Introduction to Launch Vehicle Analysis and Design

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Lecture - 07

Trajectory under Gravity

Hello and welcome. In this segment, we will look at the implication of including the effect of gravity on the ascent performance of a typical rocket. Further, we will also look at some of the issues involved and how to take care of them towards the end of the lecture. So let us begin the discussion.

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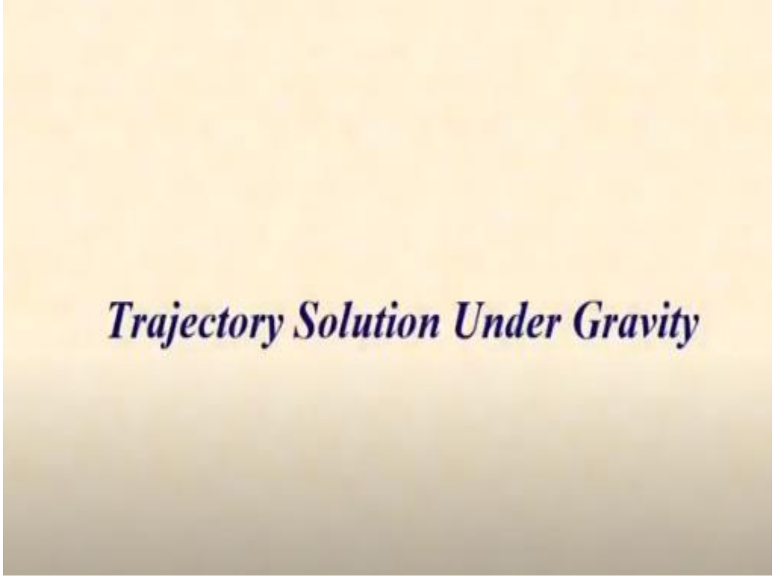

Let us look at the solution for trajectory under the impact of gravity.

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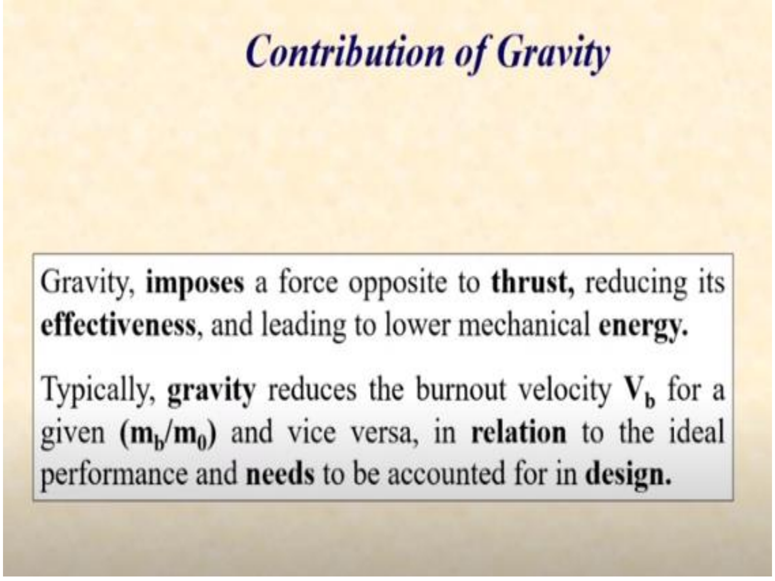

Gravity as we know imposes a force opposite to thrust and thereby reduces its effectiveness and leads to lower total mechanical energy at the end of the burnout. Typically, gravity uses the burnout velocity for a given $\frac{m_b}{m_0}$ and vice versa in relation to the ideal performance that we have seen and needs to be accounted for in the design.

