Introduction to Launch Vehicle Analysis and Design

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Lecture – 20
Lagrange Solution

Hello and welcome. In continuation to our last lecture, we will introduce the ideas of optimal staging. We will now look at the basic technique of optimal multistage design through the Lagrange's method which provides optimal solution using one extra variable called the Lagrange Multiplier. And we will probably also look at the possibilities of alternate ways of arriving at the optimal solution. So, let us begin.

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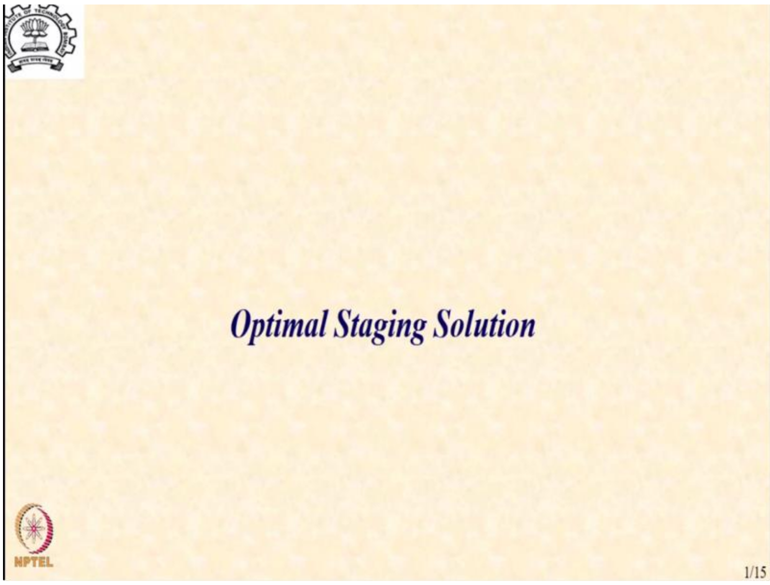

So, let us begin our discussion on the optimal staging solution.

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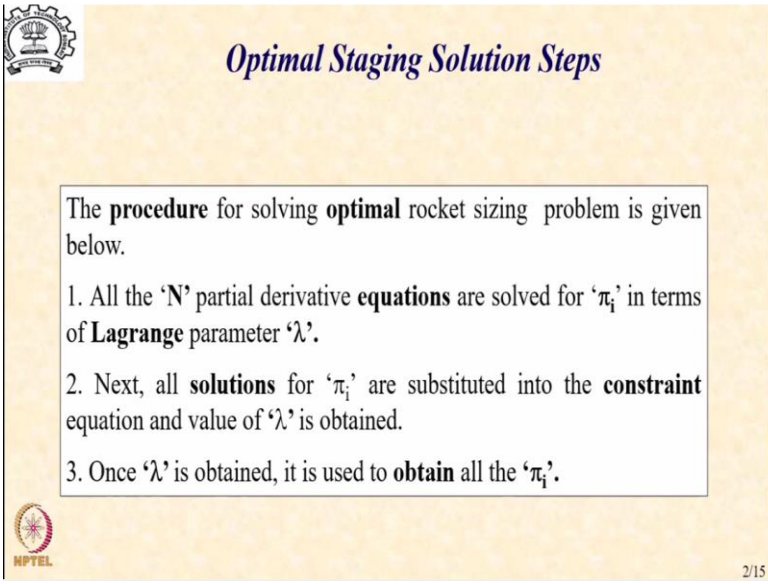

Given below is a broad procedure for solving rocket sizing problem in the present context. So, the first step is that we solve all N partial derivative equations for individual π_i's in terms of the Lagrange parameter λ. So, all the N design variables π_i's are expressed in terms of the Lagrange parameter λ. Next, all these solutions of π_i's which are in terms of λ are substituted into the constraint equation which then becomes an equation in λ.

We can solve this equation it is an algebraic equation and the solution of λ so obtained is then substituted back into the π_i's that we have already expressed in terms of λ and we obtain all the π solutions. So, we see that in this procedure we first have to express all π_i's in terms of λ which is essentially an algebraic substitution and then we solve an N^{th} order algebraic equation in λ which is arrived from the constraint relation.

And once the λ is obtained we go back to those expressions and simply substitute the value of λ and evaluate π_i's.

