Introduction to Launch Vehicle Analysis and Design

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Lecture – 13
Constant T/m Solution

Hello and welcome. In this lecture, we will look at the gravity turn trajectory solution under the constraint of specific thrust or constant T / m. So, let us begin.

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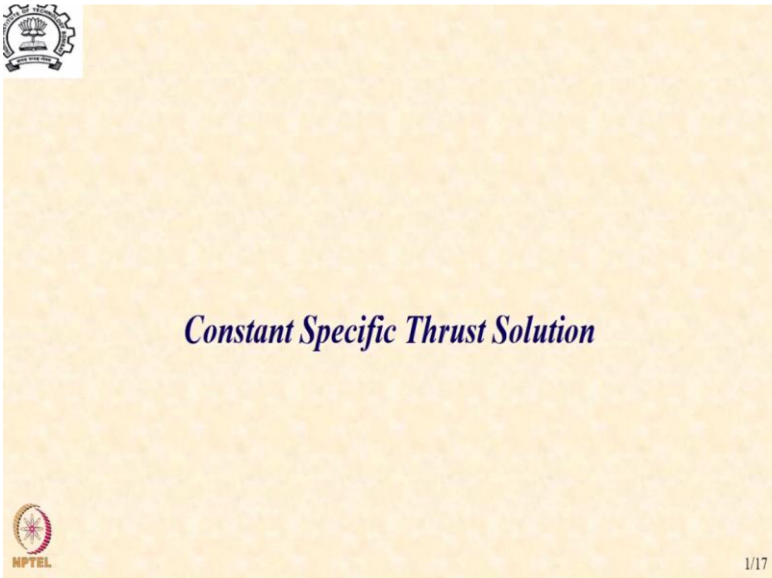

So, constant specific thrust solution is an important trajectory design strategy that provides important practical benefits as we will see now.

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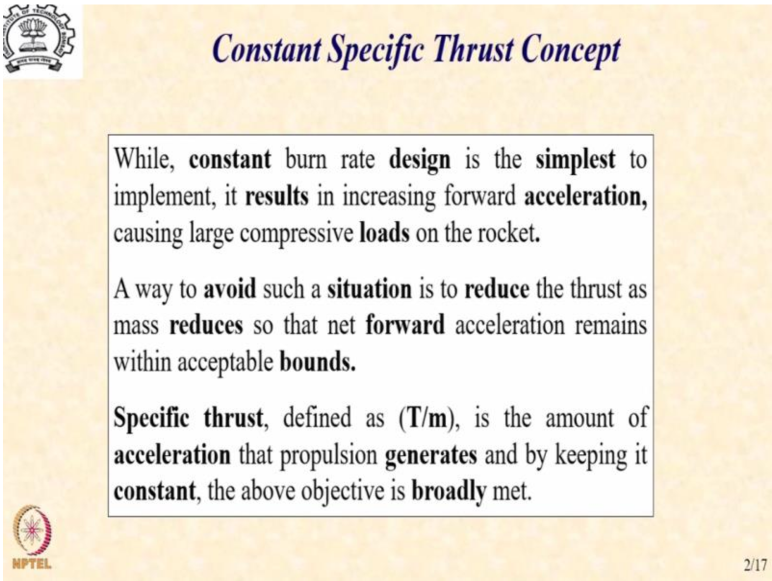

So, to understand that let us first recall the constant burn rate design that we have seen in our rectilinear trajectory solutions that we have obtained earlier. We know the constant burn rate design is a simplest to implement, but there is a drawback that we have not noted earlier, but now it is important to bring it to the front that a constant burn results in constant thrust. But as you know the mass is continuously depleting.

So, as the vehicle becomes lighter and lighter the same thrust generates a larger and larger forward acceleration. Now, because the thrust is constant if we look at force equilibrium that is acting on the rocket. The same force is acting all the time even though its mass is reducing continuously, it is becoming lighter, but the force which is acting on it remains the same. The impact of this is that we continue to get large compressive force on the rocket at all points on the trajectory.

One way to avoid this problem is reduce the thrust as mass reduces so that net forward acceleration remains within acceptable bounds. There is another aspect which is useful from practical perspective that a large acceleration would also result in much larger velocity increases which you may want to limit, you may not want to achieve a very high velocity at a particular point on the trajectory.

So, from that perspective as well there is a need to reduce the thrust as the mass reduces and this is the basic philosophy of constant specific thrust based trajectory in which the specific thrust which is defined as $\frac{T}{m}$ and is the amount of acceleration that propulsion generates is kept constant and thereby the two objectives of keeping the compressive loads for higher stages at a lower magnitude and keeping the velocity increments within bounds are broadly met.