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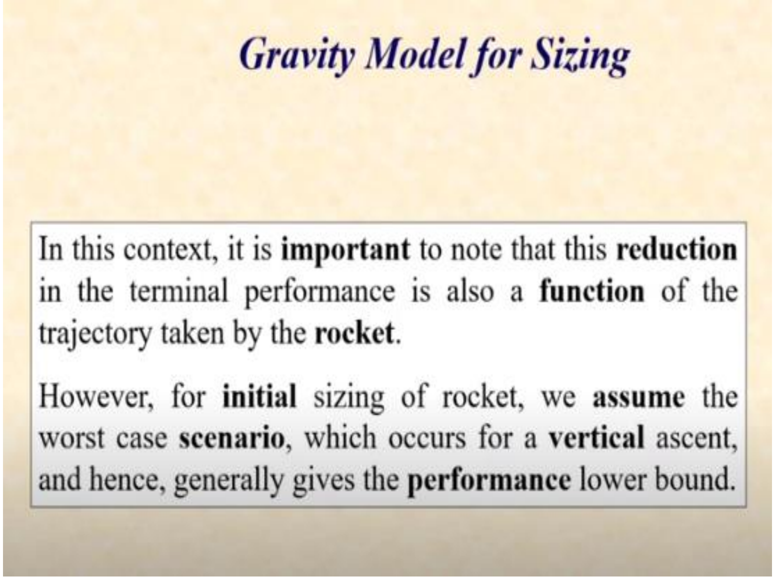

Let us now look at a typical gravity model for initial sizing of the rocket. In this context, it is important for us to note that this reduction in terminal performance is also a function of the trajectory taken by the rocket. However, for initial sizing purposes we assume the worst-case scenario particularly in the context of effect of gravity, which occurs for a vertical ascent case and hence generally gives the performance lower bound. This is same as say that the rocket moves along a radial line till it reaches its terminal point.

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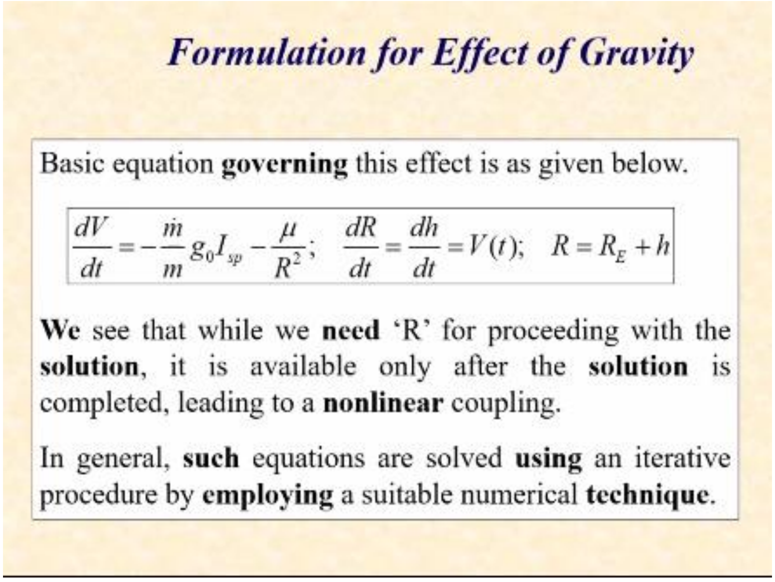

Under this condition, we can now formulate the effect of gravity and consider the appropriate equations which are $\frac{dV}{dt}$ as $-\frac{\dot{m}}{m} g_0 I_{sp}$ which was the thrust term that we have seen earlier for our ideal performance calculation, $-\frac{\mu}{r^2}$ which is the gravitational acceleration term. Here μ as you have already seen earlier is a gravitational parameter and for different planets including earth, we can find out its value.

Further, we can also write down the corresponding kinematic equation that is rate of change of radius that is $\frac{dR}{dt}$ as the radius of earth we are assuming it to be spherical, is constant. We get only $\frac{dh}{dt}$ and that is equal to the velocity along the radial line. And now, we note an important point regarding the solution process for this differential equation.

We see that we need R for proceeding with this solution in the sense that unless we know the R on the right-hand side, we cannot integrate this differential equation. But we also know that this R will be available only after you have completed the solution and this results in a typical nonlinear coupling in the differential equation.

In general, such equations are solved using an iterative procedure by employing a suitable numerical technique.