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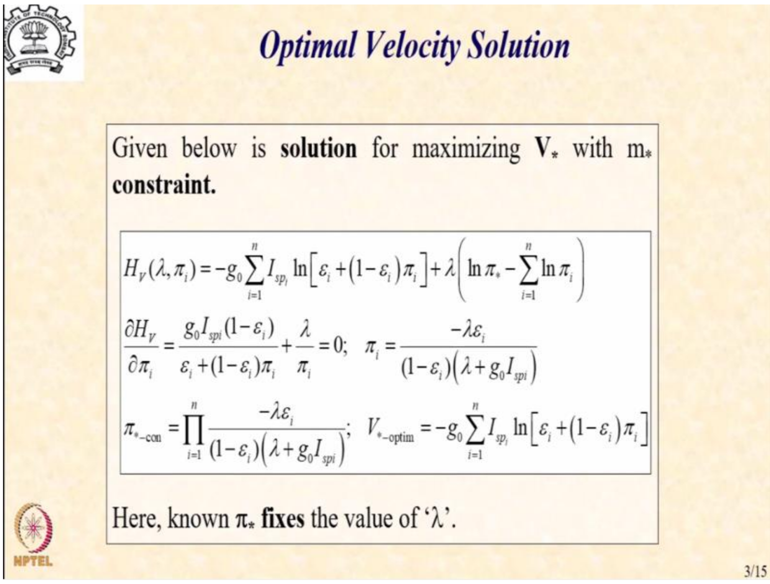

So, let us look at this technique through the two options that we have established that is in one case the V_* will be objective function and the π_* will be constraint and in the other case π_* will be the objective function and V_* will be the constraint. So, let us first look at the case where V_* is the objective function. So, we use the augmented function H_V as we have seen earlier which is nothing but $-g_0 I_{sp_i} \sum_{i=1}^{N} I_{sp_i} \ln[\varepsilon_i + (1-\varepsilon_i)\pi_i]$ that is the objective function part.

And then we have the constraint part that is $\lambda\left(\ln\pi_* - \sum_{i=1}^{N}\ln\pi_i\right)$. Now, we construct the N partial derivatives of the above augmented objective function by differentiating H_V with respect to π_i's and please note because these are partial derivatives, we use the basic strategy of partial derivative that all terms involving only π_i's will be non-zero.

All the terms which involve π_{i-1} or π_{i+1} they all go to zero. The moment we do this we realize that this partial derivative will contain only terms corresponding to π_i. So, we get this derivative as $\frac{g_0 I_{sp_i}(1-\varepsilon_i)}{\varepsilon_i+(1-\varepsilon_i)\pi_i} + \frac{\lambda}{\pi_i} = 0$ and this is an algebraic relation from which we can solve for $\pi_i = \frac{-\lambda\varepsilon_i}{(1-\varepsilon_i)\left(\lambda+g_0 I_{sp_i}\right)}$.

So, these are the relations for all the π_i's in terms of the fixed parameters ε_i and the I_{sp_i} and the Lagrange Multiplier λ. Now, the next step is to substitute these solutions of π_i's into the constraint relation. So, we write down the constraint relation as this product that is pi star constraint relation is $\prod_{i=1}^{n}\frac{-\lambda\varepsilon_i}{(1-\varepsilon_i)\left(\lambda+g_0 I_{sp_i}\right)}$.

We also know that once we obtain the λ_i's the optimum velocity will be the $-g_0 \sum_{i=1}^{n} I_{sp_i} \ln[\varepsilon_i + (1-\varepsilon_i)\pi_i]$. And we immediately realize that a known value of π_* which is the constraint is going to fix the solution of λ and because it is a product on the right-hand side it is clearly visible that we will get an algebraic equation of power n in λ.

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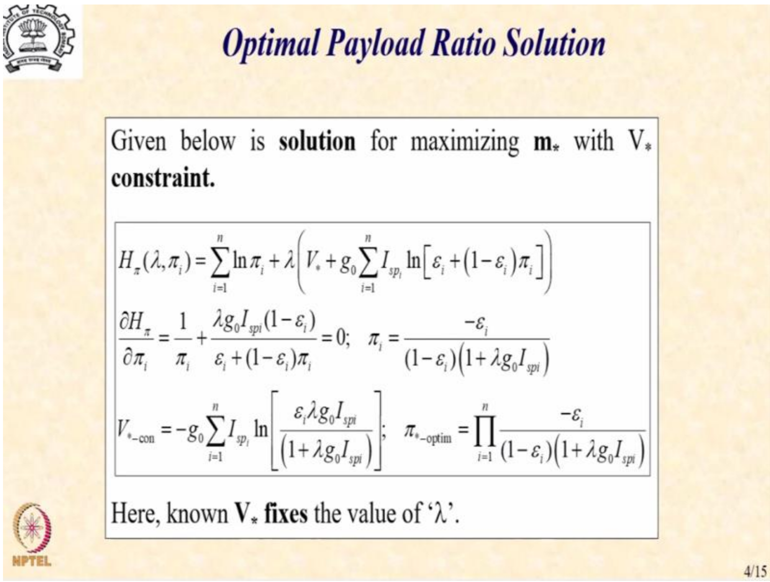

