Introduction to Launch Vehicle Analysis and Design

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Lecture – 21
Approximate Staging Solution

Hello and welcome. So, in this lecture we will explore the possibility of an approximate solution strategy which still provide a reasonably optimal solution, but with smaller computational effort. We will also look at the overall implication of the methodology and its relation to the loss of accuracy in the given solution. So, let us begin.

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Let us proceed with the approximates staging solution procedure and understand the elements that provide us with this particular simplification.

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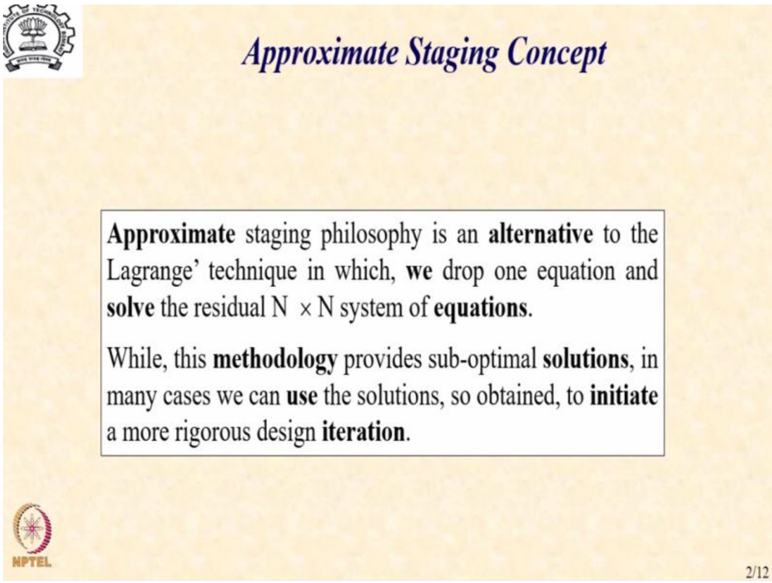

As was mentioned in the previous lecture it is an alternative to the Lagrange technique which we have seen earlier in which we drop one equation and solve the residual $N \times N$ system of equations. This is the first step which means that among the $N+1$ equations and $N+1$ unknowns. We drop one equation so that we do not have to drop the or we do not have to actually include the Lagrange variable λ that we have been doing earlier.

We realize that this methodology is going to provide suboptimal solutions because now it is no longer an exact formulation. However, you will find that in many cases we can use these solutions to initiate a more rigorous design iteration which in many cases require an initial guess of the solution.

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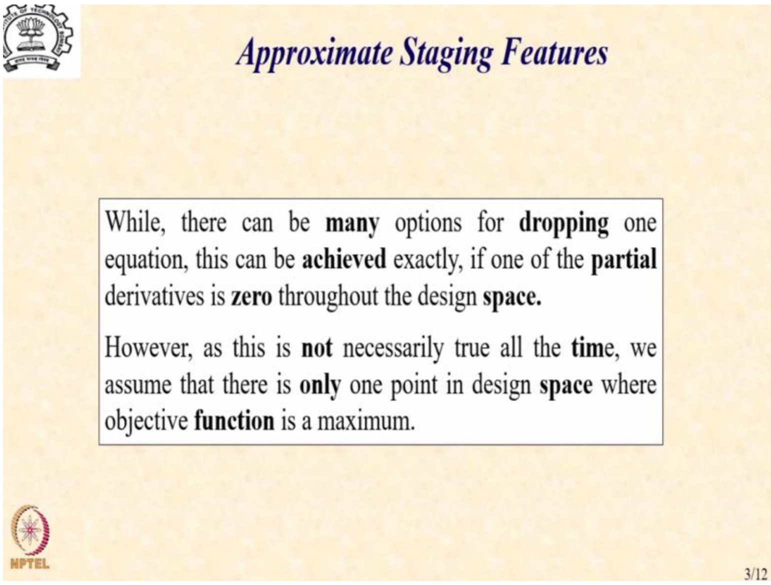

The question is how does one decide on the equation to drop? While there can be many options for dropping one equation. This can be achieved exactly if one of the partial derivatives is zero throughout the design space. Now, this is something which is not really very clearly visible at the stage of formulation, but if such a thing happened and we were able to pick that particular partial derivative which was zero throughout the design space.

