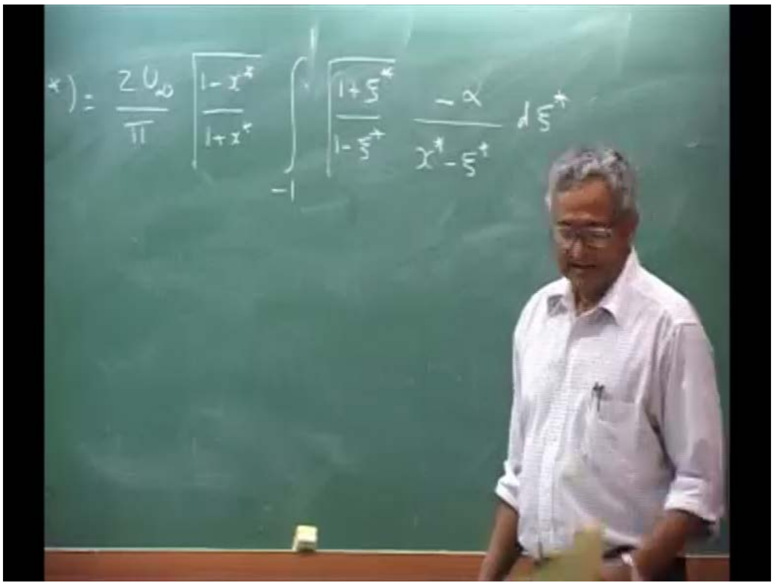

Now, you see I am writing, this is two u infinitely over pi root of 1 minus x star 1 plus x star minus 1 plus 1 1 minus 1 plus, so psi stars 1 minus psi star minus alpha, because u infinitely I have taken it outside into x star minus psi star d psi star. This is what you have, now this is the torsion integral one, which we will apply that is nothing but that is given by I will give you one integral, because this is a important I now I do not thing that is necessary I will write that integral.

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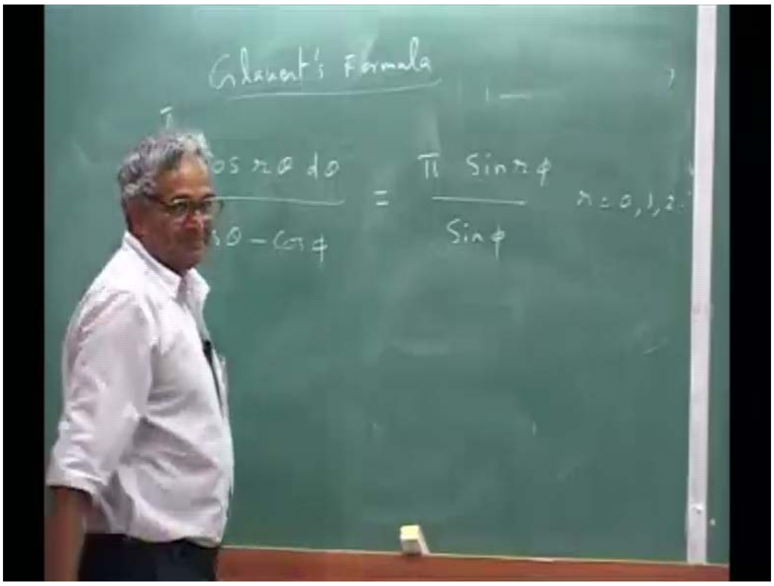

That integral is 0 to pi, this is the ((Refer Time: 46:07)) integral, the principle value cosine r theta d theta over cosine theta minus cosine phi, this is given by pi sin r phi over sine phi, where r running from 0 1 2 etcetera. This is the equation, this is the result for cache, this is actually Glauest integral, the principle value of the cache, that is now what you do is you need here, you multiply the numerator and denominator by 1. What you do is you take the 1 plus psi; that means, numerator will become square root 1 plus psi star divided by this is 1 minus psi star square. So, what will happen is let us look at only this integral, this minus alpha you can take it here that is not a problem.