Let us now look at the counter part of this particular solution where we would like to maximize m_* or in this particular case the π_* the payload fraction with V_* as the constraint. So, in this case we use the augmented objective function H_π where the first term that is $\sum_{i=1}^{n} \ln \pi_i$ is the objective part coming from the π_* and then we have the constraint error multiplied by the Lagrange Multiplier λ.

Again, we go through the same process of differentiating this augmented function with respect to π_i and similarly we get only π_i terms in this. And by solving for π_i, we get an expression for π_i in terms of ε_i, I_{sp_i} and λ_i as $\frac{-\varepsilon_i}{(1-\varepsilon_i)\left(1+\lambda g_0 I_{s\,p_i}\right)}$. So, you can see that this expression is different from the expression that we obtain when we use V_* as the objective function then we substitute these values of π_i's into the constraint relation that is V_* constraint.

And then once we do that, from this constraint relation again we will get an n^{th} order algebraic equation in λ whose solution will give us the value of λ which will fix the solution for all the π_i's and using those values of π_i's we can then obtain the optimal value of π_*. Here, the known V_* is going to fix the value of λ. So, we have seen from these two solution procedures that in both the cases the constraint is the one which will fix the solution of the weightage λ which is the coupling parameter for all the payload ratio π_i's and then it fixes their values in relation to the constraint that is applied.

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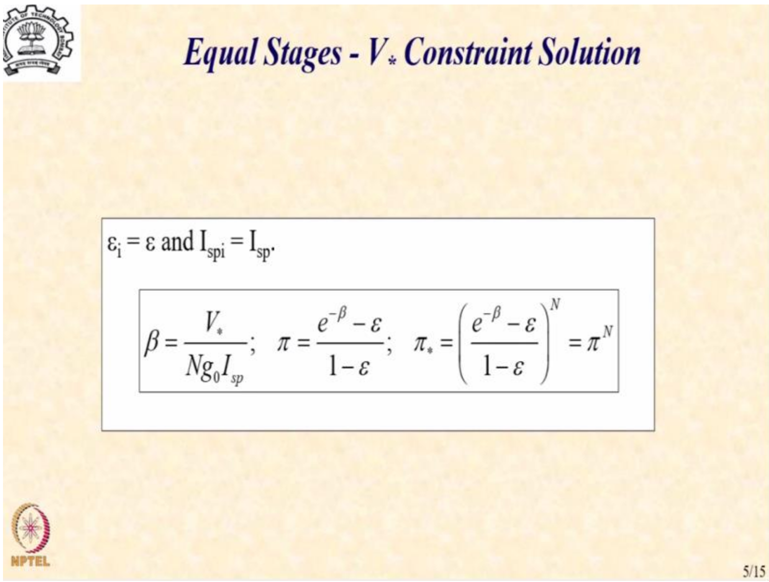

Now, there are certain special cases which we can examine. So, the first special case that is of interest is that if we had the same structural technology and the same propulsion technology to be used in various stages of the rocket what would happen? So, this is denoted as the equal stages which means all stages have equal ε and equal I_{sp}. In that case, we assume that $\varepsilon_i's$ are all epsilon and $I_{sp_i}'s$ are all I_{sp}.

And we substitute these into the expression for π_i's you will immediately notice from this that all the π_i's are going to be the same because all the π_i's are going to be the same it is now a simpler algebraic equation for λ that we get from the constraint and by putting that equation we redefine an additional parameter β as $\frac{V_*}{Ng_0 I_{sp}}$ where V_* is the velocity constraint.

And the π for every stage is $\frac{e^{-\beta}-\varepsilon}{1-\varepsilon}$ because all the $\pi's$ are the same the π_* is nothing, but π^N. We realize that this particular solution in an extremely simple representation if you have the same structure and the same propellant to be used in all the stages. Of course, if either the structure or the propellant or both are different then obviously this formula is not applicable and we must use the expressions as given in the previous two derivations.

