Introduction to Launch Vehicle Analysis and Design
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Lecture – 11
Constant Pitch Rate Solution

Hello and welcome. In the last lecture, we had established the basis for generating a curvilinear trajectory through consideration of a two-dimensional motion model. And we had also examined the set of equations for the possibility of extracting the closed form solutions. And we had mentioned that there are three possible solutions that we can obtain which also have practical utility.

So, we are now going to look at these solutions one by one. The first one that we are going to consider in this lecture is a solution that we obtain through the assumption of pitch rate being constant. So, let us begin.

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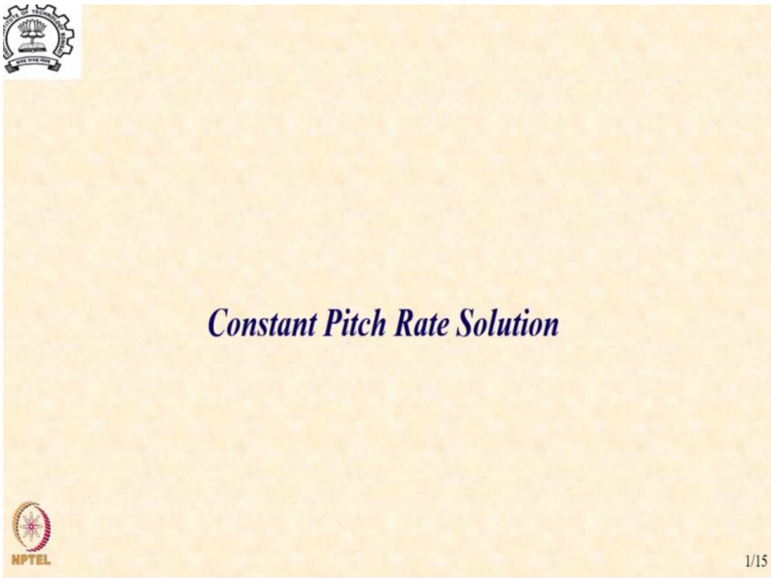

So, let us see what are the features of such a solution and how we can obtain it?

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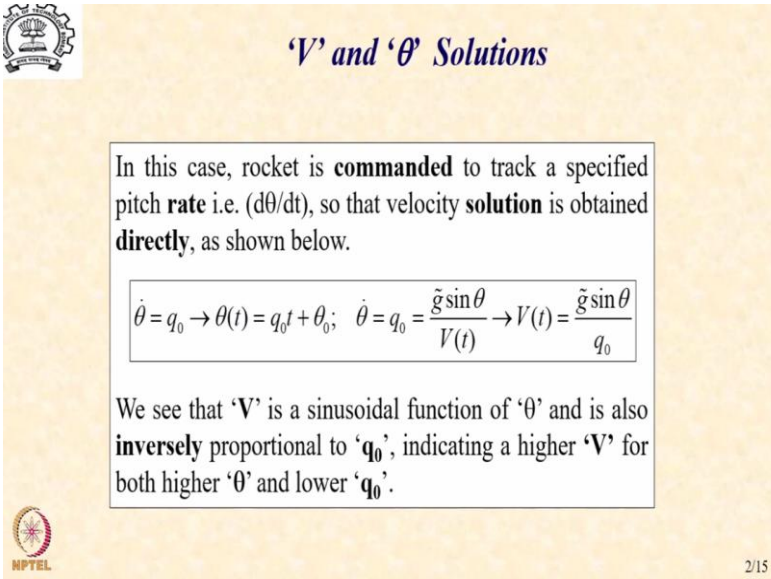

In this case that is the case in which we assume the pitch rate to remain constant throughout the trajectory. The rocket is commanded to track a specified pitch-rate $\frac{d\theta}{dt}$ so that velocity solution is obtained directly as shown below. We know that $\dot{\theta}$ which is a pitch rate is a constant. So, we give a symbol q_0 that directly gives us θ solution as $q_0 t + \theta_0$ where θ_0 is the angle at the initial time.

Then we go to the second equilibrium equation that is $\dot{\theta} = \tilde{g}\frac{sin\theta}{V(t)}$ and we directly get the V solution as $g\frac{sin\theta}{q_0}$. So, we see that V is a sinusoidal function of θ and is also inversely proportional to q_0 indicating that we are going to get a higher velocity for a higher θ and for a lower q_0 value.