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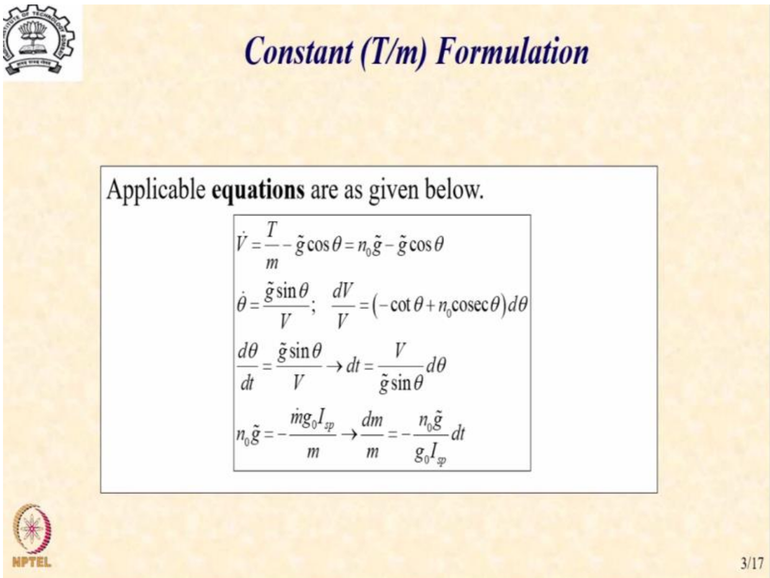

Let us now look at the applicable equations. So, now when we take the $\dot{V}$ equation we have the thrust term for $\frac{\dot{m}g_0 I_{sp}}{m}$, but that is effectively $\frac{T}{m}$ term. So, we replace that with $\frac{T}{m}$ which is going to be a constant. So, that constant is represented as $n_0\tilde{g}$ where n_0 is a real number that is indicative of the number of g or the amount of g that the propulsion is generating.

Of course, the $\dot{\theta}$ equilibrium remains the same, but now we realize that my $\frac{T}{m}$ equation gets split into two parts. The first part gets into the V_0 equation as $n_0 g - \tilde{g}\cos\theta$ and the second part of the same equation is $n_0 g = -\frac{\dot{m}g_0 I_{sp}}{m}$. And that gives directly a differential equation for mass in terms of n_0 and time and now you broke an interesting feature of this particular formulation.

That the mass is now an explicit function of time as against the earlier two solutions where mass was always an implicit function of theta which means now we have a direct control over the burn profile or the solution that we are going to get from this differential equation is going to be the variation of mass as a function of time or we can directly control the $\frac{dm}{dt}$ or $\dot{m}$.

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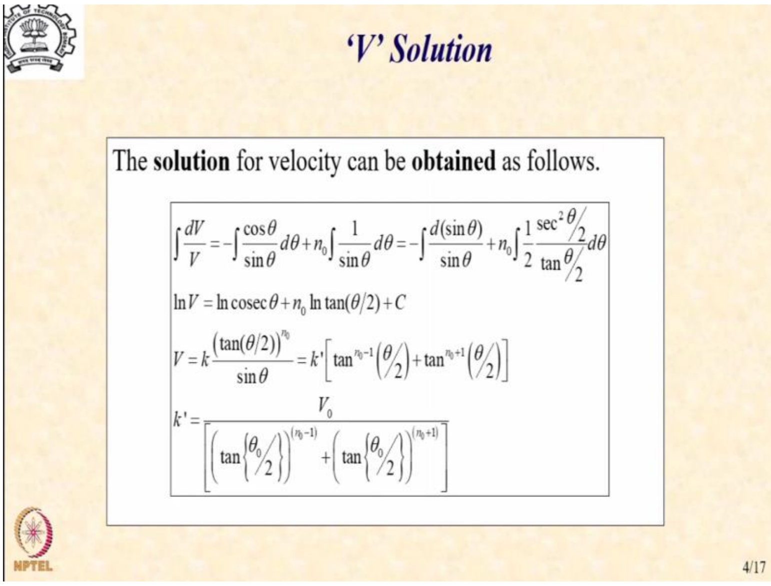

Let us now proceed with the solution of these four differential equations. So, the solution for velocity is obtained from our $\dot{V}$ equation the $\dot{V}$ as $n_0 g - g \cos\theta$. We do a little bit of calculus-based jugglery. So, we take the ratio of two differential equations which explicitly removes the n variable and we get $\frac{dV}{V}$ on the left-hand side and we then can perform the integrations which are not that straight forward.

But again, of the trigonometric form which are commonly tabulated in any books or calculus. So, you can refer to those books from which you will be able to perform these integrals and we now get a solution for velocity in terms of tan and n_0 which is your real number of g's that the propulsion is generating. So, the velocity now is a function of the angle θ and n_0. Of course, by submitting the initial condition we can evaluate the constant of integration k'.

