Introduction to Launch Vehicle Analysis and Design

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Lecture - 23
Variant Design Solution

Hello and welcome. So, in this lecture, we will look at some of the details of the trade-off ratio concept including its formulation and the solution in order to understand the implication of trade-off ratio as an important design tool. So let us begin.

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So let us proceed with our discussion on trade-off ratio for variant design.

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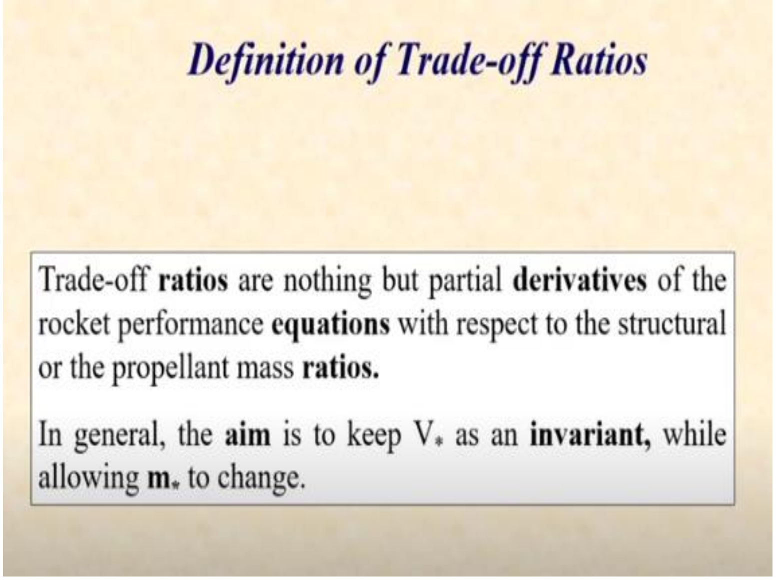

As I had mentioned in the previous lecture, trade-off ratios are nothing but the partial derivatives of the rocket performance equations with respect to two configuration parameters that is the structural mass or the propellant mass which we know directly influence the rocket performance in terms of either the burnout velocity or in terms of the mission payload mass fraction.

Now while there can be requirements on performing a different mission because same m_* in most cases, the variants are required to perform the same mission for a different mass as we have seen in the examples in the previous lecture.

So at least to understand the idea, we use the basic constraint that V_* is an invariant which represents the spacecraft mission which is to be performed while we are going to look at the implication of changes in the stage to the change in the mission payload mass or payload mass fraction π_*.

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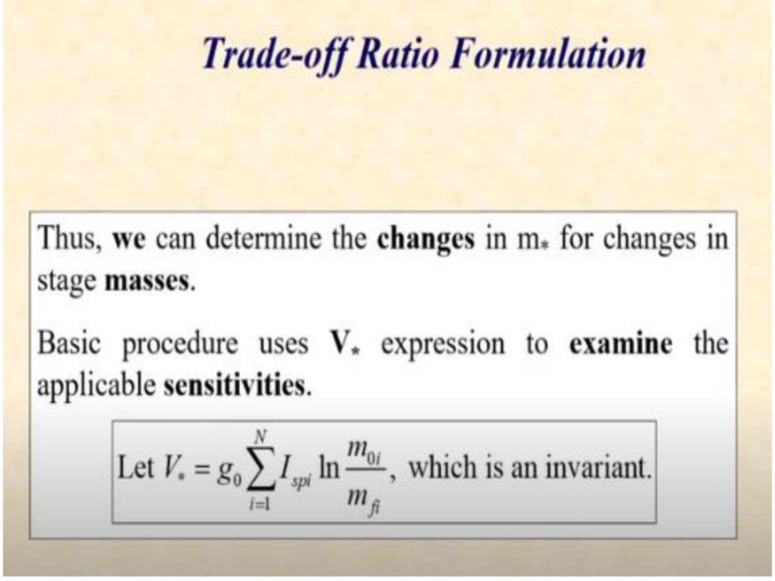

So, in the present study, we will talk about a philosophy for tradeoff ratios, which we will assume that you do not want the V_* to change and we would like to find out how a small change in the stage mass will influence the change in the payload mass. To do this, we make use of the V_* expression to examine the applicable sensitivities.

So let V_* be the $g_0 \sum_{i=1}^{N} I_{spi} \ln \frac{m_{0i}}{m_{fi}}$ is treated as a variant. Note that m_{0i} and m_{fi} have already been defined by discussing the multi-stage rocket configurations.

