Introduction to Launch Vehicle Analysis and Design

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Lecture - 34
Curvilinear Trajectories

Hello and welcome. In this tutorial we will consider problems related to generating a curvilinear trajectory using the three basic analytical procedures that is constant pitch rate, constant velocity and constant specific thrust solutions. We will assume that all these trajectories are in vacuum and that constant sea level gravity is applicable to simplify our analysis procedure. So let us begin.

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So let us first consider the constant pitch rate problems.

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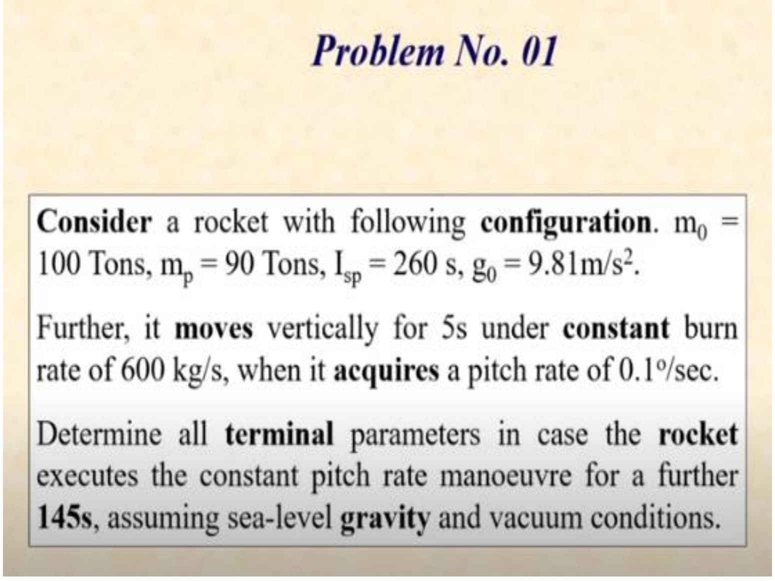

So let us consider a rocket with the following configuration that is it has a liftoff mass of 100 tons, is carrying 90 tons propellant of 260 second I_{sp}. Further it moves vertically for 5 seconds under constant burn rate of 600 kg at which point a pitch kick is given and it acquires a pitch rate of 0.1 deg/s.

Let us try and determine all the terminal parameters in case the rocket executes the constant pitch rate maneuver from this point onwards that is $t = 5s$ for a further 145 seconds assuming sea level gravity and vacuum conditions.

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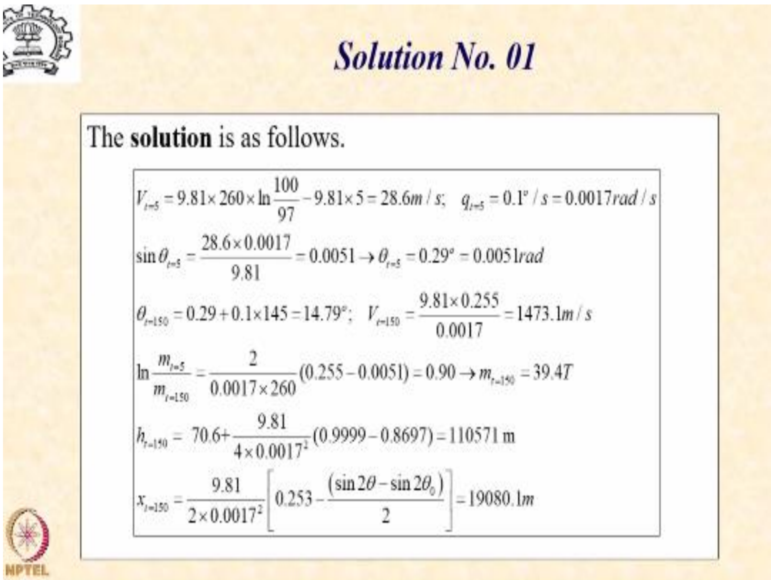

So let us look at the solution. So, we first calculate the velocity at the end of 5 seconds, which is under the vertical motion assumption and constant gravity assumption. So, we use the standard formula of $g_0 I_{sp} \ln\frac{m_0}{m_b}$. And in this case, because we are burning propellant at 600 kg/s for about 5 seconds, we burn 3 tons of propellant. So, what is left at the end of 5 second is 97 tons.