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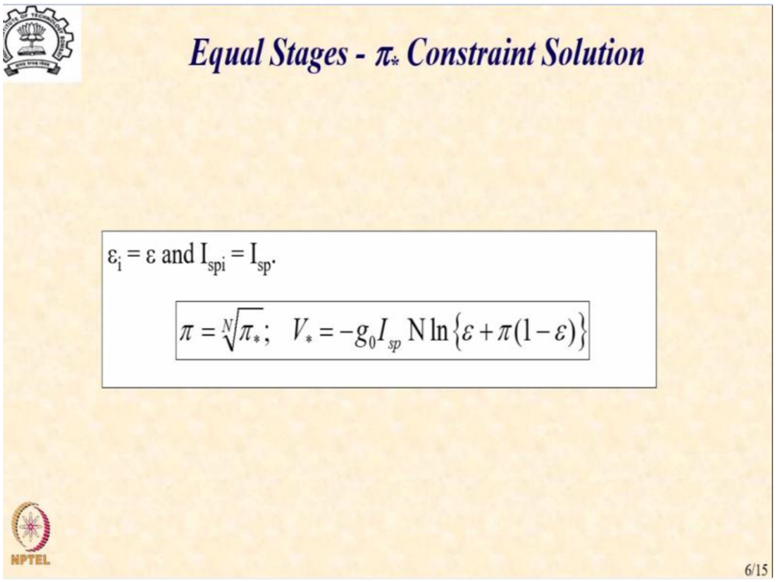

Let us look at the same thing for m_* or π_* constraint. So, in this case because π_* is a constraint it can be shown that all the $\pi's$ will be same because all the $\pi's$ are same the π will be nothing, but the $\sqrt[N]{\pi_*}$. So, directly that is the solution for a stage payload ratio and the V_* now can be obtained directly from this value of π.

So, we realize this when we use this simplification of equal stages the solution simplifies greatly.

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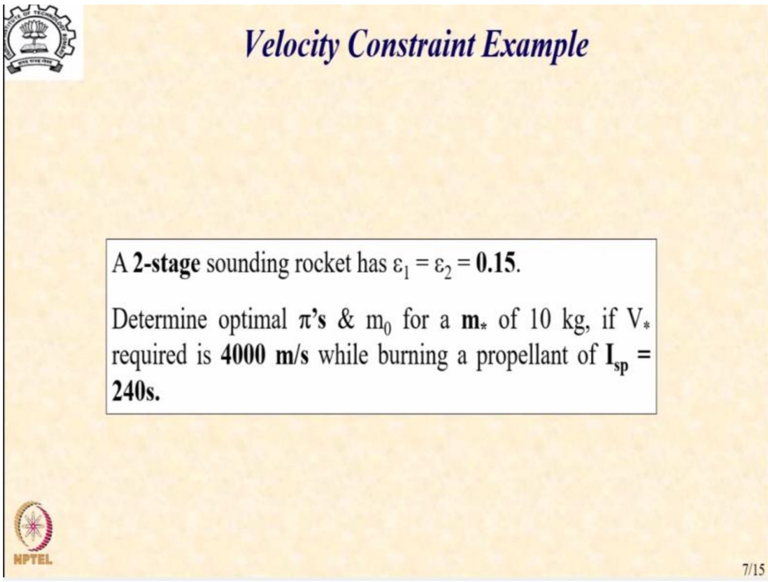

Let us now demonstrate these expressions through couple of examples. So, let us first look at the case of a two-stage sounding rocket having equal stages that is it has $\varepsilon_1 = \varepsilon_2 = 0.15$. Let us try and determine the optimal π and the lift off mass m_0 for m_* of 10 kg if V_* required is 4,000 m/s while burning a propellant of $I_{sp} = 240s$.