Then that equation is automatically exactly specified and satisfied at all the places which obviously means that if we drop this partial derivative and the equation corresponding to it there will be no error committed as far as the solution is concerned because all other points the remaining ones will together pick a point which is optimal from their perspective and it is optimal also from this particular design variable because anyway this is the minimum value that this design variable has.

There are two aspects that become bottleneck in this case. One, there is no way of knowing upfront which of those equations will result in a zero-partial derivative throughout the design space. The second point is the design space may have multiple optima in which case it is possible that we may pick a solution close to a local optimum, but may completely miss a global optimum.

However, with these two limitations still the methodology has a reasonable value in trying to reduce the computational effort and still giving a reasonably good initial guess in situations where there is only one optimum in the design space. So, which means if there are functions which are such that there is one maxima or a minima in the design space then these methodologies can still work with reasonable degree of accuracy provided, we hit upon the right partial derivative equation to drop.

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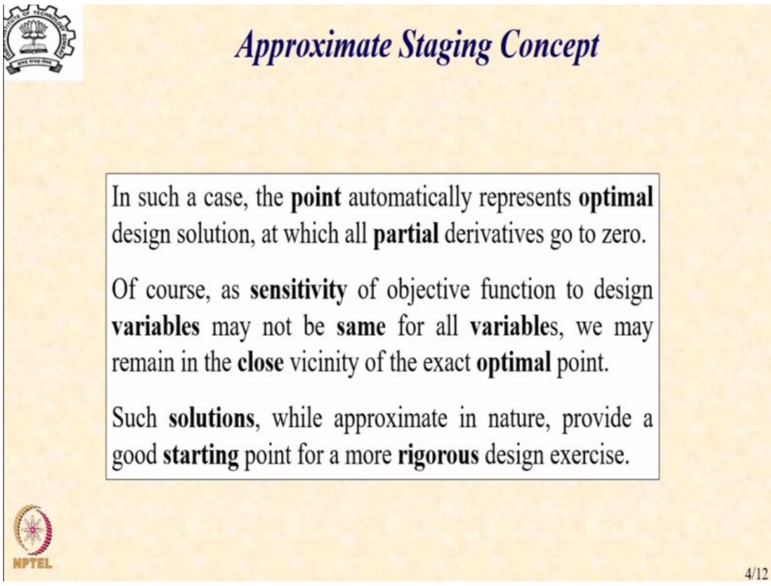

The reason why this is true is that in such a case the solution obtained from the remaining equations will automatically become optimal. But we must realize that the sensitivity of objective function to design variable is not same for all the partial derivatives particularly in a general context. So, obviously depending upon the equation that we drop we may be somewhere in the vicinity of the exact optimal point.

As I mentioned these are approximate, but in many cases can serve as good starting points for a more rigorous design exercise.

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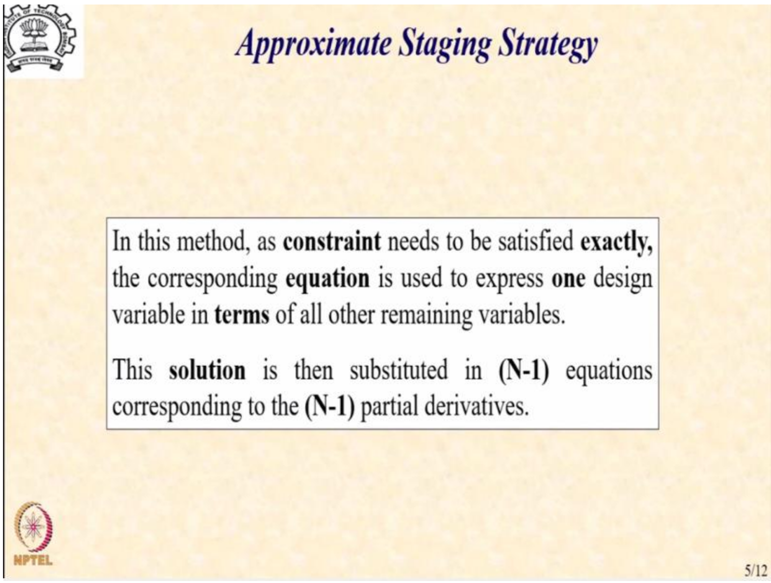

Now in order to ensure that the loss of accuracy is not excessive, we satisfy the constraint exactly which means that the constraint is always exactly satisfied by one of the partial derivatives is probably only approximately satisfied and this gives us another benefit that the constraint equation can be used to substitute one design variable as a function of the remaining $N-1$ design variables.