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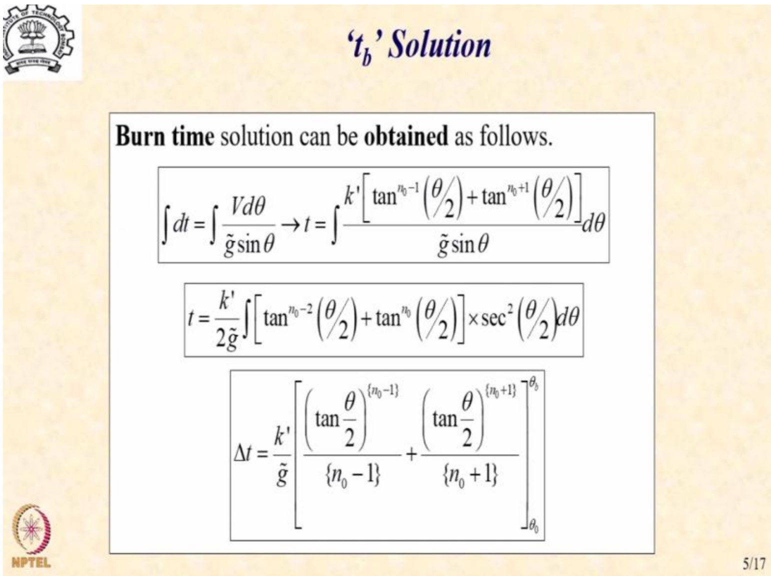

Let us now move over to the time solution. The time solution is the integral of $\frac{V d\theta}{\tilde{g}\sin\theta}$ and now you realize that V itself is a slightly complex function of θ. So, obviously this integral becomes a little bit more complicated than the integral for the velocity. However, we can still take recourse to trigonometric substitutions which are commonly done in the process of performing trigonometric integrals.

And are part of standard text books on integral calculus. We will find that the above integrals reduces to the form of the $\left[\tan^{n_0-2}\left(\frac{\theta}{2}\right) + \tan^{n_0}\left(\frac{\theta}{2}\right)\right] \times \sec^2\left(\frac{\theta}{2}\right) d\theta$ and this integral again can be performed through substitutions as per the standard integral table and now we get a solution for time in terms of θ which is now in transcendental relation.

The meaning of transcendental relation is that now it is in terms of trigonometric functions and it is an implicit relation. Of course, if θ_b and θ_0 are specified along with n_0 then this is just an evaluation for Δt which means that for specified $n_0\theta_0$ and θ_b we can find out what is the time that is going to be taken to perform its manoeuvre. However, if we want to find out what should be θ_b or θ_0 for a specified time during which the manoeuvre is to be performed.

We will need to solve this transcendental equation and iterative fashion. So, it is going to become numerically lot more intensive.

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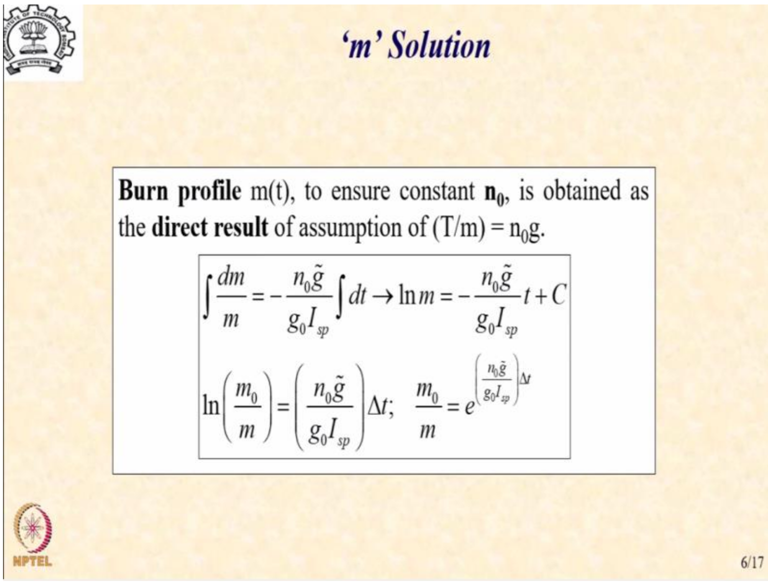

Let us now go to the burn profile solution from the 4th equation that we generated and which is a direct result of the assumption of $\frac{T}{m} = n_0 g$. So, this integral is not very difficult to perform and we get the mass fraction as an exponential function of n_0 and t and we now see that we have another root to evaluate Δt if we specify an $\frac{m_0}{m}$ and an n_0 then we can calculate Δt.

Conversely for a specified Δt and a mass fraction we can calculate n_0 which is going to satisfy this constraint relation which means we can use this as a design equation where we may design for a specific forward acceleration by specifying the time to be taken for the trajectory and the mass fraction available.