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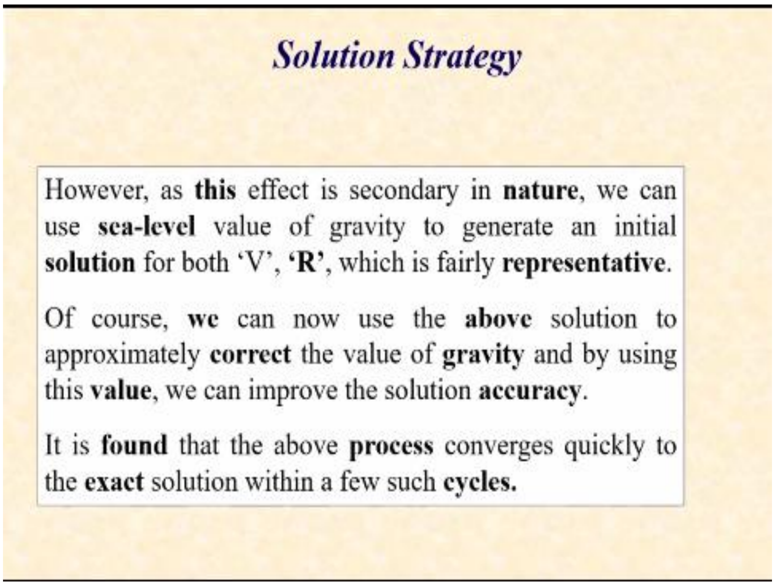

But in the present case, we adopt a slightly different approach, which is based on the fact that compared to thrust, the gravity is of a lower order in terms of its impact. So, we also term it a secondary effect. And therefore, as a first approximation, we can use sea level value of gravity to generate an initial solution for both V and R, which is fairly representative as we will see through an example later.

Of course, we can now use the above solution to approximately correct the value of gravity. And by using this corrected value, we can further improve the solution accuracy. We can do this task. It is actually found that the above process converges quickly to exact solution within a few such cycles so that we actually can get a reasonably good solution without having to directly solve the nonlinear differential equation.

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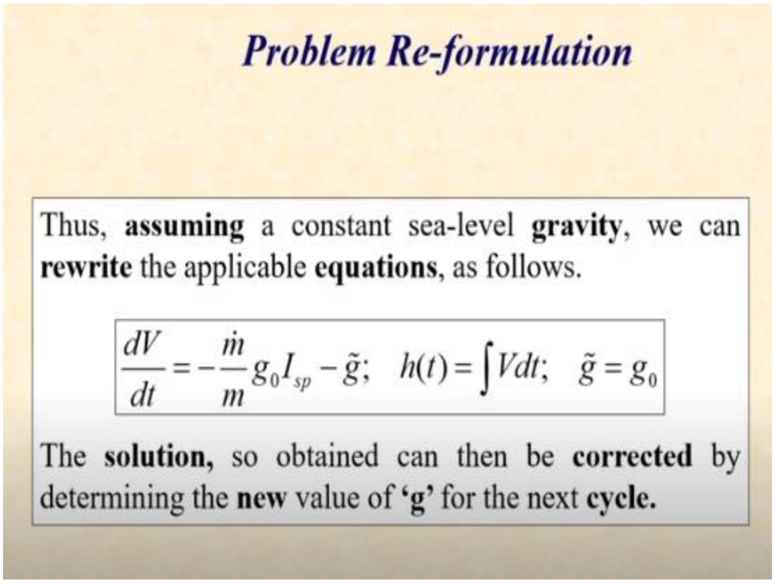

So let us now reformulate this problem assuming that the gravity is a constant term and we will use the sea level value at some point so that we can rewrite the applicable equations as $\frac{dV}{dt} = -\frac{\dot{m}}{m} g_0 I_{sp} - \tilde{g}$ and I have put a tilde on top of it to indicate that it is a constant value. And that this constant can take different values in different context as per the requirement.

In the present case, I am just proposing that let us put $\tilde{g} = g_0$, a sea level value. And then of course, the equation for h which is $\frac{dh}{dt} = V$ can be rewritten as $h = \int V dt$. We will note now, that in comparison to the ideal burnout solution, we will now get a solution for altitude or distance traveled depending upon the velocity solution.

As I mentioned earlier, the solution so obtained can then be corrected by determining the new value of $\tilde{g}$ for the next cycle, as we will see through an example next.