The benefit that this gives is that the remaining partial derivatives are only $N-1$. So, effectively we need to evaluate only $N-1$ partial derivatives and not N partial derivatives and these $N-1$ partial derivative equations in terms of $N-1$ design variables can be solved simultaneously to obtain solution for $N-1$ design variables.

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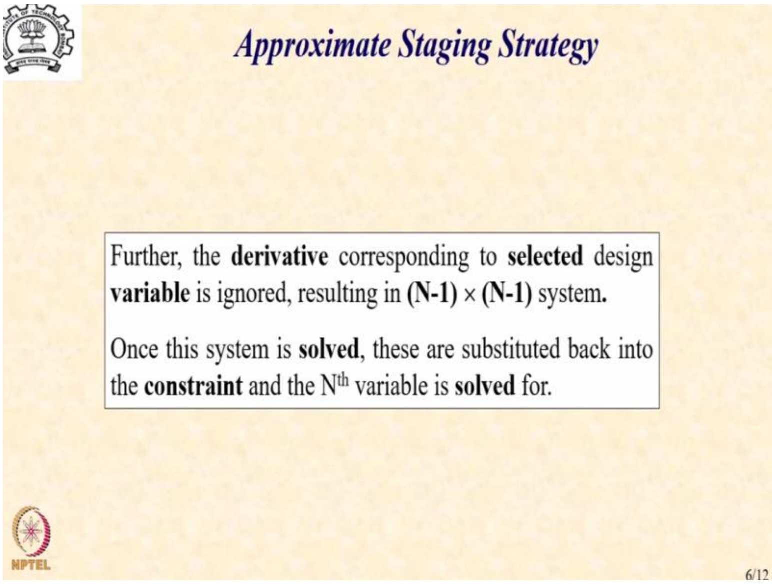

And once you do that, we substitute this solution back into the one design variable which is a function of the remaining $N-1$ design variables and in the process, we get solution for all the N design variables.

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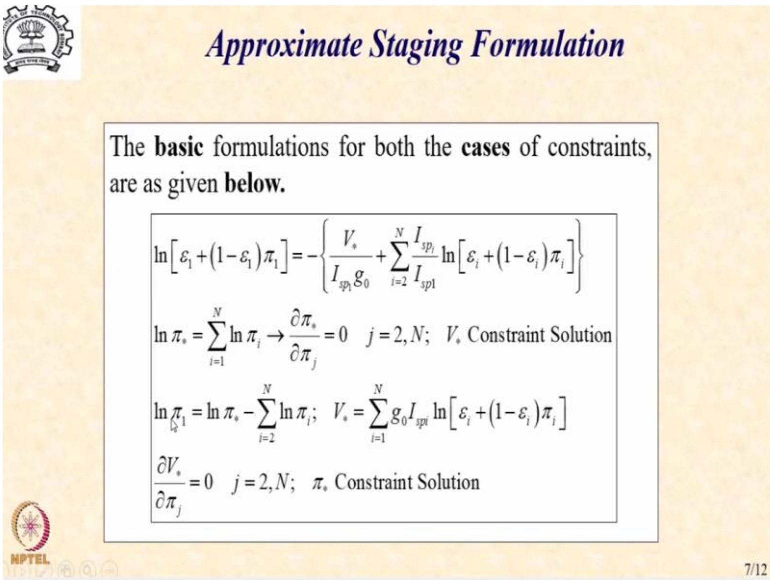

Let me show you the basic formulation that this strategy is connected to. So, let us assume that we are going to choose π_1 to be the design variable for which the partial derivative will not be evaluated. So, now what I do is I take this and express π_1 as a function of the remaining π's that is from π_2 to π_N and this is my constraint relation. Here, of course I have used the velocity constraint so V_* is coming here.