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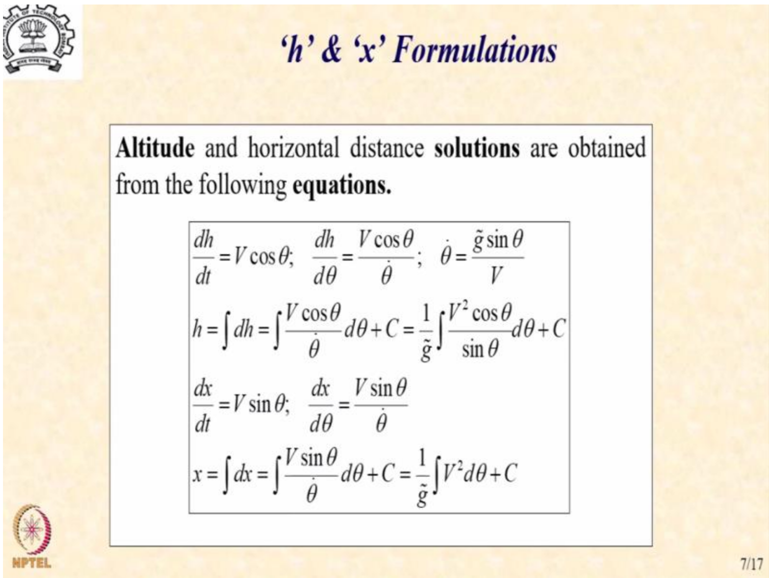

We can now go over to the altitude and the x solutions using the same two equations that is $\frac{dh}{dt}$ as $V\cos\theta$ and $\frac{dx}{dt}$ as $V\sin\theta$. So, we realize that the altitude integral now involves V^2 and similarly the horizontal distance integral also involves V^2. Obviously, these are going to become a little bit more complex than the time integral itself. So, while I have not given the solution here.

I will mention here that these integrals are possible to be performed using the same strategy that we have used earlier of trigonometric substitution of tan and sec functions. So, my suggestion to all of you that please perform these integrals and obtain the expressions for h and x just to understand the solution effort involved in generating the solution for h and x in comparison to the two other solution technique that is constant q_0 and a constant V.

So that it will give you a fair idea of the amount of effort involved in generating the solution for the case of constant $\frac{T}{m}$.

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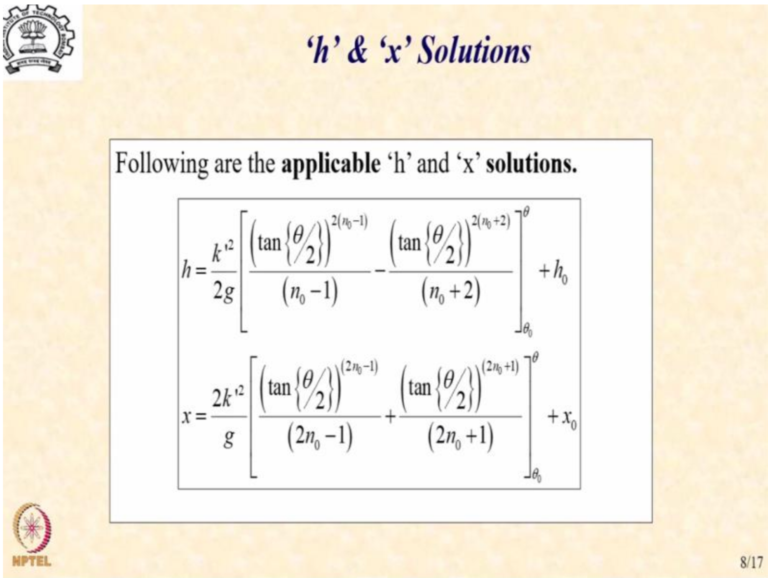

Of course, I have given you these solutions in the next slide, but my suggestion would be that please go through this exercise for your own satisfaction to understand the actual effort involved in arriving at these two expressions and that will also help you verify these two expressions. We can see that these expressions are as complicated as the expression for the time solution and the velocity.

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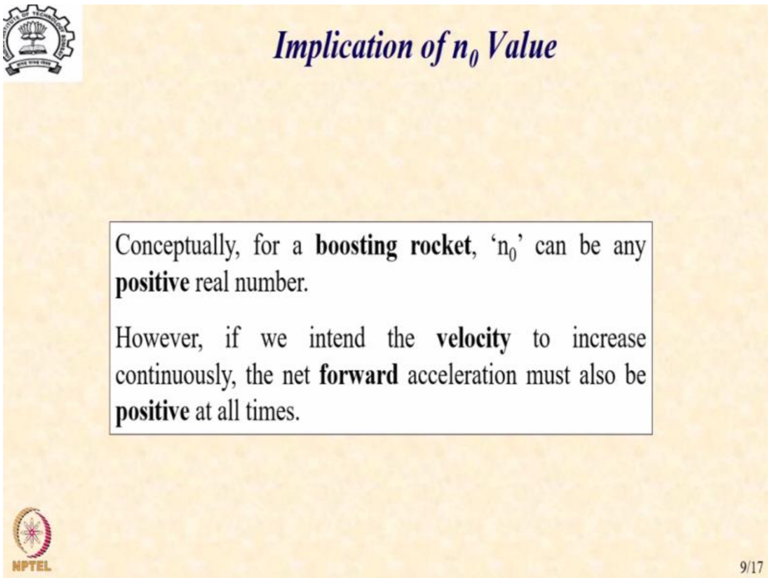

And we now come to a stage where we realize that n_0 is the primary driver for all the four relation that is the velocity the time or θ, the h and the x expressions. Now, of course at this point the equations have been obtained without any specific consideration to what n_0 is going to be. We are just saying that it can be any positive real number. In fact, you will note that may be a negative number also might be possible from a mathematical perspective that the solution obtained may also be acceptable for a negative value of n_0.