And this term corresponds to the effect of gravity. If we perform this simple calculation, the velocity at the end of 5 second is 28.6 m/s. Now, the pitch rate at 5 second is given as 0.1 deg/s which is converted into 0.0011 rad/s. And now, we employ the basic equation that $\sin\theta_{t=5}$ is $\frac{V_0 q_0}{g_0}$. When we do this, $\sin\theta$ is 0.0051.

So, we get $\theta_{t=5}$ as 0.29 degree or 0.0051 radian. So, as you would remember we said that among the three quantities that is velocity, pitch rate, and the pitch angle only two can be specified and third will be solved using the constraint relation which is what we have done in this case. So, once we know the initial angle with respect to vertical, then it is just a question of applying the basic formula.

And we can get the pitch angle at the end of 150 seconds that is additional 145 seconds at the rate of 0.1 deg/s. So, we get the angle at the end of 150 second as $14.79°$. Once we get this angle, we can then talk about the velocity using the same constraint relation that is $g_0 \sin\theta$ that is $\frac{\sin 14.79°}{q_0}$ which is 0.0017.

And we find that the velocity at the end of 150 seconds will be 1473.1 m/s. Let us now turn our attention to the mass fraction. So, we use the mass expression of $\ln m_0$, mass at $t = 5s$ that is the starting point for the gravity turn divided by the mass at the 150 instant. And we use the two angles that is $\sin 14.79°$ and the $\sin 0.29°$ and with that, we find that we are talking about a mass ratio or mass at the 150 second as 39.4 tons.

Similarly, now, we can use the terminal parameters of $14.79°$ as the angle to calculate the altitude solution and we find that the altitude is 110 km and we can do the same thing for the horizontal distance that is 19 km. So, this trajectory is such that it is almost like a straight-line vertical trajectory and the horizontal distance travelled is only 19 km.

So, it is possible that the flat earth approximation is going to be valid in this case. Let us now proceed further and consider the next problem.

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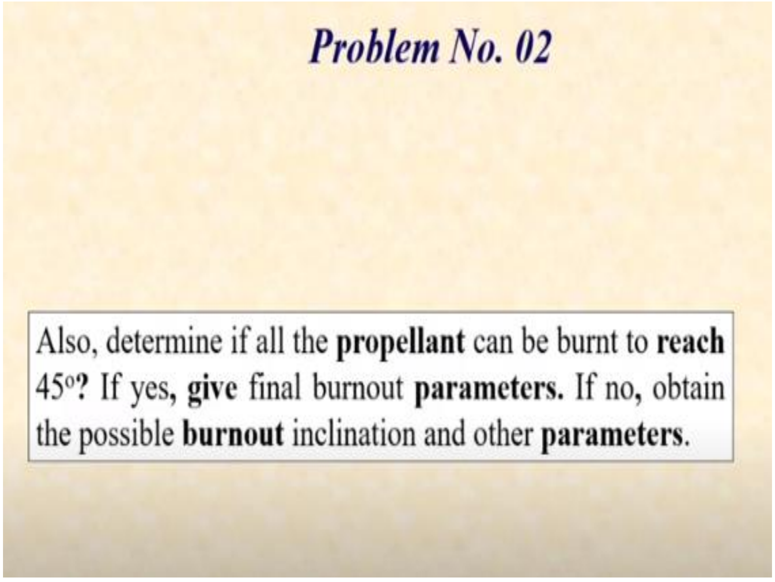

In this let us try and find out if we can burn all the remaining propellant and reach the angle of 45°. If such a thing is feasible, let us try and find out the burnout parameters. But if it is not possible, let us try and find out if we burn the remaining propellant what are the possible burnout inclination and other parameters.