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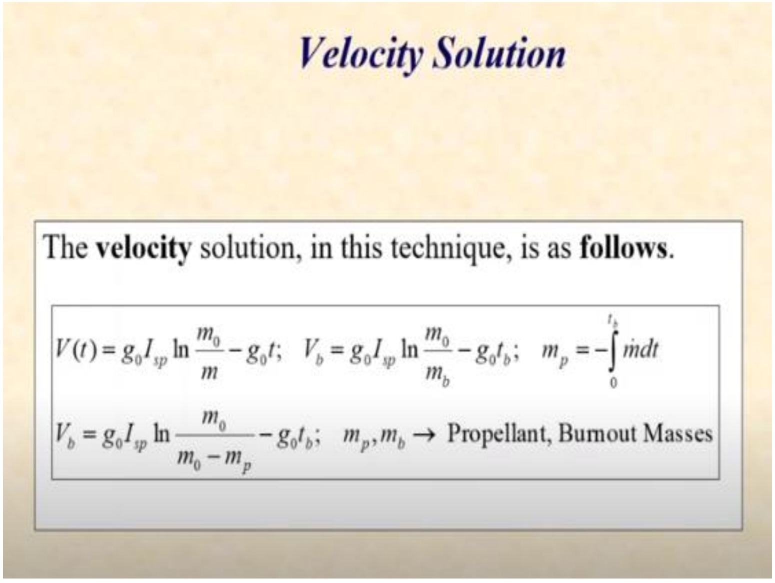

Let us now extract the velocity solution through this technique. And we find that it is now a simple matter to perform integration of the differential equation given by $\frac{dV}{dt} = -\frac{\dot{m}}{m} g_0 I_{sp} - g_0$. So, when we integrate the differential equation, the first term is same as what we have obtained for the ideal burnout case.

And now you have one more term with a negative sign $-g_0 t$. This means that as time progresses, we are going to get a velocity which is lower than the ideal velocity depending upon the value of g_0. Further, we now need to introduce another performance parameter which is given as the mass of the propellant as nothing but the integral of $\dot{m}$ or the burn rate integrated over the time interval.

Which means that these two equations together complete the solution for the velocity and the time which was not a requirement in the context of ideal burnout solution. The time did not exist explicitly, but now we have to solve for time. Here it is important to note that if we specify a time for a given $\dot{m}$, we are going to get a requirement on the propellant needed for this purpose.

On the other hand, if we specify a burned rate $\dot{m}$ then it will give us directly t_b for a specified propellant. It can work both ways and can be used effectively as a design parameter.

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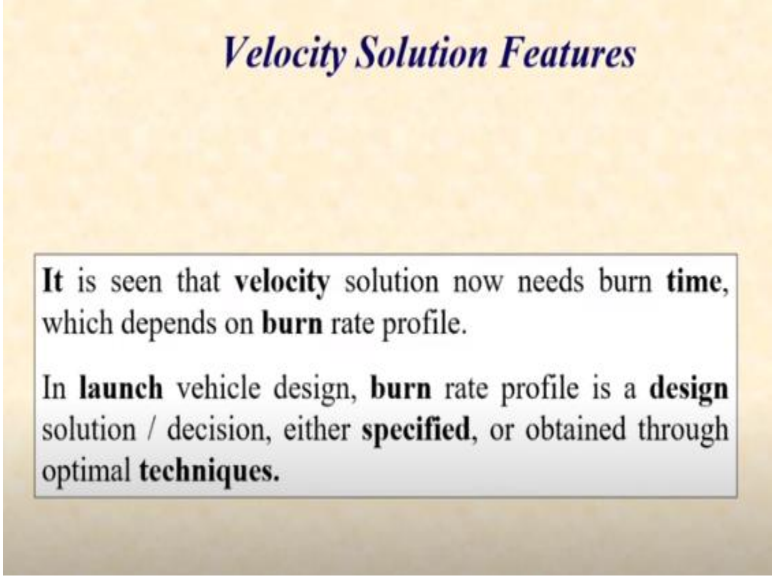

Now, let us look at some of the features. One of the important points that we have already noted is that, now we need a burn profile, which means, we now explicitly need an expression for $\dot{m}$ along with m_p to be able to solve for the time t_b which then can be used in the velocity expression to find out the burnout velocity as well as the burnout altitude which you will see later.

Here, you should note that in launch vehicle design exercise burn rate profile is generally a design solution or a design decision. What it means is that either you will specify a burn rate directly based on the nature of the infrastructure available or you would set up a separate optimization procedure to obtain the best possible burn rate profile which then you would use for carrying out the mission.