But there are certain practical constraints that we now put into the solution. The first thing we say is that we would like the velocity to increase continuously and not decrease. Of course, there could be very specific situations where you may want to use this to also decrease the velocity, but by and large you will not waste energy for decreasing the velocity. So, if we accept this primary idea that velocity is going to increase continuously.

Then the net forward acceleration must also be positive all the time. So, this is the fundamental requirement that n_0 should be said such that the net forward acceleration which is difference of n_0 and $\cos\theta$ should be positive all the time.

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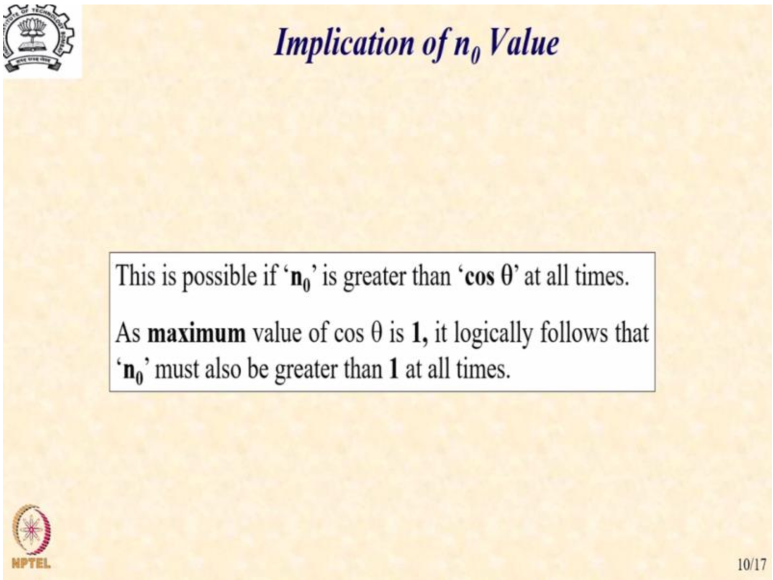

And that gives us this basic condition that in order for you to have a continuous increase in the velocity n_0 must be greater than $\cos\theta$ all the time. But here you need to note an interesting feature that as θ continues to increase from 0° to 90° $\cos\theta$ is going to decrease continuously from 1 to 0 which obviously means that as you move along the trajectory and your θ keeps on increasing the same condition will be satisfied with the lower value of n_0.

So, you can actually also talk about an n_0 profile even though the present solution has strictly not admitted that possibility we are assuming n_0 to be a constant, but in a more generic sense you would realize that it is possible for us to consider this fact that n_0 could also be varying as a function of time or a function of θ. Of course, this point we have already mentioned that if we keep n_0 greater than 1 at all times we will always be ensuring the constraint that n_0 is greater than $\cos\theta$ including the starting point.

And now we realize that n_0 being greater than 1 means that thrust must always be greater than instantaneous weight. So, this is going to be the constraint that we will have to implement in order for us to have a positive forward velocity increment at all times.

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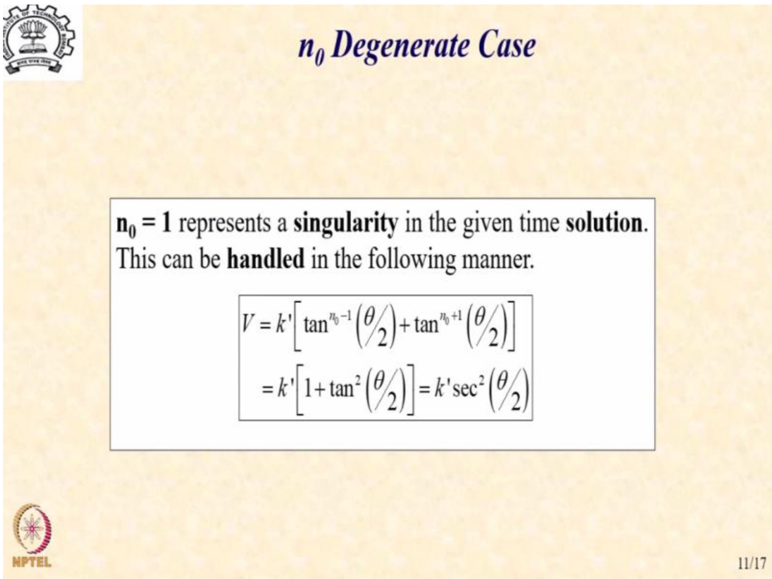

Now, there is a degenerate case that we need to pay attention to. If you go back and look at the expressions for all the four quantities that is velocity, the Δt, h and x; $n_0 = 1$ represents a very, very special case and it also leads to singularity in some of the solutions which means for $n_0 = 1$ the denominator become 0 so that the solution becomes unbounded. Obviously, that is not acceptable from the physical perspective so we need to kind of resolve this singularity which is done in the following manner.