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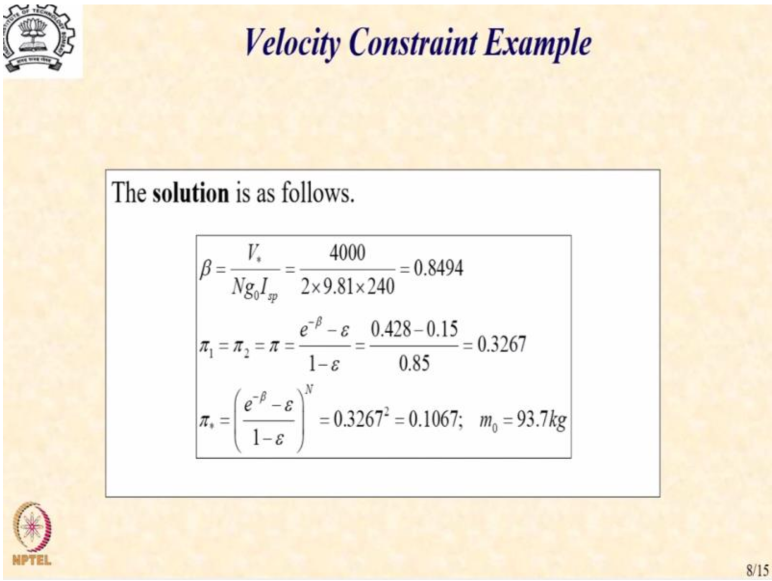

So, the solution is as follows. Let us go through the steps one by one. So, let us first calculate β which is $\frac{V_*}{Ng_0 I_{sp}}$ as it is a two stage it is 4,000 which is the V_*; $\frac{4000}{2\times 9.81\times 240}$. So, we get a β value of 0.8494. Substituting this into the expression for π which is $\frac{e^{-\beta}-\varepsilon}{1-\varepsilon}$ we get π_1 as 0.3267. Now this is the value which is common for both the stages.

So, π_* becomes the (0.3267^2) which is nothing, but 0.1067. So, our payload fraction in this case which is maximizing π_* is 0.1067 and for a payload of 10 kg the rocket must weight roughly around 94 kg. So, now we have designed an optimal sounding rocket which has a payload fraction of 0.106 and a 94 kg rocket will be able to launch a 10 kg payload.

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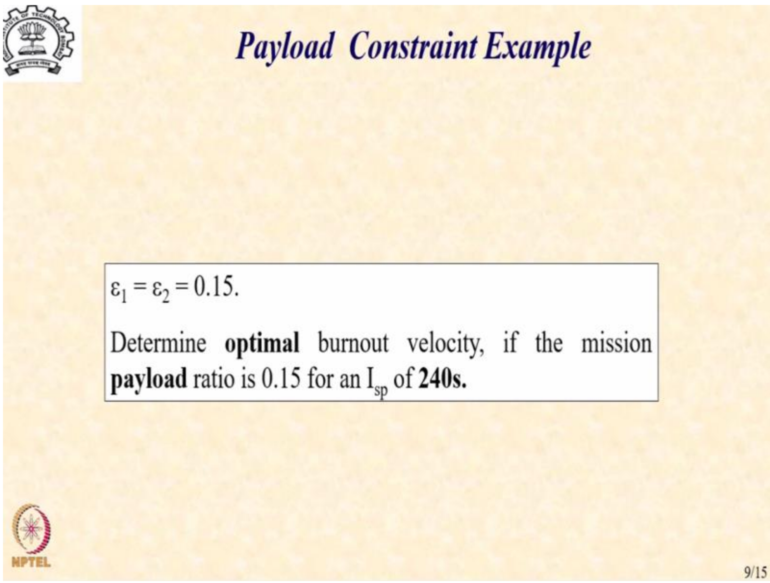

Let us now flip the problem and look at when we want to put a payload constraint and see what is the solution that we get and what is the velocity that we are going to get. So, in the previous case the payload fraction that we had got was 0.106. Let us try for a slightly higher payload fraction of 0.15 and let us see what happens to the solution for the same set of structural ratios and same I_{sp} of 240.

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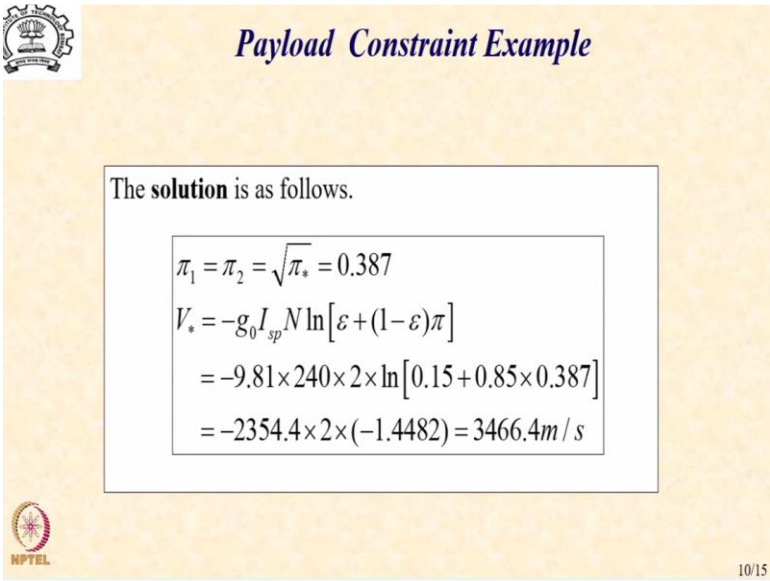