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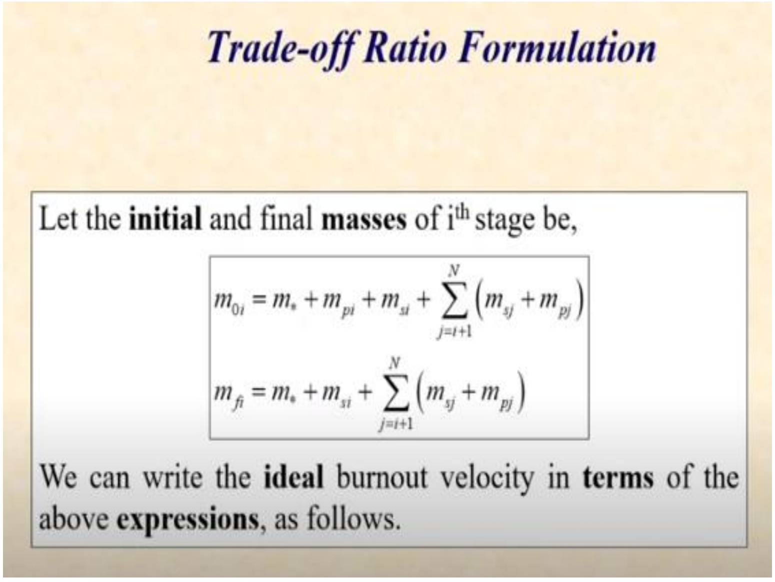

So, we just recall those definitions for m_{fi} and m_{0i} for i^{th} stage as $m_{0i} = m_* + m_{pi} + m_{si}$, that is this configuration of the stage plus $j = i + 1$ which means the stages which are higher than the i^{th} stage right up to the end stage and their mass configuration in terms of $m_{sj} + m_{pj}$. We know that m_{fi} is going to be $m_{0i} - m_{pi}$ so that it is $m_* + m_{si}$ + the same sum.

Now we can write the ideal burnout velocity in terms of the above expressions as follows.

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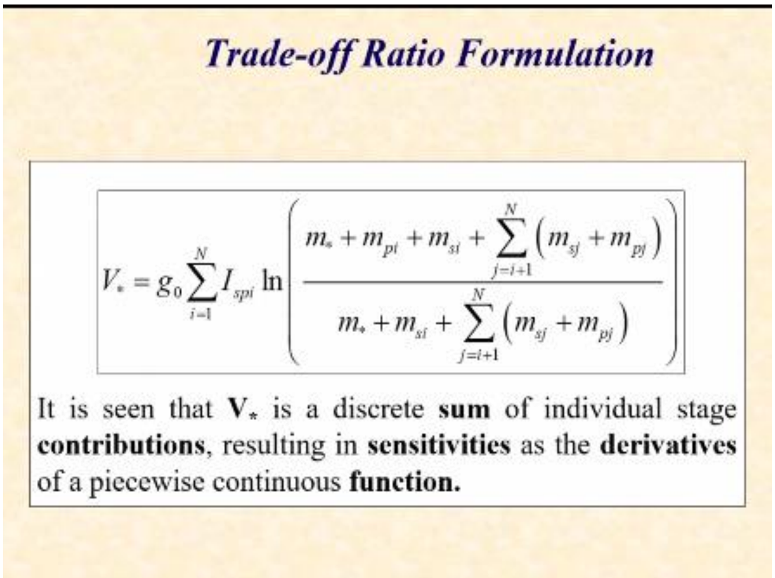

So, V_* can be expressed as $g_0 \sum_{i=1}^{N} I_{spi}$ and now I open up the expression for m_{0i} and m_{fi}. It is the ratio of m_*, m_{pi}, m_{si}. What you realize is that now this V_* we would like to keep in variant by making changes to m_{si}, m_{pi} & m_*. And this is the primary objective of this whole exercise.

But this is not a straightforward exercise because of the fact that V_* is a discrete sum of individual stage contributions. This is the first thing that we need to note that it is a discrete sum. So, what it means is that, this is essentially a piecewise continuous function. It is not a continuous function.

Which means, as you go from stages $i = 1 \ to \ N$, the numbers change and because of which there are sudden jumps in the masses at the interface points and we need to generate the partial derivatives to understand the sensitivities of a piecewise continuous function. Of course, from the calculus, you would have probably dealt with these kinds of functions for arriving at their derivatives.