And then I write my π^* which is product of all the π_i's, but because I have removed π_1 it will be a function on only π_2 to π_N and that is why I need to evaluate only those $N-1$ partial derivatives going from $j=2$ to N. So, these represent only $N-1$ algebraic equation and the N^{th} algebraic equation is my constraint relation in which once I evaluate the $N-1$ design variables π_2 to π_N.

I will substitute that into this and evaluate π_1. When I am talking about the π^* as the constraint then again, I use this expression that is $\ln \pi_1 = \ln \pi^* - \sum_{i=2}^{N} \ln \pi_i$ and this is the equation which now I am going to substitute in my objective function for π^* and the objective function which is my V_*. So, V_* will contain only π_2 to π_N and then I differentiate this again j going from 2 to N.

So, I only have $N-1$ partial derivatives. So, I solve for these $N-1$ partial derivative to solve for π_2 to π_N and substitute back into this constraint relation to obtain π value.

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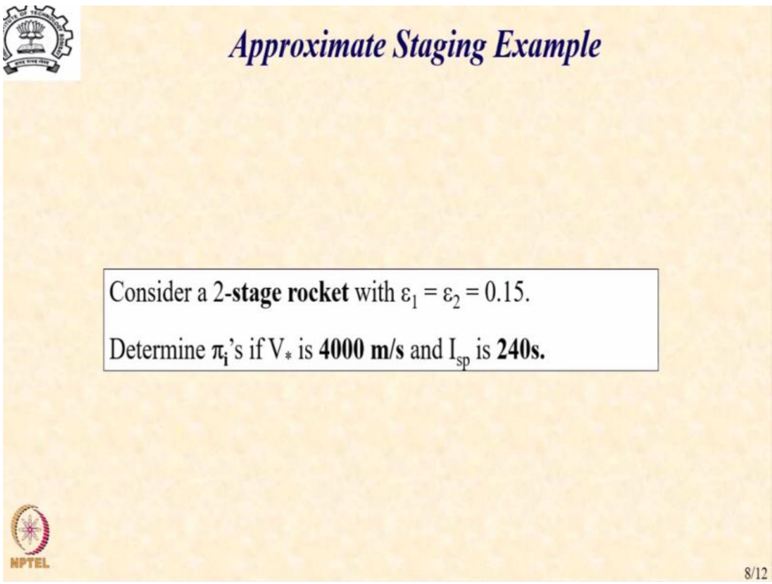

Let us demonstrate this idea through the example that we have seen in the previous lecture. So, again let us consider a two-stage rocket with the same ε that is 0.15 and the same I_{sp} of 240s and I would like the velocity to be $4{,}000\ m/s$. We recall that we had done this exercise for a two-stage sounding rocket and we have obtained a solution.

Please keep those solutions in mind because now we are going to see what this particular methodology gives as far as those solutions are concerned.

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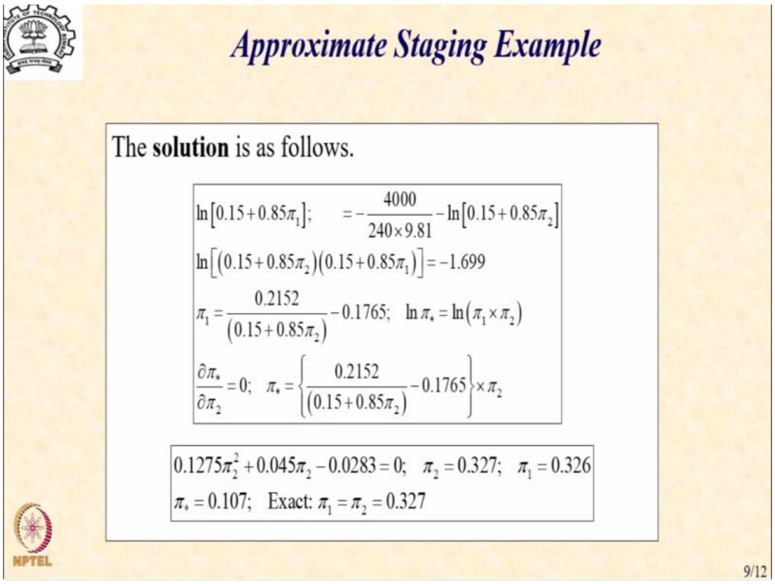