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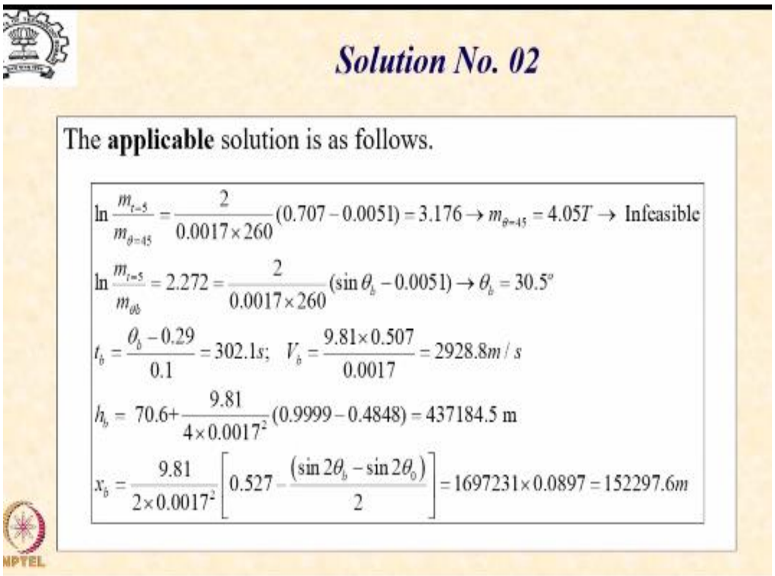

So, in this case the applicable solution starts like this. We start with the mass ratio. We assume that our angle is 45°. So, $\sin 45°$ is 0.707. So, we substitute this and from this we find that the residual mass expected at $\theta = 45°$ is 4.05 tons. Now we realize that we are carrying only 90 tons of propellant. Which means, there is 10 tons of mass which is inert or cannot be burned.

But this solution is expecting that the inert mass would only be 4.05 tons. So obviously, it is not a feasible solution. Which indirectly means that by burning all the propellants we cannot reach an angle of 45° because we do not have that much of propellant. So let us now invert the problem to say what is the propellant that we have.

So, we have only now 87 tons of propellant left once we have finished the 3 tons of propellant in the vertical motion. So, with this propellant let us try and find out what is the inclination that is possible. So, we solve the same equation but now we assume θ_b to be an unknown. And with this mass fraction which is known to us, we find that we can reach only about 30.5° of inclination from vertical, we cannot reach 45°.

And to perform this mission itself, it will take 302 seconds and at the end of this phase of burning all the propellants the velocity will be 2928 m/s. Now let us look at the corresponding altitude solution and we find that now altitude is going to be 437 km and the horizontal distance is going to be 152 km. Now if you remember we had mentioned that every 110 km of horizontal distance is equal to 1° change in the gravity direction.

So, in the present case, we find that there is going to be a change of about 1.5° in the direction of gravity and we will find that because of that the gravitational acceleration will change marginally and to that extent, there will be a correction to the final trajectory parameters because there is only a small component of the gravity which is acting in horizontal direction.

So maybe you can think in terms of finding out what is the impact of this 1.5° change in the direction of the gravity.