So, I presume you know how this can be done. In case there are certain gaps, I would suggest that you go through and refresh this material again just to understand the implication of taking a derivative of a piecewise continuous function.

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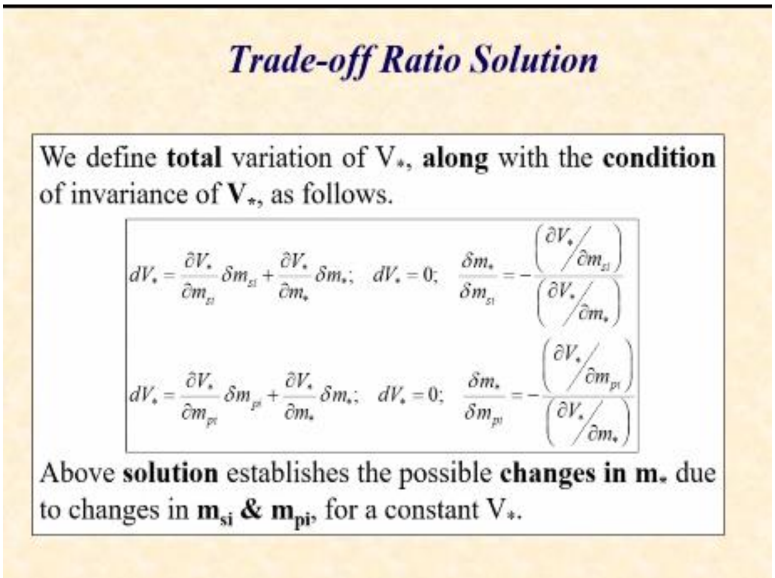

There is another calculus concept that we are going to use. As V_* is a function of many variables, instead of talking about a derivative of V_*, we talk about variation of V_*. So, we introduce the concept of a variation which is kind of a total derivative. Total derivative in the calculus is defined as a linear combination of the individual partial derivatives with respect to all the variables of the function.

In the present case, there are three variables that influence the V_* that is m_*, m_{pi} and m_{si}. And that is why our V_* will be defined in the context of variation as partial derivatives corresponding to these three quantities. So let us bring in this idea through this simple formulation. So, we define dV_* as a small variation in the V_*, the total variation, as a linear combination of the partial derivative of V_* with respect to m_{si} into a small change in m_{si}.

Plus, a partial derivative of V_* with respect to m_* for a small change in the m_* that is Δm_*. Now even though there are three variables, what we will do is that we will take two at a time which means that we will take m_* and m_{si} as one group and m_* and m_{pi} as another group. And because we are using partial derivatives it is a linear formulation.

So that in case we need to change both m_{si} and m_{pi}, it would just be a linear combination of these two separately. Once we make use of this philosophy, we just say that if V_* is a constant, then this small variation in V_* must be 0, which means that the dV_* is equal to 0.

And that gives us a relation between a small change in m_* with respect to a small change in m_{si} as the ratio of two partial derivatives $\frac{dV_*}{dm_{si}}$ and $-\frac{dV_*}{dm_*}$. So, we immediately see that the sensitivity of the payload mass for a small change or a unit change in the structural mass of i^{th} stage is nothing but negative of the ratio of the two partial derivatives, one with respect to m_{si} and other with respect to m_*.

We now repeat this exercise for the m_{pi} and we similarly find that the sensitivity of m_* with respect to m_{pi} is again negative of the ratio of the two partial derivatives of V_* that on the numerator with respect to m_{pi} and in the denominator with respect to m_*. The above expression for sensitivity establishes the possible changes in m_* due to the change in m_{si} and m_{pi} one at a time for a constant V_*.

Here it is also worth noting that I could easily have done this exercise for keeping m_* equal to constant and then take in the m_* derivative with respect to m_{si} and m_{pi} and then I could have defined the partial derivatives for the sensitivities which would give me a different V_* for keeping the m_* constant.

So, the same formulation strategy is entirely applicable for the other case as well. Now we need to understand how we can evaluate these partial derivatives.