So, in the case of velocity when we implement $n_0 = 1$ the first term becomes 1 or all values of theta and the whole expression reduces to $1 + \tan^2\left(\frac{\theta}{2}\right)$ which is $\sec^2\left(\frac{\theta}{2}\right)$. So, your velocity expression degenerates to $k'\sec^2\left(\frac{\theta}{2}\right)$. So, this is the expression that you will use in case of $(n_0 = 1)$ for velocity.

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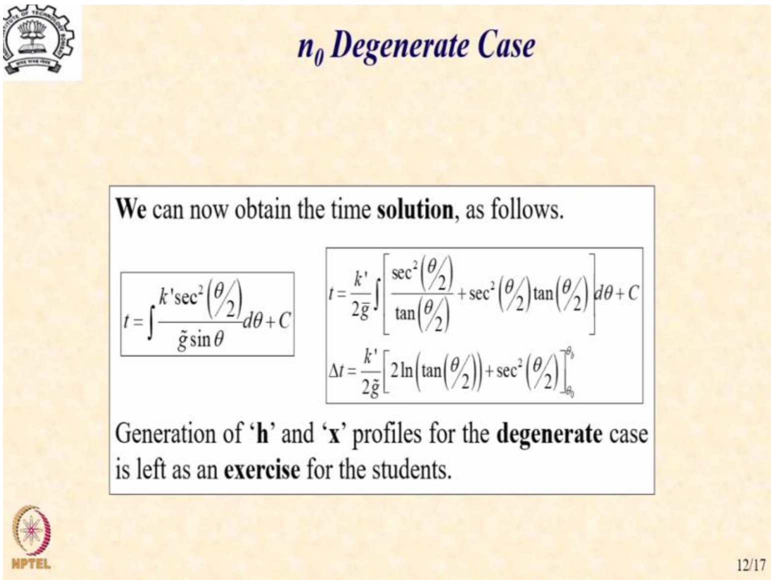

Now, we can obtain the time solution using this velocity expression. So, instead of trying to use the original expression and taking its limit as $n \to 1$ we now use this is as the velocity expression and perform the integration in a fresh. Now this is the integral that is $\frac{\sec^2\left(\frac{\theta}{2}\right)}{\tilde{g}\sin\theta}d\theta$ is the integral that we need to perform for time. And I will leave you to use all the trigonometry substitution possibilities.

And arrive at the solution as I have given here which is nothing, but $\left[2\ln\left(\tan\left(\frac{\theta}{2}\right)\right)+\sec^2\left(\frac{\theta}{2}\right)\right]$ is the Δt solution. Same thing can be done for h and x profiles that you can use this modified velocity expression now to integrate for altitude and horizontal distance. I will leave this exercise to you all to perform and arrive at those expressions. The point which now I would like to mention is that if you take the original expression with n_0 and try to take the limit of $n_0 \to 1$.

There is a methodology in calculus which you might be familiar with your hospital rules where the similar functions their limits can be obtained by applying a particular procedure. My suggestion would be that you can also take that route on those expressions. Take the limit as $n_0 \to 1$ and see if the limit reduces to the expression that we have obtained for Δt, h and x in the present case.

That would tell you the different ways in which we can arrive at the solution for the degenerate case of $n_0 = 1$. It is only degenerate case from a mathematical perspective, but from a physical perspective n_0 should always be greater than 1 which will ensure that we have positive velocity increases.

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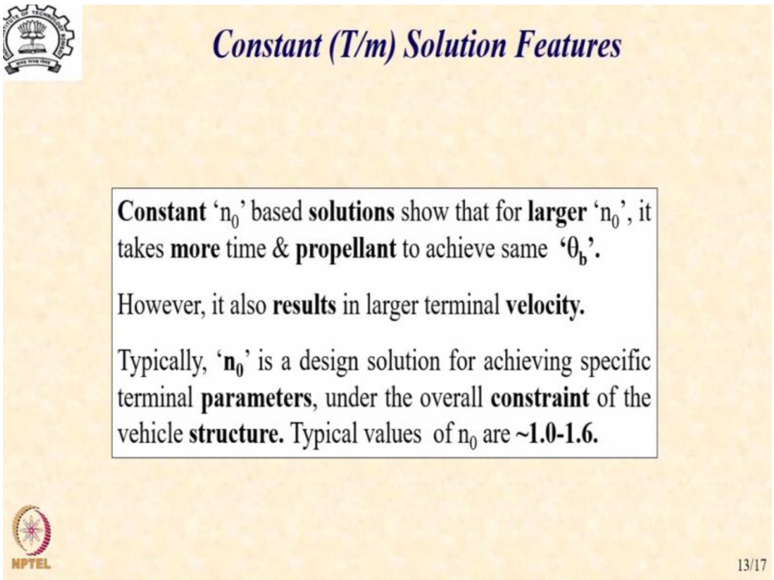

So, now let us look at some of the features of the solution which we have obtained. So, we find that a larger n_0 will require more time and more propellant to achieve the same burnout inclination. So, which means if you are interested in generating larger velocity you will use a larger n_0, but then it will also require a larger propellant and will last longer. So, if your desired requirements are larger terminal velocity you will go for a higher n_0.