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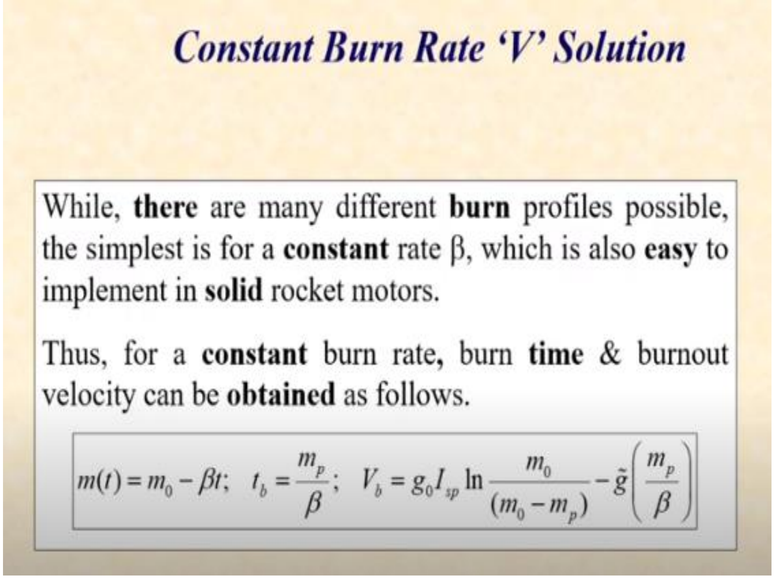

Of course, as we have already seen, there are many such possibilities that one can think of for burn profiles. But the simplest that we can at this point make use of is the constant rate β which indicates that the propellant burns at a constant mass flow rate which is also easy to implement in solid rocket motors. In fact, you will find that by and large solid rocket motors burn at a constant rate of burn.

Same thing can also be implemented in other propellant as well. So, it is an extremely useful burn profile which can give us a fairly good idea of the terminal performance under the impact of gravity. Now, if we assume that our burn profile is such that it is consuming the propellant at a constant rate, then we can write the expression for mass at each time instant as the lift of mass $m_0 - \beta t,$ a simple linear expression.

And with that expression, we automatically realize that the total burn time will be nothing but the ratio of the propellant carried and the burn rate $\beta.$ Taking this expression for $t_b,$ we now go back to our velocity expression at the terminal point that is $V_b.$ And we now can write down the expression of V_b as the first term corresponding to the ideal velocity $-\tilde{g}t_b,$ which can be written as $\frac{m_p}{\beta}.$

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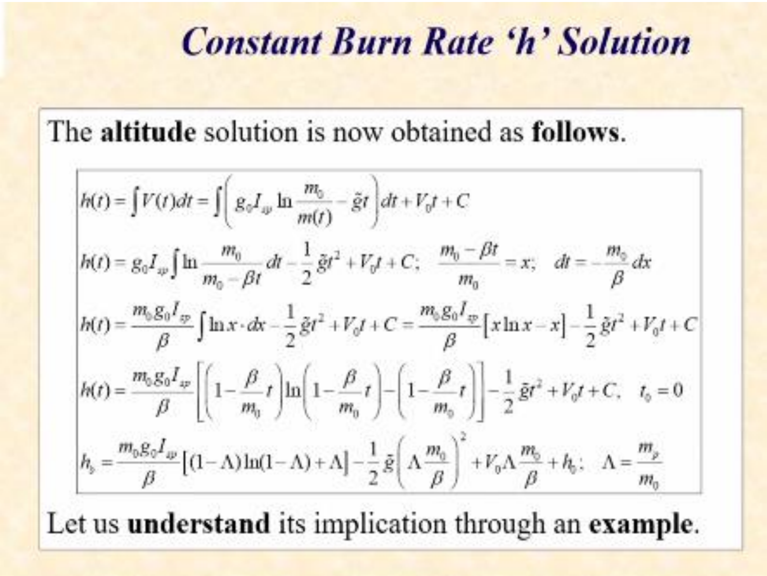

With this let us now go to the altitude solution that is let us integrate the velocity expression over 0 to t_b to obtain the altitude. So, I am showing you here are the steps involved. So, we take the expression for velocity, put it under the integral. Also put the initial conditions and then perform this integral. I will not go through the steps in detail. My suggestion to you would be that please verify these steps and familiarize yourself with how the whole process has been completed.