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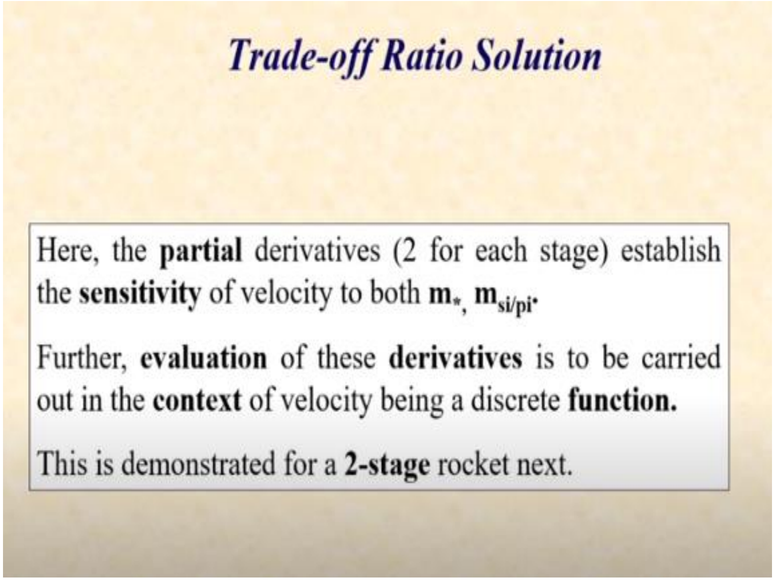

Of course, we realize that these partial derivatives, two for each stage please remember, that for each stage there are two partial derivatives, one with respect to the structural mass, one with respect to the propellant mass. And if there are n stages, there are going to be $2n$ such parameters which will establish the sensitivity of the velocity to both m_* and m_{si} and m_{pi}.

And again, let me reemphasize that the evaluation of these partial derivatives has to be carried out in the context of velocity being a discrete function or a piecewise continuous function of m_{si}, m_{pi} & m_*. So let us try and demonstrate this for a simple case of a 2-stage rocket to understand the procedure involved and how these piecewise continuous functions can be differentiated to give the sensitivities.

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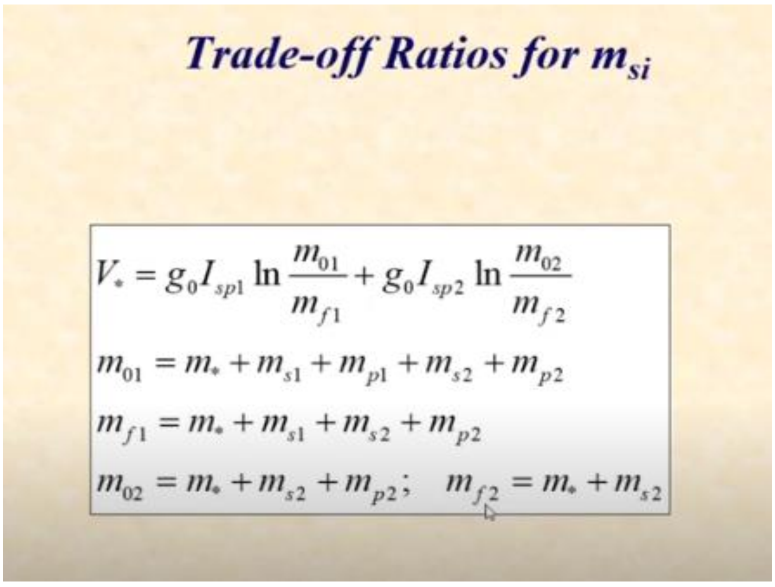

So let me just for the sake of demonstration, open up the sum and write V_* as the sum of velocities coming from the two stages that is first stage and the second stage. Similarly, let me write m_{01}, which is the starting mass of the first stage in the long hand fashion of $m_* + m_{s1} + m_{p1} + m_{s2} + m_{p2}$.

The reason why I am writing like this is that my process of differentiating a piecewise continuous function will become simpler when I express it in this form and will be clearly visible as to what the steps are involved. Similarly, I write m_{f1} by subtracting m_{p1} from m_{01} so that I will get $m_* + m_{s1} + m_{s2} + m_{p2}$. And similarly, the starting mass for the second stage that is m_{02} as $m_* + m_{s2} + m_{p2}$.

And m_{f2} as $m_* + m_{s2}$. As there are only two stages the final mass of the second stage would be just the m_* and the structural mass of the second stage at which the final velocity we would have achieved.