Typically, n_0 is a design solution which is derived from specific terminal parameters and this is under the overall constraint of the vehicle structure. Typical values which are commonly employed in the context of constant specific thrust solution are between 1 and 1. 6 rarely you will go beyond 1.6 to 1.7 when you want to perform this particular trajectory manoeuvre more often than not, they will be closer to about 1.1, 1.2 and they provide good solutions for trajectory in the ascent mission.

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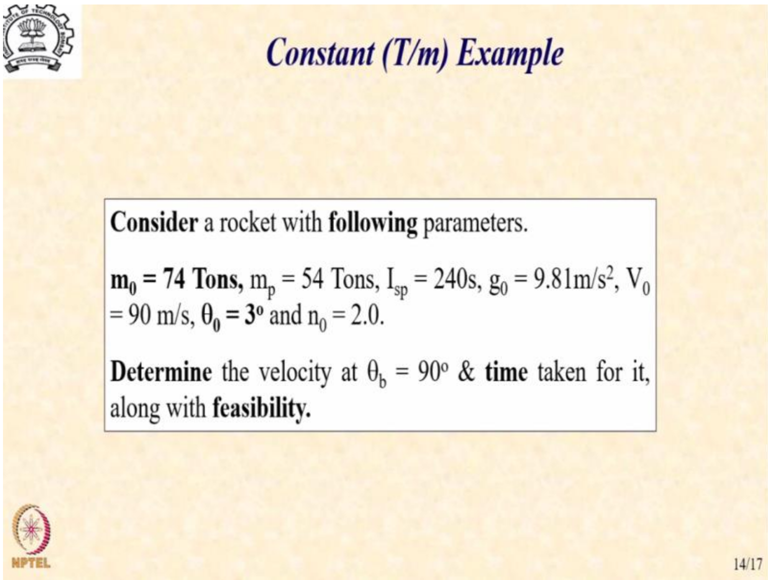

Let us now understand the features of this solutions that we have obtained through an example that we have been considering. So, let us consider the following parameters for a rocket having 74 tons of starting mass, 54 tons of propellant and the initial velocity at 19 m/s and initial inclination of 3D view. And as we have no better information let us put n_0 as 2.

We know that this is outside the bound of the range that we have given of 1 to 1.6, but let us try and experiment with this number just to find out what happens if we give such a value. And see why the constraint on the n_0 value is being implemented explicitly. Let us try and determine the velocity when the vehicle becomes parallel to local horizon. Let us try and find out the time taken along with a possible feasibility of performing such a mission because we are not sure that this mission is going to be feasible it might be.

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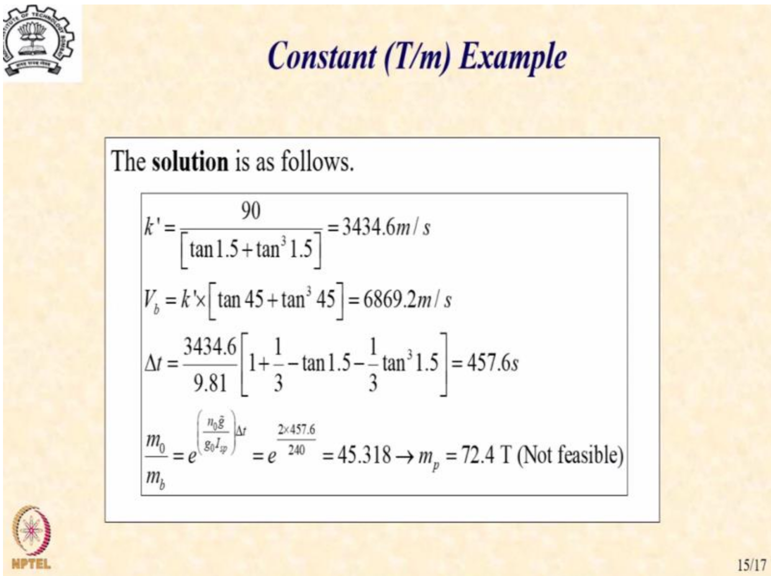

So, the solution is as follows. So, we start with calculation of k' which turns up to be 3434.6 based on that we calculate V_b. So, that turns out to be $6{,}869\ \mathrm{m/s}$ fairly high velocity starting from $90\ \mathrm{m}$. So, you are able to go from $90\ \mathrm{m/s}$ to almost $7{,}000\ \mathrm{m/s}$ using this option. The time taken is $457s$ because n_0 is a large value we have already set there it is going to take lot of time.

But when we come to the mass fraction, we hit a roadblock. The mass fraction solution in this particular case says that you must have had 72.4 tons of propellant to complete this mission. But you only have 54 tons which obviously means that this mission is not going to be feasible. Unless you now go back and modify the rocket and say that I am going to carry so much of propellant maybe it is going to be feasible.