So, the solution is as follows. We first express π_1 in terms of π_2 because that is the only other unknown. So, I can solve for π_1 from this. So, the following is the expression for π_1 that is $\frac{0.2152}{(0.15+0.85\pi_2)} - 0.1765$ that is the solution of π_1 in terms of π_2. My $\ln\pi_*$ is $\ln\pi_1\pi_2$. So, this $\ln\pi_*$ is function of only π_2 because it is only function of a single variable.

I need only one partial derivative of π_* with respect to π_2 which I evaluate and then I get the constraint relation saying that my π_* is this expression and with that expression I get a quadratic equation for π_2. Again, you will realize that because your π_* derivative is going to give us the quadratic equation only one of the π_2 solutions will be consistent other π_2 solution will be inconsistent. I have not mentioned the inconsistent solution, but I am mentioning the consistent solution and you immediately see that the π_2 turns out to be 0.327.

And correspondingly first take this π_2 and substitute into my constraint solution of π_1 I get π_1 as 0.326. I realize that I have got an exact solution. The exact solution for this problem through the Lagrange Multiplier Method was 0.327 for both the stages. Here, you can see there is a marginal leakage possibly because of the truncation error in the decimal places.

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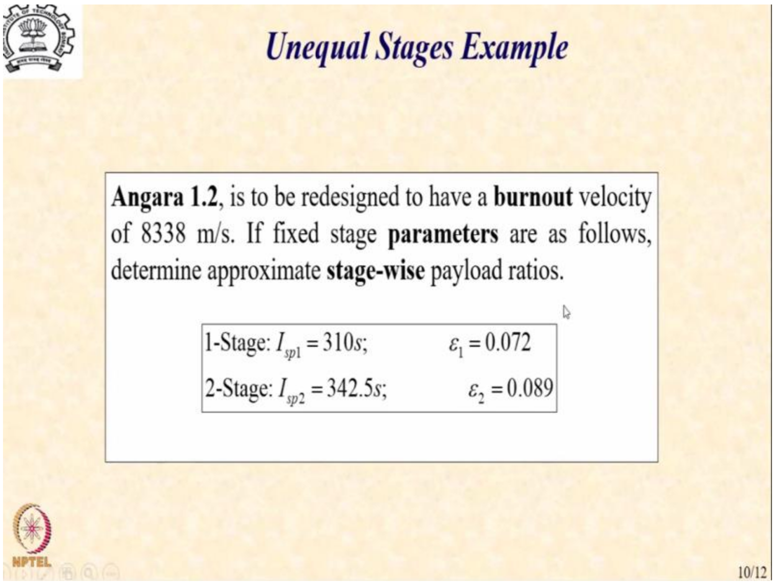

Let us now do the same exercise for unequal stages. So, we have the problem of Angara 1.2 and let us redesign this to have a burnout velocity of 8338 m/s with the fixed parameters so it is a velocity constraint problem. Let us determine the approximate stage wise payload ratios.

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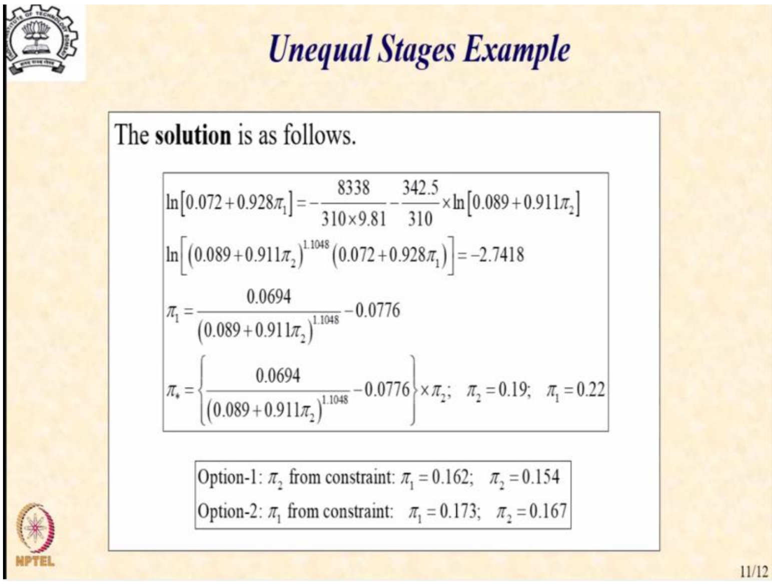