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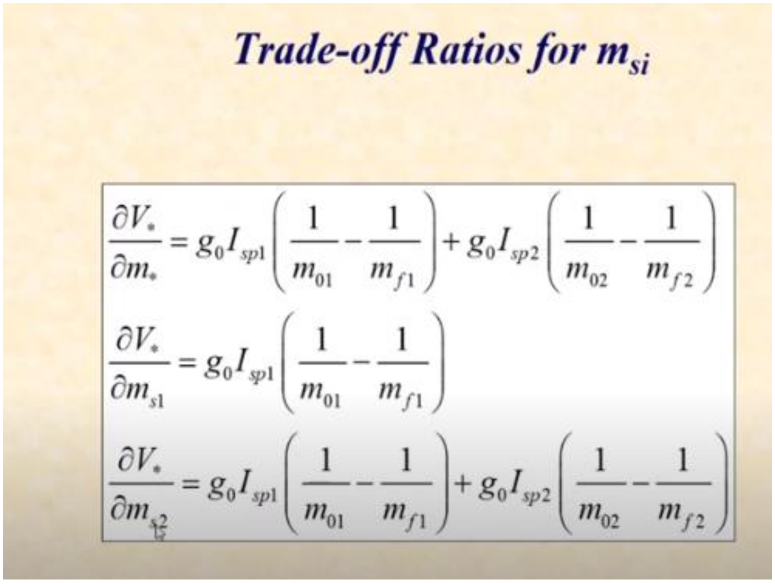

Let us now take the partial derivatives of the velocity equation the way it is written. And immediately we realize that as both m_{01} and m_{f1} and m_{02} and m_{f2} contain the m_*. And it is a logarithmic function of m_*. We know that when we differentiate a logarithmic function, we get 1 by that function. So, I can easily write $\ln \frac{m_{01}}{m_{f1}} = \ln m_{01} - \ln m_{f1}$.

And then when I differentiate these two, $\ln m_{01}$ when I differentiate, I get $\frac{1}{m_{01}}$. And m_* differentiated with respect to m_* gives me unity, so no problem. And similarly, I get $\frac{1}{m_{f1}}$ when I differentiate $\ln m_{f1}$. Similarly, when I differentiate m_{02} and m_{f2}, I get $\frac{1}{m_{02}}$ and $\frac{1}{m_{f2}}$. And this becomes my $\frac{dV_*}{dm_*}$ expression.

When I look at V_* expression, I immediately realize that when I use m_{s1}, the variation is only with respect to the structural mass of first stage. So, the structural mass of second stage is constant. So, the partial derivative terms will contain only the term corresponding to m_{s1} which appears in m_{01} and m_{f1} because in m_{02} and m_{f2}, m_{s1} does not exist.

Because we have removed that mass and this is the implication of a piecewise continuous function being differentiated. I hope you have understood the philosophy. But maybe you can work with this a little more just to understand the step. Once we understand this, we do the same thing for m_{s2}. And now you realize that m_{s2} is part of both m_{01} and m_{02} as well as part of m_{f1} and m_{f2}.

So, you get the same expression as what you would get for differentiating with respect to m_*. So, you can see that dV_* by m_{s2} is same as $\frac{dV_*}{dm_*}$ while $\frac{dV_*}{m_{s1}}$ is different.

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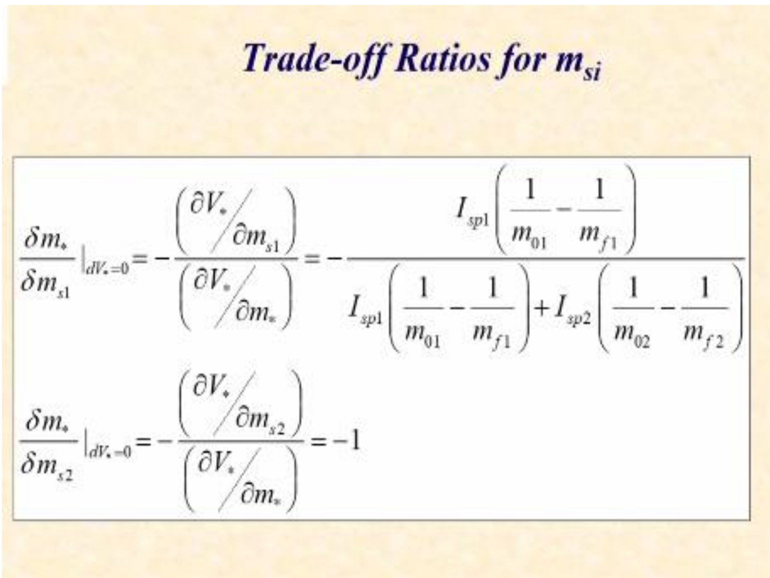

With this, let us now go back to our sensitivity definition that is ratio $\frac{\delta m_*}{\delta m_{s1}}$. And we find that the ratio is as ratio of two partial derivatives. And when I take the ratio of $\frac{\delta m_*}{\delta m_{s2}}$, I suddenly find that both numerator and denominator are the same so I get -1. We will talk about this idea a little more. But at this point it is sufficient to mention that a small change in m_{s2} will generate an equal and opposite change in m_*.