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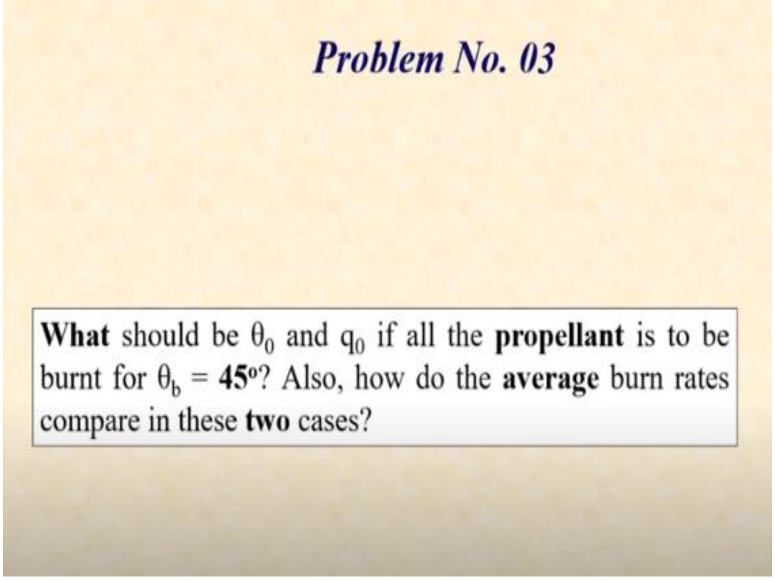

Let us now move forward to the problem number 3 where we are now putting one more constraint and the constraint is that I want to burn all the propellant and I want to reach 45°. And now can I design the pitch kick. So, this problem concerns designing the pitch kick and let us see if by burning the remaining 87 tons of propellant can we achieve 45° and in the process what should be the pitch kick parameters.

And then also we will compare the burn rates in these two cases. The last case where we have reached the 30.5° with burning all the propellant and the requirement that we should reach 45° by burning all the propellant.

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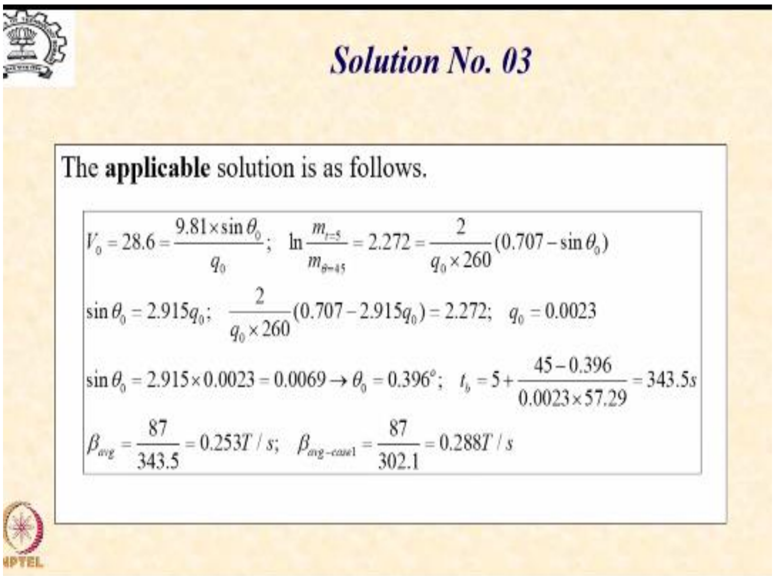

So, in this case what we do is we have the velocity which is a constraint of 28.6 and we use the constraint relation, which is now a function of both theta naught which is an unknown and q_0 which is also an unknown. But what is known is this mass ratio which is 2.7272 and this should be equal to $\frac{2}{q_0}(0.707 - \sin\theta)$. So now we have two simultaneous equations in two unknowns' θ_0 & q_0.

We solve this equation, the first equation for $\sin\theta_0$ in terms of q_0 and substitute that into the second equation, which now becomes an equation only in q_0. And by solving this we get a solution for q_0 as 0.0023 rad/s as compared to 0.0017 rad/s, that is what we had assumed.

So, the first change that has occurred is that instead of 0.1 deg/s pitch rate, we would probably need a slightly higher pitch rate. And when we substitute this value of $q_0 \times$ the $\sin\theta_0$ expression, we find that θ_0 instead of being 0.29° is now it is $0.396°$.

And with that, we find that this combination of θ_0 and q_0 and the pitch kick parameter will ensure that we will be able to burn all the remaining 87 tons of propellant and that we will reach the angle at the end of this mission phase as $45°$. Of course, we find that in order to do this the time taken would be 343 seconds. And this 343 second gives us an average burn rate of 0.253 T/s.