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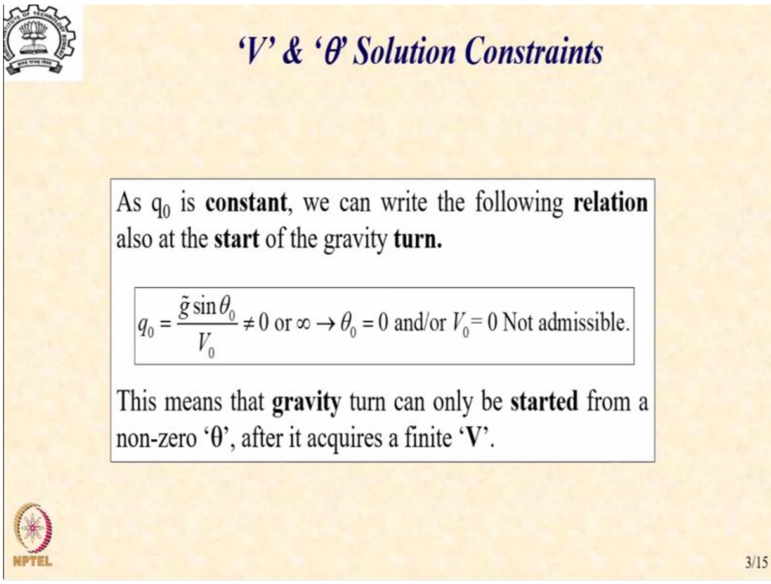

So, what are the features of this solution that we have obtained just now? So, we note that if q_0 is a constant throughout the trajectory including the starting point at $t = 0$. It has to valid at all the points. So, obviously the same relation is also valid at the initial point or the start of the gravity turn. So, we can write down the applicable relation for velocity at the start of the gravity turn as shown below.

And this relation is nothing, but $q_0 = g\frac{sin\theta_0}{V_0}$, where V_0 and θ_0 are the initial values of velocity and the pitch angle θ. Now, as q_0 is a finite quantity it automatically means that there are going to be certain restrictions that θ_0 and V_0 will have to satisfy. For example, if you start from $\theta_0 = 0°$ even though $V_0 \neq 0$ the q_0 will become 0 and if $q_0 = 0$ that is not an admissible solution.

Similarly, if we start from a finite θ_0, but if we make $V_0 = 0$, then it results in infinite q_0 which is also not an admissible solution. So, we come to the conclusion that both V_0 and θ_0 have to be nonzero values before we can start our gravity turn. Of course, the reason for this is not far to see because our gravity turn depends on the normal component of gravity to the velocity vector.

And for $\theta_0 = 0°$ the gravity vector does not have any component normal to the velocity vector for a vertical motion which obviously means that the gravity turn can also not be started when the vehicle is vertical and that it must have some finite inclination from vertical before the gravity turn can start. Moreover, we also note that not only gravity turn can be started only after it acquires a certain attitude with respect to vertical.

It also needs to acquire a finite velocity. And this is where we need to take a view of how such a thing is going to be achieved particularly because our lift off is basically vertical. The velocity vector is pointing the vertical direction even though once the rocket clears the launch tower it will have a finite velocity, but it does not have an inclination with respect to vertical you cannot execute a gravity turn manoeuvre as we have established.

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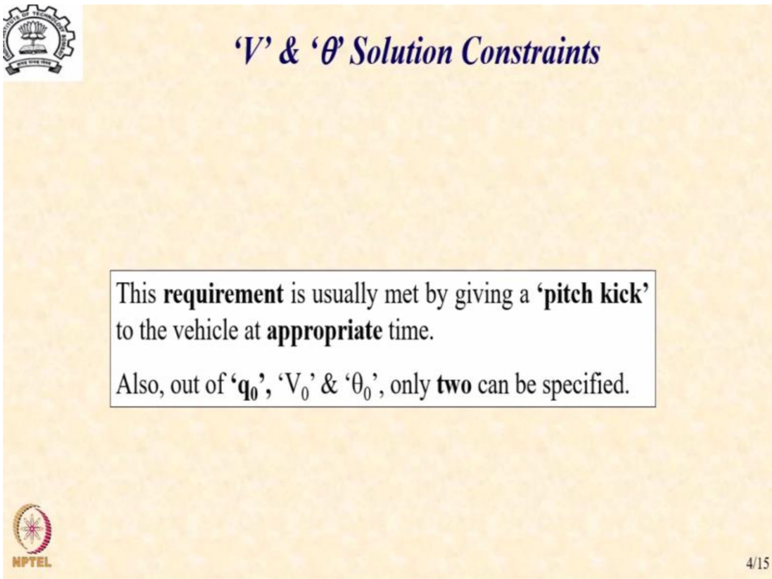