Which means that if I reduce the structural mass by 1 kg, it immediately allows me to add 1 kg in the payload mass and that tells you that this is the best possible efficiency that one can get. That you are trading off 1 kg of structural mass with 1 kg of payload mass. Conversely, if you want to increase 1 kg payload mass you must reduce 1 kg of structure from the second stage. Of course, this number is going to be less than 1 in case of m_{s1}.

So obviously, the efficiency is lower. So, for the same change desired in m_*, you may need to make larger changes in the structure of the first stage as compared to the second stage. And this establishes the fact that from a structure perspective, the second stage is 100% efficient on the final stage. Now the same logic could be extended to n^{th} stage so that the last stage is the most efficient from structural point of view.

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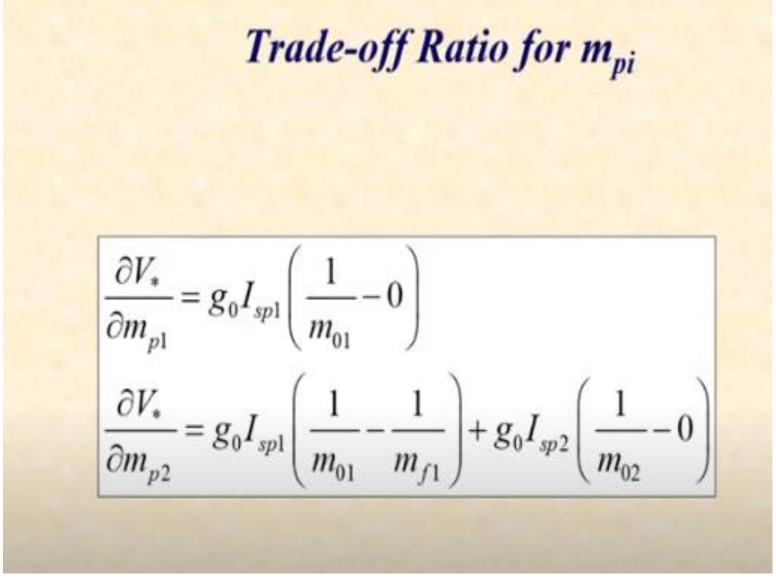

Let us now do this exercise for the propulsion mass. So, I am not going to go through the details. We can again generate the derivative $\frac{dV_*}{dm_{p1}}$ and $\frac{dV_*}{dm_{p2}}$, and we already have $\frac{dV_*}{m_*}$ and so we can take the ratios directly as we have seen earlier.

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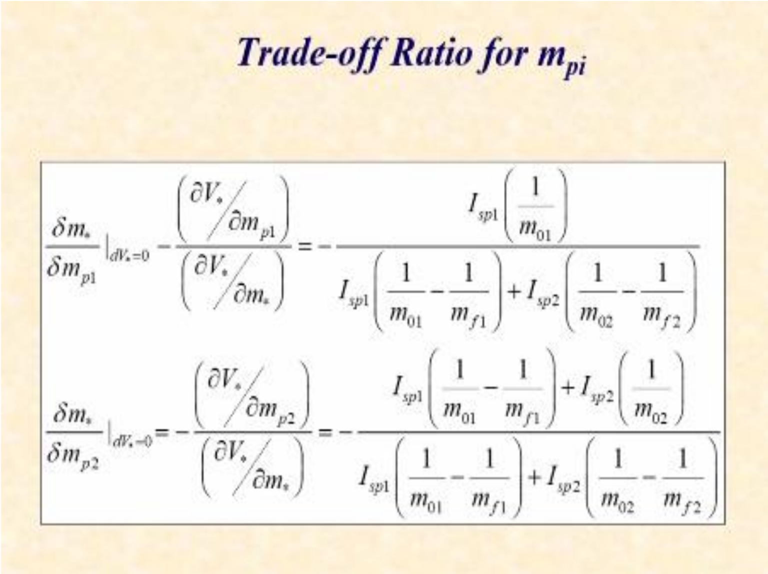

So now we are going to get the two efficiencies that is $\frac{\delta m_*}{\delta m_{p1}}$ and $\frac{\delta m_*}{\delta m_{p2}}$ in terms of the $m_{01}, m_{f1};\ m_{02}, m_{f2}$. And an interesting feature is that I_{sp} also directly appears in these two expressions. Now of course, I could have independently generated the same sensitivities with respect to I_{sp} by keeping both V_* and m_* in a particular manner.