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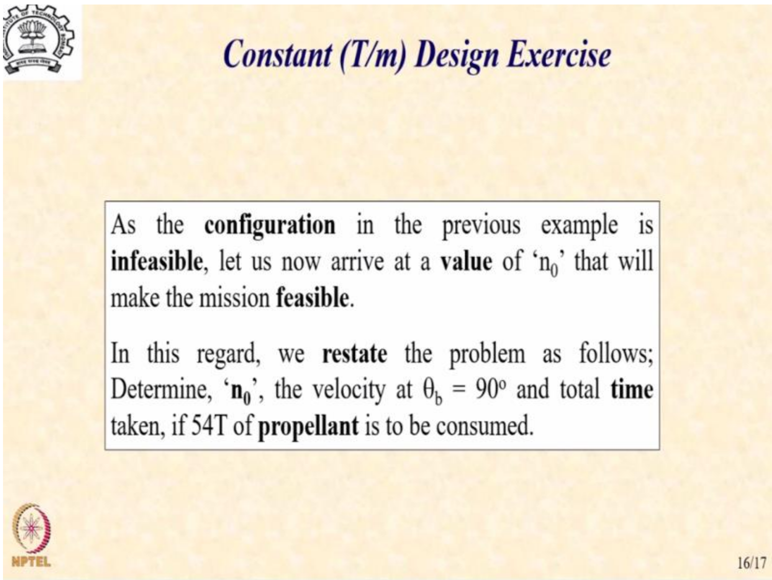

But let us now turn this problem around and see for what value of n_0 the mission becomes feasible. The reason why the problem is posed in this manner is that n_0 was an arbitrarily chosen figure. And more importantly it was taken to be significantly higher than the range specified. So, obviously it means that this is where the problem is. The solution is not feasible not because of any other issue.

But because n_0 is not a consistent parameter in the given mission. So, let me restate the problem as follows. Let us try and determine n_0 the velocity at this point and the total time if 54 tons of propellant is to be consumed. So, now I am saying that in place of n_0 I am specifying the amount of propellant that I want to burn and let me see what is the value of n_0 I can use which will make this happen.

Now, I am not going to do this exercise. My suggestion is you now set up this problem based on what we have presented in the previous example and try and arrive at the value of n_0 which is going to be consistent with the parameters of the problem. I will only mention two points. One you are going to require to solve a set of nonlinear algebraic equations and the second point you might have to resort to an iterative procedure to arrive at the correct solution for n_0.

I will give you only this much of hint. I suggest that you do this at some point I will be uploading the solution for this so that you can verify whether you are thinking along the right lines

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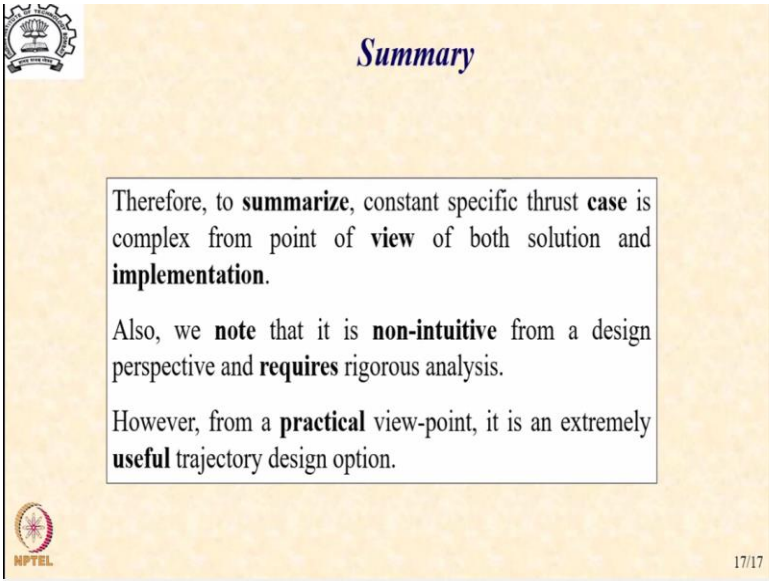

So, to summarize the constant specific thrust case is complex from the point of view of both solution and implementation. It is not only not very easy to solve, but also not going to be very easy to implement. Also, we note that it is non intuitive from a design perspective what it means is that it is not straight forward to interpret what is likely to be the impact of change in n_0 on the trajectory behavior because the mathematical relations involved are complex trigonometric functions and they cannot be directly interpreted.

So, obviously you are going to require rigorous analysis. However, from a practical perspective it is going to be an extremely useful trajectory design option because it is going to give you a handle on managing your structural mass which if you remember was one of the benefits that was mentioned when you were talking about the various options for gravity turn trajectory.

So, we said that we are going to be able to manage the structural mass better when we control your n_0 because it is a compressive force is maintained you can maintain your structural mass which will support the compressive weight.