While in the previous case it is 0.288 T/s. So, it is marginally lower, but they are of the same ballpark. With this we have come to the end of the problems that we would like to consider for a constant pitch rate case. My suggestion is that you try these problems independently to verify some of the solutions. And in case there are anomalies, you may please report them to the TA's.

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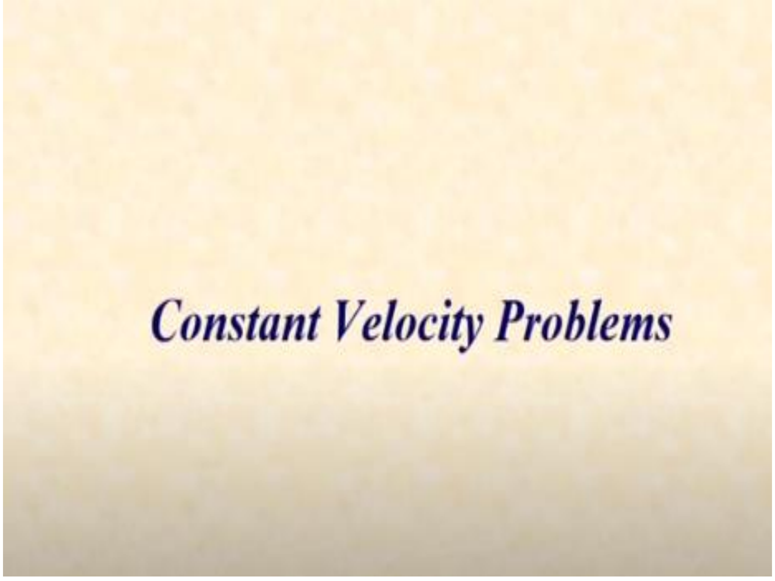

Let us now look at the constant velocity problems.

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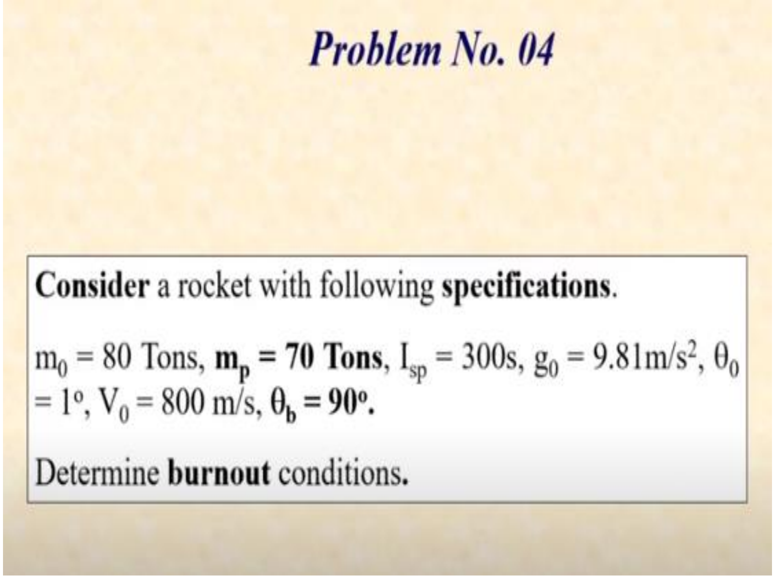

And in that let, us consider a rocket with the following specifications. So, we have a rocket with 80 tons of liftoff mass, 70 tons of propellant having an I_{sp} of 300 seconds. And the initial conditions are such that you already are having an inclination of 1° from vertical. The rocket is firing at a velocity of 800 m/s.

And it is desired that at the end of a constant velocity maneuver the burnout final angle becomes 90°. So, our terminal constraint is that the angle becomes 90° from vertical or that the velocity vector is parallel to the local horizon. Let us try and now determine the burnout conditions.