So, this solution is as follows. Now we know that both the π's are same which are nothing, but $\sqrt{\pi_*}$ sand it is 0.387. So, now you can see in the previous case the π was 0.32, but now the π has become 0.38. So, the payload ratios are higher because the payload ratios are higher now, I substitute these into my V_* expression and what I get as V_* is slightly lower.

Instead of 4,000 m/s I get only 3,466 m/s and here there is now an important result that we need to note. There is a tradeoff between the burnout velocity and the π_*. If you want a higher π_* you must accept a lower velocity or if you want a higher velocity, you must accept a lower π_*.

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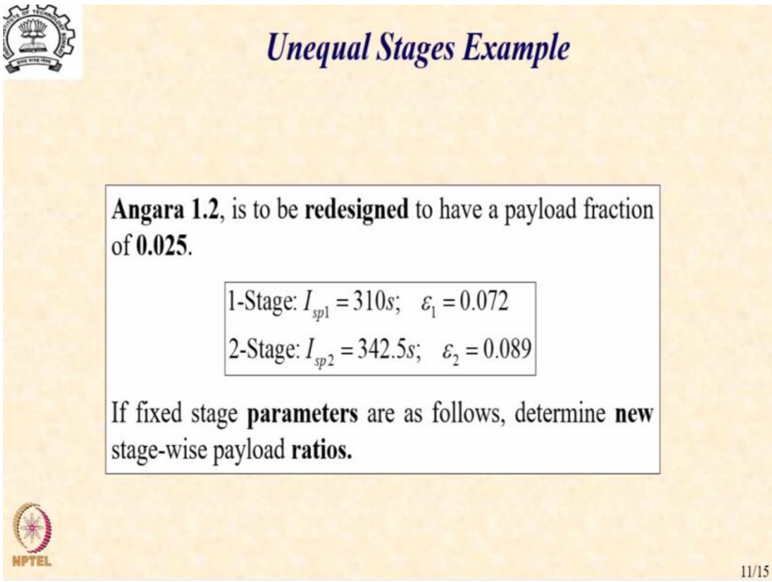

Let us now go to the general problem where we have stages which are not equal and it is useful to recall the example that we saw in the last lectures about Angara 1.2 and let us say that is the rocket that we want to redesign so that we get a payload fraction of 0.025 which means I want to use that rocket to achieve a higher payload fraction. So, my π_* has been fixed at 0.025.

And let me see what should be the optimal staging and what will be the corresponding optimal velocity which I am going to get. For the first stage the I_{sp} is given as 310 and the structural ratio is 0.072. For the second stage the I_{sp} is 342.5 and the structural ratio is 0.089. Let us now try to determine a new stage wise payload ratios and the corresponding ideal optimal velocity.

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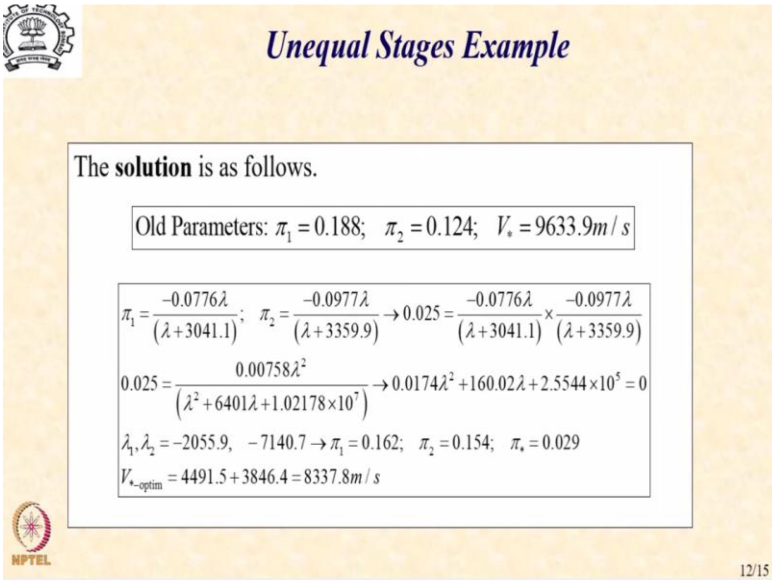