But even without doing that, I realize that if m_{01}, m_{02} these quantities do not change. Which means if my structural mass does not change, then if there is a change in I_{sp}, I can use this partial derivative expression directly to show how a small change in I_{sp} will affect for a constant propulsion mass the change in m_*. It is possible for me to reinterpret these algebraically.

I suggest that you try this on your own, just to understand the implication of the discussion.

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Trade-off Ratio Generalization

$$\frac{\delta m_*}{\delta m_{si}}\Big|_{dV_*=0} = -\frac{\sum_{j=1}^{i} I_{spj}\left(\frac{1}{m_{0j}} - \frac{1}{m_{fj}}\right)}{\sum_{k=1}^{N} I_{spk}\left(\frac{1}{m_{0k}} - \frac{1}{m_{fk}}\right)}; \quad \frac{\delta m_*}{\delta m_{sN}} = -1; \quad \text{Always} < 0$$

$$\frac{\delta m_*}{\delta m_{pi}}\Big|_{dV_*=0} = -\frac{\sum_{j=1}^{i-1} I_{spj}\left(\frac{1}{m_{0j}} - \frac{1}{m_{fj}}\right) + \frac{I_{spi}}{m_{0i}}}{\sum_{k=1}^{N} I_{spk}\left(\frac{1}{m_{0k}} - \frac{1}{m_{fk}}\right)}; \quad \text{Always} > 0$$

Once we have that, we can now directly talk about the sensitivities. And in a generic sense, we now see the $\frac{dm_*}{dm_{sN}} = -1$, is always less than 0. And $\frac{dm_*}{dm_{pi}}$ this expression, is always a positive quantity. And this is another point that you need to understand that the trade-off ratio for structural mass is inversely related that in order to increase the payload mass, I must reduce the structural mass that is the relation.

Whereas in order to increase the payload mass, I must increase the propellant mass. So, if I increase the propellant mass or if I increase the I_{sp}, I will get a higher m_*. Whereas, if I decrease the structural mass, I will get a higher m_* or if I increase the structural mass, I get a lower payload mass.

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Two-Stage Example

Consider a rocket with the following **mass** configuration.

$$m_{p1} = 21087kg; \quad m_{s1} = 1296kg; \quad I_{sp1} = 261s$$
$$m_{p2} = 3854kg; \quad m_{s2} = 360kg; \quad I_{sp2} = 324s$$
$$m_* = 668kg$$

Let us just understand these relations through a simple example for a rocket which has the following mass configuration. So, it is 2-stage rocket with heavy first stage propellant of 21,000 kg and a structure of close to 1300 kg with an I_{sp} of 261 seconds. The second stage has propellant of 3850 kg, the structure of 360 kg, an I_{sp} of 324 seconds, and the payload mass of 668 kg. Let us try and obtain this trade-off ratios.

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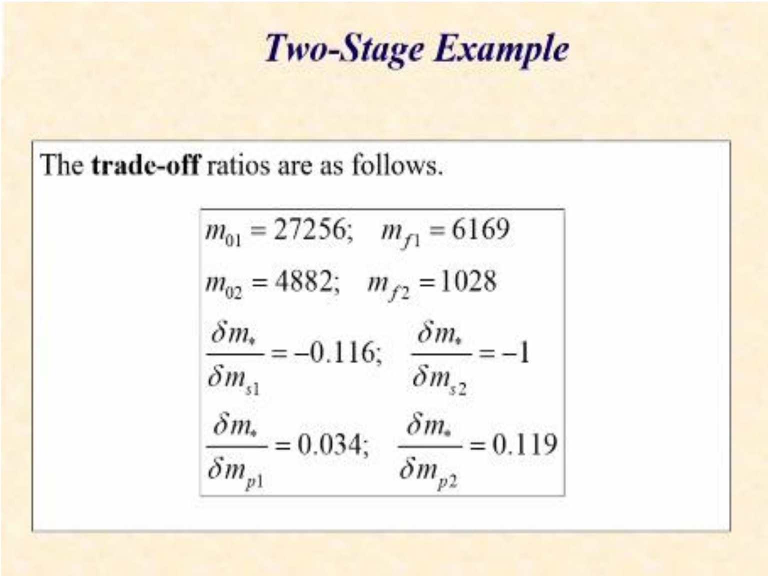