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Solution No. 04

The **burnout** solution is as follows.

$$\Delta t = \frac{V_0}{g}\ln\left(\frac{\tan\frac{\theta_b}{2}}{\tan\frac{\theta_0}{2}}\right) \rightarrow \Delta t = \frac{800}{9.81}\ln\left(\frac{\tan 45^o}{\tan 0.5^o}\right) = 386.6s$$

$$\frac{m_b}{m_0} = \left(\frac{\sin\theta_b}{\sin\theta_0}\right)^{-\frac{gV_0}{g_0^2 I_{sp}}} \rightarrow m_b = 80\times\left(\frac{1.0}{0.0087}\right)^{-0.2719} = 22.02T$$

$$\Delta h_b = \frac{V_0^2}{\bar{g}}\ln\frac{\sin\theta_b}{\sin\theta_0} \rightarrow \Delta h = \frac{800\times800}{9.81}(4.741) = 309325m$$

$$\Delta x_b = \frac{V_0^2}{\bar{g}}\Delta\theta_b = \frac{800\times800}{9.81}(1.5709 - 0.0087) = 101915m$$

So, the burnout solution is as follows. First, let us talk about the time taken to complete this maneuver which is given in terms of the velocity V_0, the burnout angle θ_b and the initial angle θ_0. And we find that the time taken to complete this maneuver is going to be 386.6 seconds. And in this time the burnout mass can be obtained using this formula.

And when we apply this formula, we get the burnout mass of 22 tons which means that from 80 tons only 22 tons are left. So, which means about 58 tons of propellant have been consumed out of 70 tons of propellant. So, we may note that a 12 ton propellant is still there and of course then we can calculate the altitude.

So, if you look at the altitude it is 309 km and the horizontal distance traveled during this time is 101 km. So again, there is a change of roughly about 1° in the direction of gravity. You may assess its impact on the overall performance in terms of the change in the gravitational vector direction and its impact on the solution.

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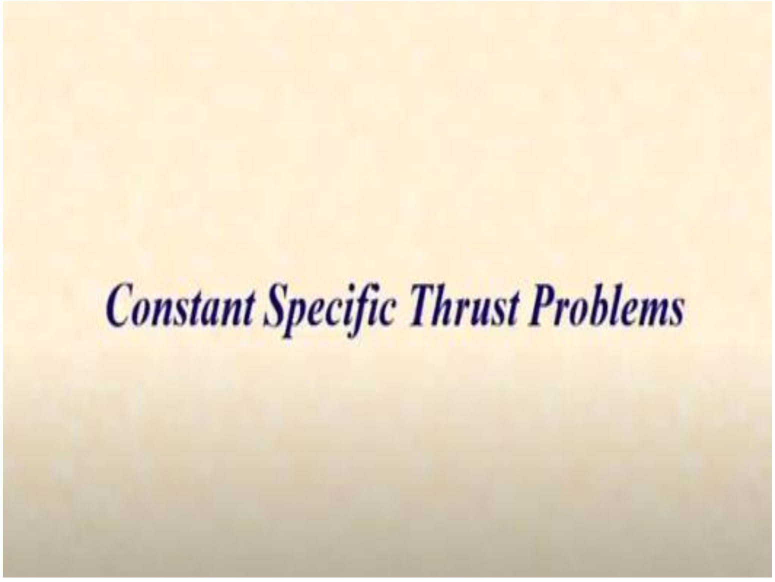

Let us now consider the constant specific thrust problem.