And this requirement is usually met by giving a pitch kick or executing what is called a pitch down manoeuvre on the vehicle at an appropriate time. The implication of this pitch kick or a pitch down manoeuvre is to give a lateral impulse which generates a pitching moment instantaneously and this generates a pitch disturbance that starts a pitching motion and because of the pitching motion it acquires a finite θ.

It is given in the form of an impulse actually so that we acquire a constant pitch at the end of the manoeuvre which becomes our initial pitch angle at that velocity V_0 which is what we treat as our initial time t_0. We also need to note from the relation that we have seen for $q_0\theta_0$ and V_0 that as this is a constant relation only two of these three can be actually specified.

And third one will be determined from the constraint. For example, if you specify q_0 and V_0 you can determine θ from that relation or if you specify V_0 and θ_0 you can determine q_0. So, this also has to be kept in mind by setting up the configuration for a gravity turn with the constant pitch rate manoeuvre as we have discussed just now.

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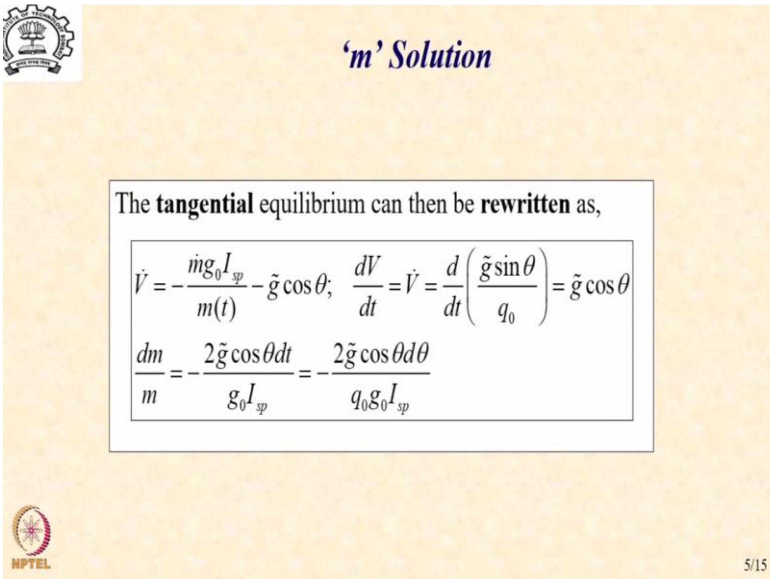

Now, let us go to the next equation in the context of the gravity turn that is the velocity equation, but we immediately realize that you already have the velocity solution. So, what is it that the velocity solution is going to give us? The velocity solution now is going to give us the mass profile because from the second equation we already have the velocity for a given pitch rate and we already have θ solution as a function of time.

So, the only unknown now left is the mass solution and the tangential equilibrium is going to be used to write a differential equation for mass as shown below. So, we are going to convert this into a differential equation for mass through basic calculus-based manipulations, change of variables. So, by doing that we can show that the tangential equilibrium differential equation can be rewritten as $\frac{dm}{m} = -\frac{2\tilde{g}cos\theta dt}{g_0 I_{sp}}$.

And what we do now is also the change of variable of dt which is converted to $d\theta$ through the basic expression of $\frac{d\theta}{dt}$ So, using $\dot{\theta}$ expression we change the variable from dt to $d\theta$. Now this differential equation if we integrate, we are going to get the solution of mass as a function of θ and not an explicit function of time, but we also realize that as θ is already an explicit function of time.