So, the old parameters if you remember you can go and check the π_1 was 0.188 the π_2 was 0.124 and corresponding to these two the V_* was 9,633.9 m/s. This was the solution that we had obtained when we were looking at the mass configuration. So, basically, we are having overall payload fraction which is not very large. Now, let us formulate this problem in the context of the solution that we have obtained.

So, let me just go ahead and substitute the value of ε_1 and $I_{sp_1} \times \pi_1$ expression and similarly ε_2 and $I_{sp_2} \times \pi_2$ expression and then I say that π_1 and then I say that π_1 and π_2 which is π_* must be equal to 0.025 that is the constraint, so this is our constraint relation. This results in with some amount of algebraic manipulation. A quadratic equation in λ for a two stage whose solution actually results in two values of λ_1 and λ_2.

One is – 2055.9 other one is – 0.7140.7 we will pick one of those. In fact, I will leave you to verify which one we should pick because I will give you a hint that the other value will be an invalid value. It will give you an inconsistency in your solution which you should independently verify. So, I am not saying which one of these I have used, but using one of those I get two solutions π_1 and π_2 as 0.162 and 0.154.

And I get π_* as 0.029. Let me make a comment here we had started with the specification of a payload fraction of 0.025, but we have ended up with a value of 0.029. Kindly note that this is essentially because of the truncation errors which are part of the solution process that we do not use all the decimal places and particular when there are large numbers where manipulations results in smaller numbers.

And if we ignore the higher digits, it is possible that we will result in little bit of error. You can actually verify this by doing a more accurate calculation and show that your pi star will be close to 0.025 which is the constraint that we have put and for these values of π_1 and π_2 you get V_* as 8,337 m/s and now we make a comparison. Originally where π_* was smaller.

But the V_* was 9,600, but now because you want a higher π_* where V_* reduces to a smaller value.

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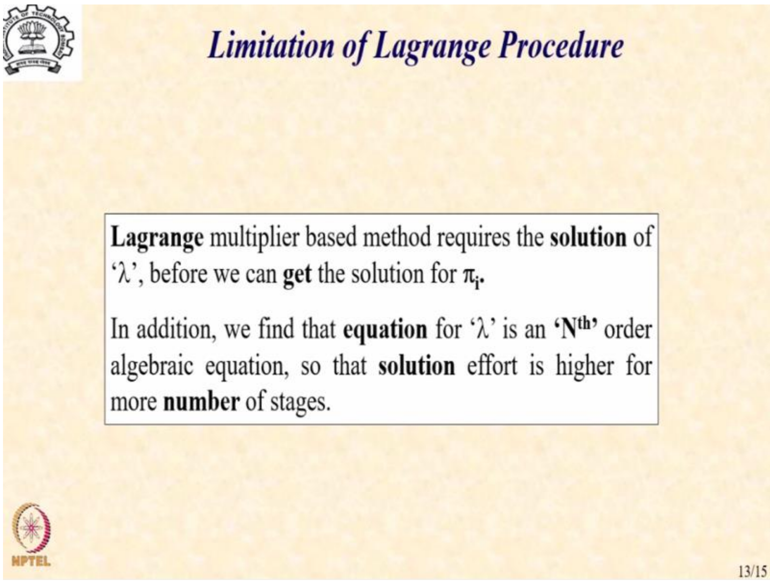

Let us now look at whether this particular technique has an issue. It is a good technique we have already seen, but there are certain drawbacks that we must take note of. So, the first thing that it is seen is that we first need to get a solution for λ before we get a solution for π_i at least for the unequal stages. For equal stages we are in a position to eliminate λ so that it is a simpler solution.

But more often than not we are not going to get equal stage configuration. So, obviously it is going to require lot more computational effort. And then of course your λ is an N^{th} order algebraic equation. So, there are two issues involved with it. As you increase the number of stages to 3, 4, 5 the order of algebraic equation is going to replace. So, you are going to get that many roots for λ.