I will come to the last step which is the expression for the burnout altitude given in terms of the liftoff mass, the propellant I_{sp}, the propellant burn rate β and one additional parameter gamma, which is the ratio of the total propellant and the liftoff mass. This particular parameter is also called propellant loading fraction for a rocket and it is an important figure of merit, which decides the quality of a mission and also the nature of a launch vehicle that we are going to use.

Let us try and understand the implication of both the velocity expression and the altitude expression under the constant gravity and constant burn rate assumptions that we have made so far through an example.

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Constant 'g' & 'β' Example

m_0 = 80T, m_p = 60T, I_{sp} = 240 s, **g_0** = 9.81m/s^2, R_E = 6371 km, β = 600 kg/s.

Determine V_b, h_b, for **sea-level** 'g' and compare the **values** with the ideal burnout **solution**.

$$V_{ideal} = 3264 \text{ m/s}; \quad \Lambda = \frac{60}{80} = 0.75$$

$$V_b = V_{ideal} - \tilde{g}\left(\frac{m_p}{\beta}\right) = 3264 - 9.81 \times \frac{60000}{600} = 2283 m/s$$

$$h_b = \frac{80 \times 9.81 \times 240}{0.6}[0.25 \ln 0.25 + 0.75] - \frac{1}{2}9.81(100)^2 = 77600 m$$

So let us consider the same problem that we had considered for ideal burnout solution that we have a liftoff mass of 80 tons, the propellant mass of 60 tons, same I_{sp} of 240 seconds and the sea level gravity of 9.81 and we introduce two parameters which we are going to additionally need. One the burn rate β and currently it is chosen as, these may be arbitrary number of 600 kg/s just to understand the implication.

And we will introduce the radius of earth as 6371 km as I will tell you later how and where we are going to use and get this information. Let us try and determine the burnout velocity and the burnout altitude for sea level value of gravity and compare these values with the ideal burnout solution. So let us go through the steps. First, let us write down the value for the ideal burnout velocity that we have already obtained.

So, I am just reproducing here and then introduce the propellant loading parameter that is $\frac{60}{80}$, that is about 0.75, that is the propellant loading factor. Now, we know that V_b is going to be the ideal $-\tilde{g}\,\frac{m_p}{\beta}$. Now because we are using the sea level gravity, we replace $\tilde{g}$ with 9.81. Our m_p is 60,000 kg, while β is 600 kg.

And if you perform this simple arithmetic operation, we will find that the burnout velocity now is 2283 m/s against 3264 m/s, effectively a reduction of 981 m/s. That is the amount of reduction which has happened because the burn time is 100 seconds. Let us now go to the altitude expression. I suggest that you try this yourself later. But I have shown the important steps in the last line.

And if you perform this calculation, you will find that the altitude that is reached is about 77,600 meters or 77.6 kms. Now obviously, a point is to be noted here. We have started this calculation by assuming that the gravity value throughout this trajectory is corresponding to sea level. But when we arrive at the solution, we find that rocket actually at the end of the trajectory or the burnout will be at around 78 km altitude.

So obviously, the gravity that value we are using may not be the value that will be actually applicable at this altitude and that we may need to correct our gravity value. So let us see how we can do that.

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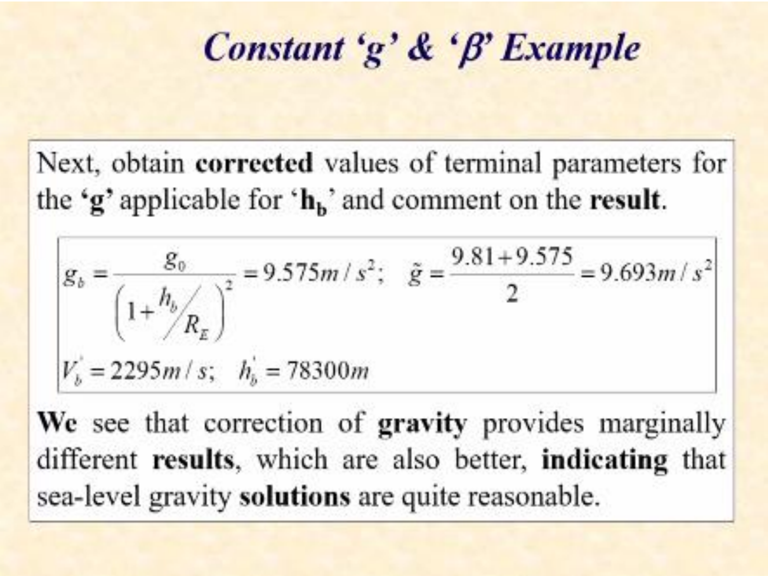