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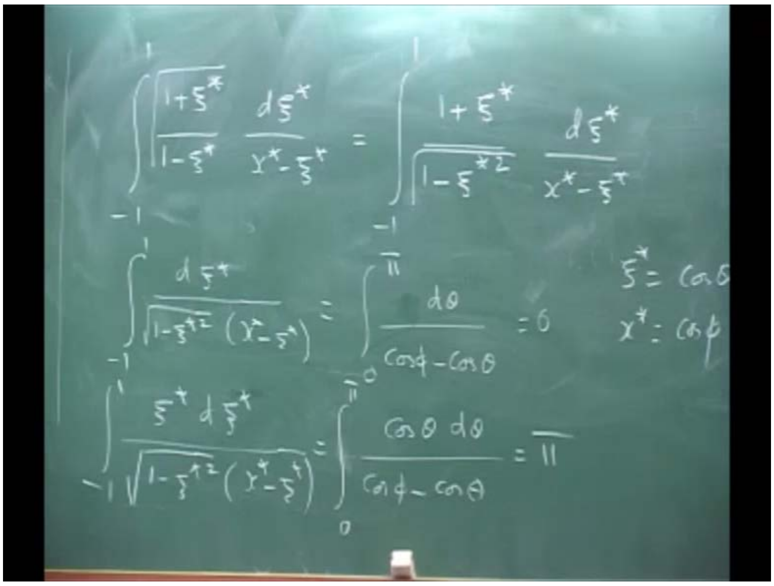

We will be writing only that integral that is minus 1 plus 1 minus 1 plus 1 minus psi star d over x star, this you write it as minus 1 plus 1 1 plus divided by 1 minus square with a undergo. Because, I am multiplying by the same numerator denominator, I will get 1 minus a square kind of a thing into d over x here what I will do is I will write 2 integrals. One is I can split it into two parts, one is 1 divided by this whole thing, another one is psi star divided by this whole thing, so you will have two integrals the integral this one divided by root of 1 minus square x minus x star.

This is actually 0 to pi minus d theta over which is essentially 0, how you get it is you write this cosine theta change of variable. If I write this is cosine theta, when you differentiate this will be d psi star will become minus psi theta d theta, here you will have one theta that psi and psi will cancel out living behind only then you put x star as cosine theta. Now, this integral looks exactly when r equal to 0 the integral is 0, but when you use the another one it is 1 minus plus 1 another one.

Second integral which is psi star d psi star root of 1 minus psi star square into x star minus psi star, this becomes because you know this is cos theta numerator that will become directly cos theta. So, you will have here cos theta d theta over, we take is what is this cos theta this is at minus, so you will have minus sign what.

Student: Length

Which length no this will become what limit is always this will change to pi.

Student: minus 1 to 1 can become pi to for minus 1.

You put a see what is that what the psi is minus 1 is pi to 0 I will change it, I put a see this is minus and I am put what 1 is 0 then this minus should not be there. Now, here this will become cos theta d theta cosine phi minus cosine theta, this will become again 0 pi, because this particular terms comes and sits there. Now, this quantity from here sin phi sin phi will go out, you will get pi, now what I have is this integral is nothing but is there a minus pi I should get a minus pi, how I am getting this. I should get a minus pi, please understand this is cos phi cos theta, this is cos theta minus cos theta, that is the difference, so you will get a minus pi.

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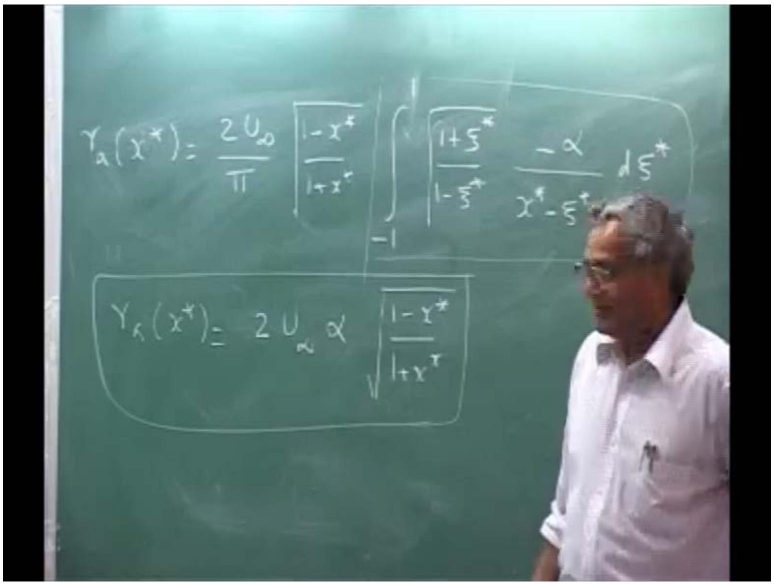

Now, this one quantity become gamma a x star become, minus and minus you will get plus pi and pi will cancel out you will get alpha, so you will get 2 u infinitely alpha root of 1 minus x star over 1 minus x star this is my. That means, I have you now see it automatically satisfied the condition, when x star is equal to 1 which is the straining edge gamma is 0, but only thing is that x star is equal to minus 1 which is the leading x gamma a is infinity. This is the penalty, but the result match very close to the experiment except for few small distance from the leading x, because that is why the leading x it will have very high stagnation pressure point.