And then you will have to pick the one which is going to give the feasible solution so that is going to be an additional effort to pick among the λ the value which will give you the correct and this can only be done by actually checking for all the λ values. This can become a tedious exercise if all the λs are real numbers. If in some cases, some of the λs appears as complex conjugate.

They can straightaway discarded because λ has to be a real number that is the original interpretation with which this whole formulation has been done. So, it cannot be complex, but it can be a real number. So, if all the 5 roots for a 5th stage rocket is a real then you will have to check for all those λs before discarding saying which one of them is consistent solution and remaining are inconsistent solution. So, it becomes lot more computationally intensive.

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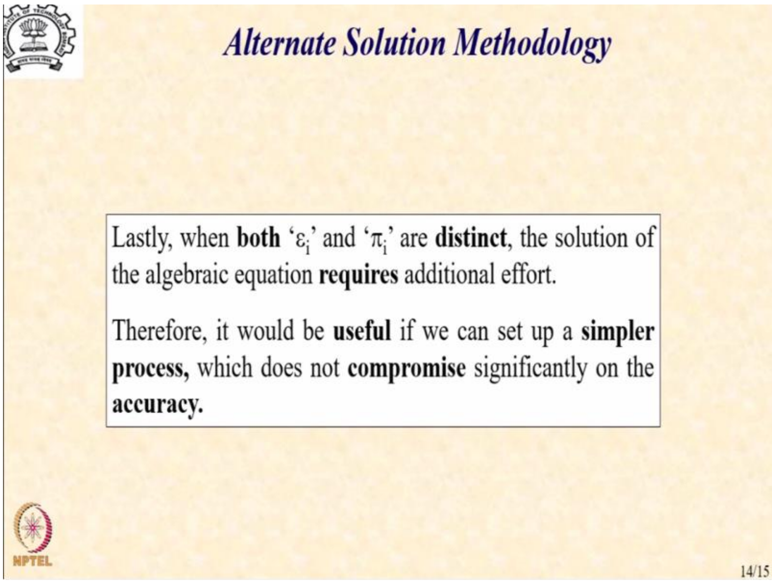

So, is there an alternate way in which we can do this? The alternate way should be such that it simplifies the process of solution as compared to the procedure that we have used here, but should not compromise significantly on accuracy which means in some initial design stages we maybe in a position to sacrifice a bit of accuracy for computational comfort and simplicity of the solution process.

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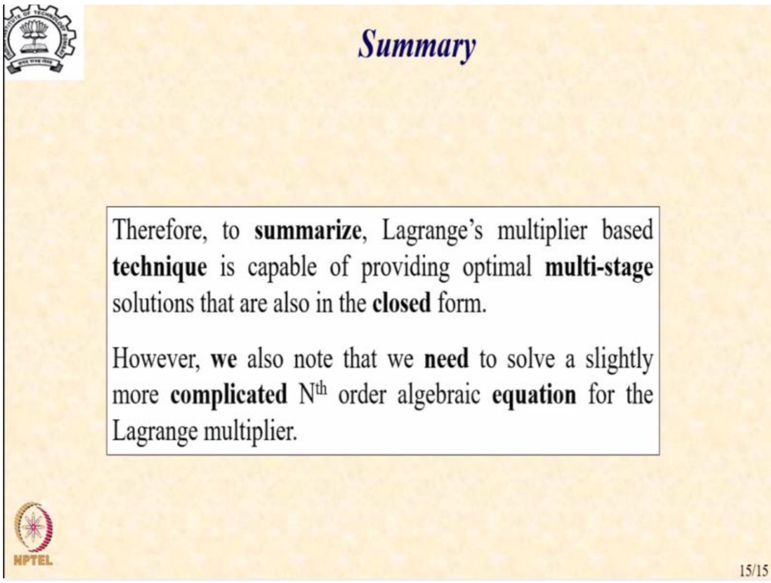

So, to summarize Lagrange Multiplier based technique is capable of providing optimum multistage solutions that are also in the closed form that is a great benefit, but we also note that we need to solve a slightly more complicated N^{th} order algebraic equation for the Lagrange Multiplier. So, we have seen in this lecture the mechanization of the basic procedure proposed by the Lagrange for extracting optimal solutions of a constraint optimization problem.

And we note that it becomes extremely simple in the context of equal stages and we have also seen that in the context of unequal stages the numerical effort is going to increase almost exponentially as number of stages are increased and that there is a need to look at an alternate methodology that will simplify the process without losing the accuracy which is what we will look at in the next lecture. So, bye see you in the next lecture and thank you.