One way, there are number of ways of course one can do that, but one way of doing this is to use a value of an average gravitational acceleration between the sea level and the 78 km point. We can use a simple arithmetic average and see in what manner our solution changes. Let us see what is the implication of such a hypothesis of using an approximate gravitational value, which is an average of the gravitational value at sea level and 78 km altitude.

So, we go to our nonlinear expression for the gravitational acceleration with subscript b indicating burnout altitude h_b. And we find that at that altitude the gravitational acceleration is $9.575\ m/s^2$ as against 9.81 at the sea level. The difference is roughly around $0.25\ m/s^2$, not a very large number, but still a different number.

So, we say that now, instead of g_0 as $\tilde{g}$, we are going to use an average value of the gravity as $\tilde{g}$. So, we take the arithmetic average of 9.81 by 9.575, which results in the average value of gravity $\tilde{g}$ as 9.693. I will leave you to verify that, if we use this value of gravity, the revised value of the velocity will be 2295 m/s. And the revised altitude will be 78,300 m instead of 77,600 m.

So, which means that we have roughly about 700 m of higher altitude and roughly about 15 m/s higher velocity. Now if you just check the percentages, you will find that this is a very small percentage change in the velocity as well as the altitude which obviously means, that based on even the average gravity the change in the solution in comparison to the sea level value is only marginal.

And that brings us to an important point that the sea level gravity solutions are quite reasonable. So as a first cut design exercise, it is actually possible for us to get a fairly good estimate of the impact of gravity by just bringing in the sea level value of gravity without worrying about the gravity at higher altitudes. And then if necessary, we can always make the corrections.

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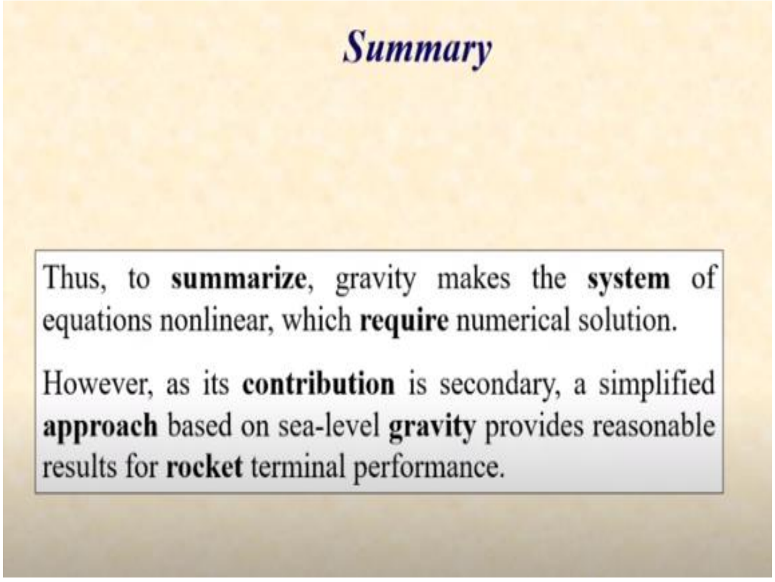

So, to summarize, the gravity makes the system of equations nonlinear, which obviously require numerical solution. However, as its contribution is secondary, a simplified approach based on sea level gravity provides reasonable result for the rocket terminal performance. Of course, please note that the results that we have obtained so far are for a constant burn rate.

But I would also like to mention here that even if we had a different burn profile, the result in the context of impact of gravity would not be significantly different except in some special cases that we are going to discuss next.

Hi, so we now are in a position to consider the implication of the solution that we have obtained in terms of what is the nature of the solution in relation to the ideal burnout performance and what are the features that we must note in order to improve the quality of our modeling and the solution. So, bye and see you in the next lecture and thank you.