So, the solution in this case as follows. So, again I take the π_1 and substitute into the constraint relation with the velocity of 8338 m/s and get an expression of π_1 in terms of π_2 as $\frac{0.0694}{(0.089+0.911\pi_2)^{1.1048}} - 0.0776$. We define π^* as product of these two differentiate this will π_2. I have skipped those intermediate steps, but my suggestion is that you verify those steps.

And once you verify those steps you can obtain the solution for π_2 as 0.19 and if you take this 0.19 as π_2 and substitute into π_1 relation you can show that π_1 will be 0.22. Now, how do these compare with the actual values? I leave this exercise to you, but you will find that these numbers are not exactly equal to the π_1 and π_2 numbers that we had obtained for this problem using the Lagrange Multiplier Method.

Now option 1 and option 2 as you can see are by interchanging the variables and you will see that there is now a marginal difference between these two solutions. In one case if you get π_2 in terms of π_1 as a next constraint and then differentiate with respect to π_1 you are actually ignoring the partial derivative corresponding to π_2 whereas in the other case you are ignoring partial derivative with respect to π_1.

You will realize that these two provide two different solutions as we had anticipated that as a sensitivity of the partial derivative for different design variables is going to be different you are going to get different solutions depending upon which particular partial derivative you drop.

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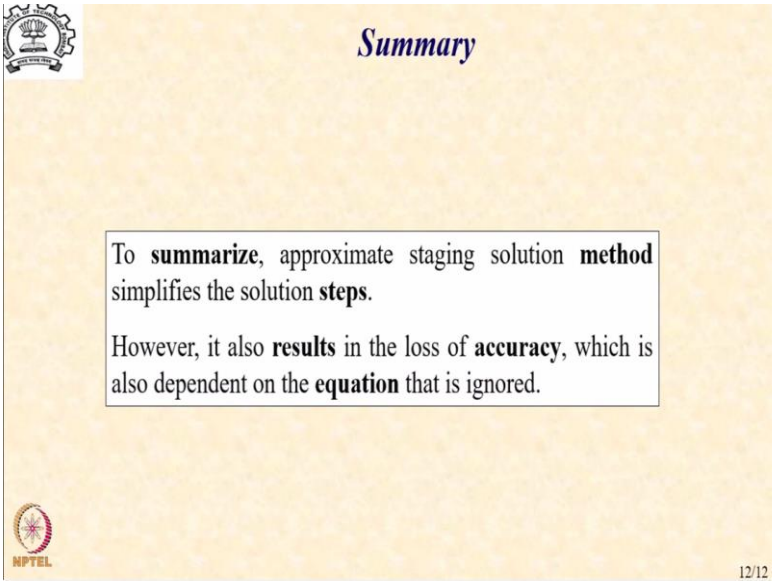

So, to summarize the approximate staging solution method simplifies the solution steps that is clear to us. However, we also see that it results in loss of accuracy and moreover that loss of accuracy is also depended on the equation which is ignored. So, in this lecture we have seen the methodology for obtaining the approximate solution by dropping one partial derivative equation.

And we realize that while we get a solution which is in the vicinity of the actual optimal solution it is not really the optimal solution and that it varies depending upon which particular design variable, we choose for dropping the partial derivative. However, we still note that within the context of the objective for this exercise we still get a reasonably good solution as the starting point for any rigorous design exercise which would ultimately give us the exact optimal solutions.

With this, we have completed our discussion on the optimal solutions for staging. In the next lecture, we will look at a related concept called the concept of variant of a launch vehicle which is an extremely important, practical concept which is commonly practiced by most space agencies to minimize their cost of inventory as well as cost of launch vehicle development. So, bye see you in the next lecture and thank you.