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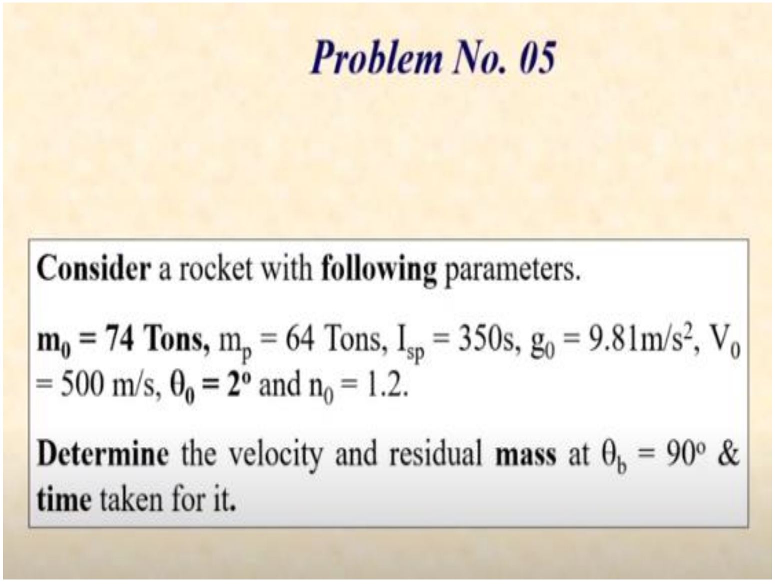

In this case, let us consider a rocket of mass 74 tons carrying a propellant of 64 tons of $I_{sp} = 350s$. It has an initial velocity of 500 m/s, an initial inclination of 2°. And the constant specific thrust factor is 1.2. Which means, that $\frac{t}{mg}$ is 1.2g. Let us try and determine the velocity and the residual mass when it reaches the angle 90° and also the time taken for this maneuver.

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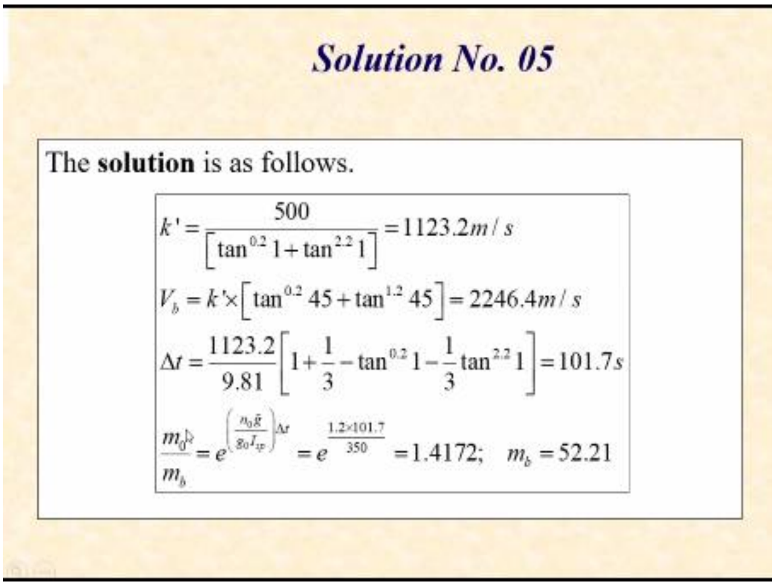

So let us first begin by calculating k' as 500 that is velocity divided by $\tan^{0.2} 1 + \tan^{2.2} 1$. If you perform this calculation, you will find that the k' turns out to be 1123.2 m/s units. And your V_b at the end of burnout which is defined as $\theta_b = 90°$ will be $k' \times [\tan^{0.2} 45 + \tan^{2.2} 45]$.

Now you will realize that because $\tan 45$ is 1, no matter what is the power raised it will remain 1. So, this is $1 + 1$ which is 2. So, the V_b is just twice of k' and that is 2246.4. Similarly, we can now calculate the time taken for this maneuver, which is 101.7 seconds. And then we can talk about the mass fraction which in present case leads to the burnout mass as 52.21 tons.

Hi, so in this particular tutorial, we have considered the three cases of gravity turn maneuver that is constant pitch rate, the constant velocity and constant specific thrust or t by m case and we have looked at the application of various formulae in different forms to arrive at different solutions. And you will note that it is very easy to convert these relations into a territory design context where you can arrive at a design solution based on certain terminal requirement.