Now we just do the substitution in the expression. I suggest that you do that exercise yourself. Just to confirm, I am just giving you the m_{01}, m_{f1}, m_{02} & m_{f2} which can be obtained from the data that is given in the previous sheet. And with those I can evaluate the partial derivatives and then take the ratio of the partial derivatives. And I find that $\frac{\delta m_*}{\delta m_{s1}}$ is -0.116.

What it means is that, if I reduce the structural mass of the first stage by 1 kg, I will be able to add only 116 grams of payload. But if I make the same change in the second stage structural mass, I can add full 1 kg or 1000 grams. So, you can see that second stage is 100% efficient. With regard to propulsion find that the first stage is highly inefficient that for 1 kg propellant mass increase I am only able to add on 34 grams of payload.

Of course, the second stage is definitely better than the first stage. So that the same 1 kg propellant gives me 119 grams of payload. This is because of two reasons. One, the second stage is more efficient and more importantly, it also has a higher I_{sp} compared to first stage. And now we realize that most launch vehicles try to use a higher I_{sp} fuel in the higher stage and now you will understand why.

Why you do not use solid propellant in the higher stages? Because it is a lower I_{sp} fuel, it will give you a lower efficiency of the stage so that in case you want to do a trade-off, it is not going to be a very efficient design. Whereas, if you use the cryogenic fuel in the higher stages, then even a small saving of structural mass or a small saving in case in the propulsion mass is going to significantly add to the payload mass and becomes an extremely useful way of creating a variant.

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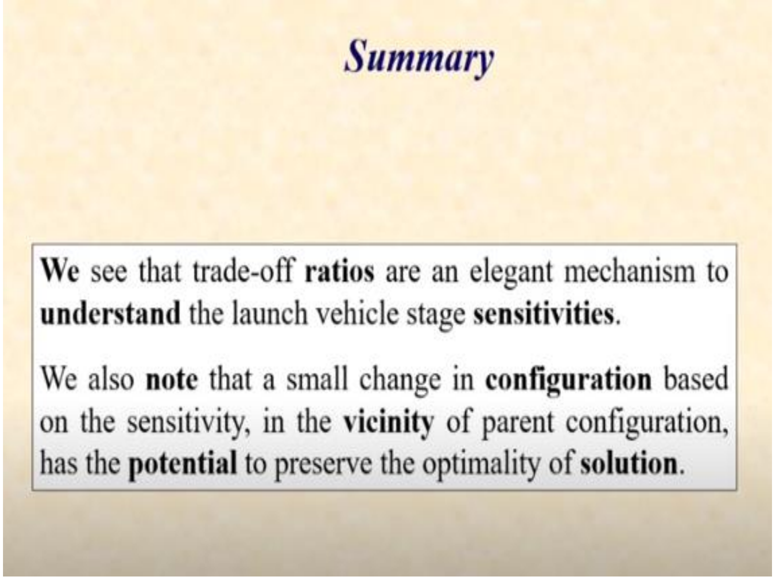

So, to summarize, we see that trade-off ratios are an elegant mechanism to understand the launch vehicle state sensitivities. We also note that a small change in configuration based on the sensitivity in the vicinity of the parent configuration has the potential to preserve the optimality of the solution. Hi, so in this lecture, we have seen a simple mechanism through which we can set up the solution for the sensitivities of the stage under the constraint that the V_* is a constant.

And we have obtained the solutions and understood that within the limitations of the assumptions that we have made, it is still an extremely useful idea for designing a modified mission with minimal computational effort just by looking at the stages which are more efficient and the amount of saving in the mass can lead to improvement in the payload mass capability.

With this, we close our discussion on the serial, the series or what is also called the restricted staging as a concept for multi-stage rocket design. We will now conclude this idea in the next lecture by looking at the concepts of parallel staging. That is if you add a booster stage, then in what way the configurations change, what are the issues involved with the booster stage, and what kind of benefits we can derive by making use of parallel staging as compared to a serial staging or a restricted staging. So, bye. See you in the next lecture and thank you.