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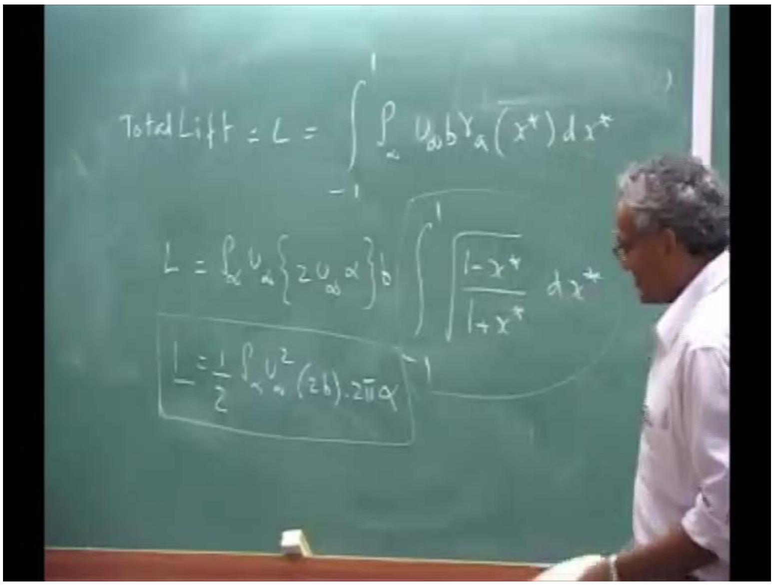

Now, what you can do is you can get the lift I erase this, total lift, this is what this is l which is equal to minus 1 to plus 1, we got it as because you put p u minus p l rho u gamma everything rho u gamma a. So, rho infinitely you will have what, u infinitely into gamma a, you can have it like this, now substitute that expression you will get lift equals rho infinitely u infinitely gamma a is 2 u infinitely alpha. And there is b also, because this integral is this because there is a b because non dimensional, so I will have a b and then minus 1 to plus 1 root of 1 minus x star over 1 plus x star d x star.

Now, this term you can write it as, see this integral again you can integrate it this is very simple x is cos theta you can substitute, this integral value is actually pi. So, your lift will become I am going to take half rho infinitely u infinitely square, then I am taking 2 b, since this is pi I have taken a 2, I am taking 2 pi into alpha his is my lift on the airfoil.

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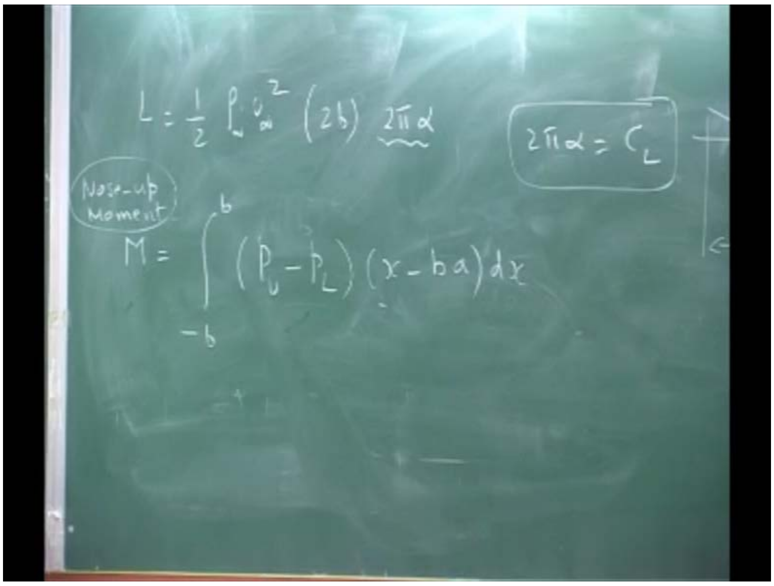

You got your put a lift for unit length by this expression, lift is half rho infinity u infinity square this is the dynamic pressure, card is 2 b and then c l, c l is 2 pi alpha this 1 infinitely is alpha c l lift coefficient. This you call it as which is directly on all time into two types pi, so this is the theoretical value. Now, if you want moment again it is minus b to plus b, you take it as p u minus p lower and moment about what point you are taking, because I put a diagram like this.

I want to take momentum what my usually call it as elastic access, so this is my d a and this is my b momentum about this point; that means, will I have x minus b a upper minus lower. So, and this moment I am writing it as nose up, please note down this is nose up moment into d x, now what we can do it you can substitute again non dimensionalized substitute. And then p u minus p l is again you are given rho u gamma this term gamma a is that, put it back in the integration, then you can get your moment, let me write the expression I will erase this part and write to full the expression.

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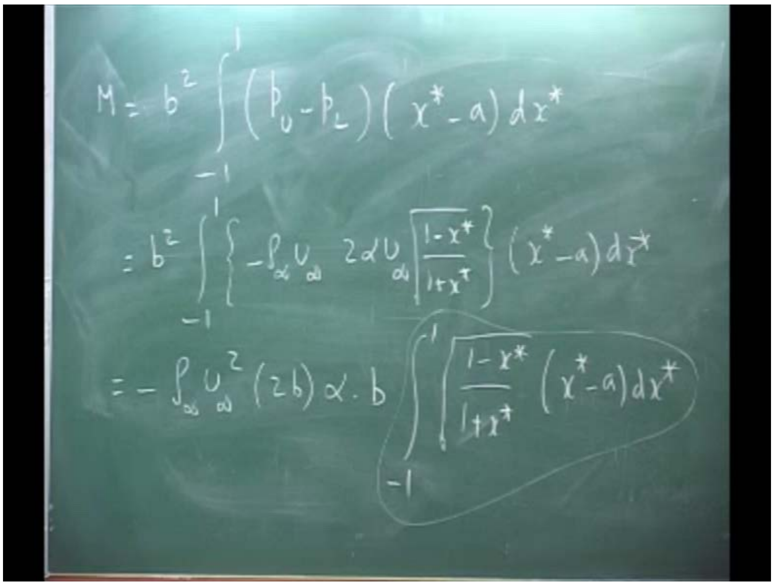

And this is looking like t, so your moment is if you non demontialized this is minus 1 to plus 1 p u minus p l x star minus a d x star, and this you write it as b square minus 1 to plus 1, because you know p u minus p l. This will be minus rho infinity u indefinity then gamma a that is again 2 alpha u infinity y root of 1 minus x star over 1 plus x star multiplied by x star minus a d x star. Now, on simplification this will become minus rho infinity u infinity square 2 b alpha into b, because 2 b I am taking in the area alpha I am taking this b, because I am splitting it into b. This minus 1 plus 1 1 minus x star by 1 plus x star minus a d x star, again you see of integration.

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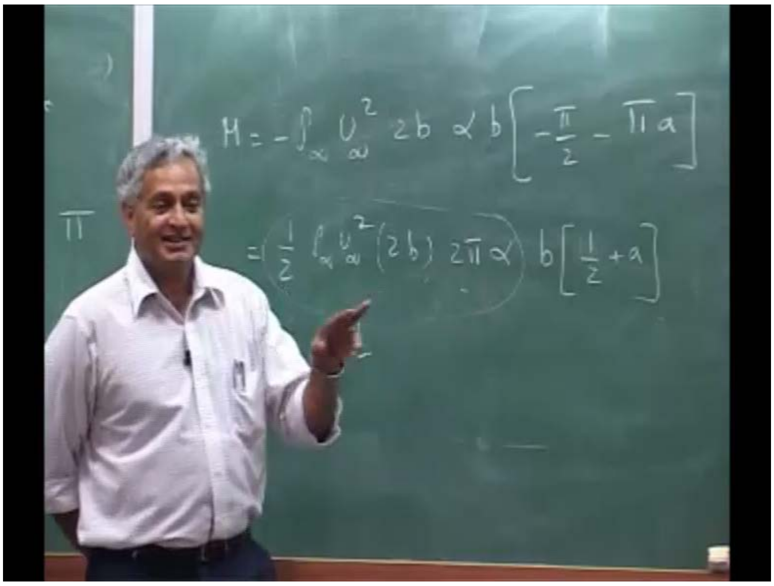

If you integrate this you will get I will write the final result, because that integration is same as what we did earlier moment become minus rho infinity u infinity square 2 b alpha b minus pi over 2 minus pi a, because this integral it will split into 2 parts. If a you will get minus pi, this will be minus pi b, now this is you can write it as, because the minus and minus you will get plus. So, you will have half rho infinity u infinity y square this is dynamic pressure area, I am taking a pi out that will be 2 pi into alpha, and taking t b here, which I take it half plus a.

Now, you know that as per this diagram, which I got this is b b, this is b a and this is my x, moment about this point is this quantity is lift, and this is the nose up moment. So, nose up moment means some lift is acting somewhere, that into the distance that is distance is b a and b by 2 is here, so you have lift is acting here this is b by 2 this is my quarter card point aero dynamic. So, incompressible steady flow gives lift expression is this moment expression that is lift is acting, but please note that if I take moment about if a is minus half; that means, I am taking moment about aero dynamic center.

Aero dynamic center moment a minus half means my moment is 0; that means, moment is independent of my angle of diagram only at that point, you say that is an aero dynamic center. Where, the moment is independent of angle, where as any other point moment will be a function of alpha, so the aero dynamic center is basically moment is independent of angle, and that is the quarter r card point for subtonic.