By substituting that solution into the mass relation, we can also generate an explicit solution for mass as a function of time. So, we are going to have the velocity and mass and θ all as a requisite function of time.

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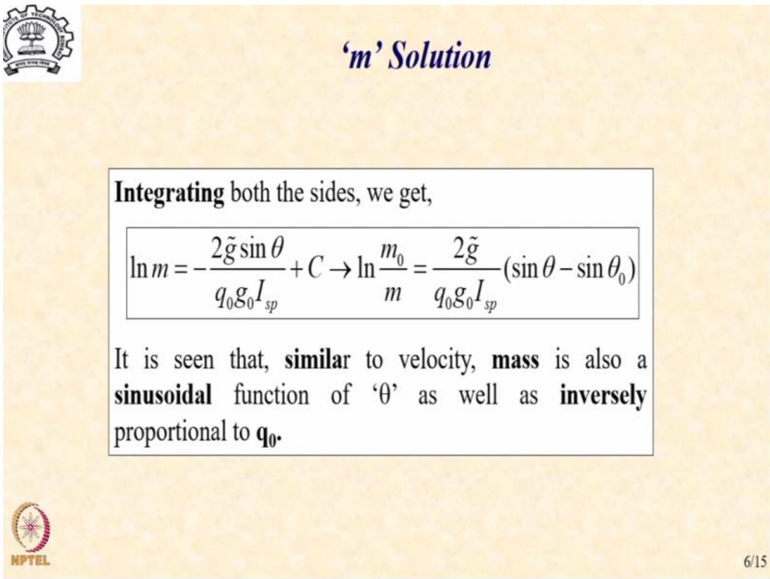

So, let us go through with the integration process and we find that this leads to the basic result that $\ln\frac{m_0}{m} = \frac{2\tilde{g}}{q_0 g_0\, I_{sp}}(\sin\theta - sin\theta_0)$. So, this is the solution for mass fraction that is executed under a constant pitch rate manoeuvre. So, we find that similar to velocity mass is also a sinusoidal function of angle θ and also inversely proportional to q_0.

So, if q_0 is small the mass fraction is going to be larger. Now, what is the implication of $\frac{m_0}{m}$ being larger? The implication is that $\frac{m}{m_0}$ will be smaller so that we will require a larger propellant to support a larger velocity which is going to automatically result for a smaller q_0 that we have already seen.

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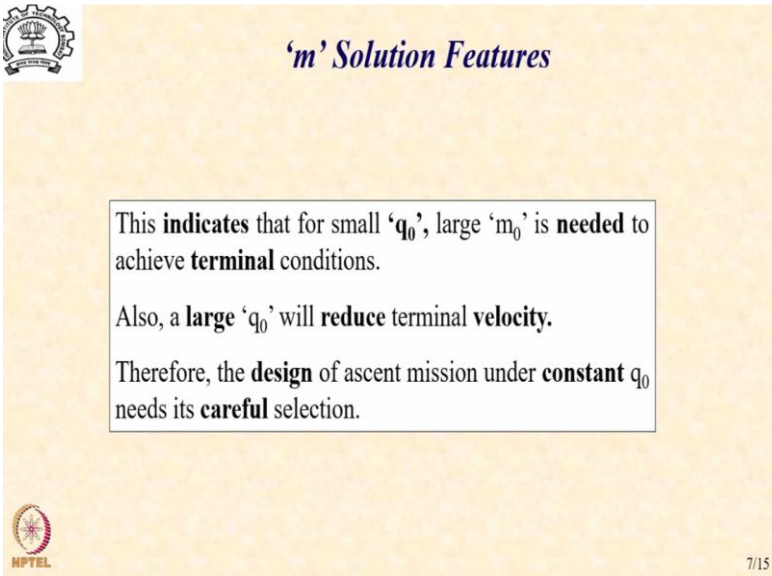

So, we also note the converse of it that if you use a large q_0 we are going to get a smaller terminal velocity. And if you realize an important fact that by appropriately choosing a q_0 value it is possible for us to design a trajectory that is going to have different terminal conditions and conversely it means that if you want to achieve different terminal conditions you just need to tweak your q_0 value which is going to be not very difficult as they should be just a part of the control algorithm where you may want to define a different reference for a constant pitch rate solution.

But we need to also realize that as the trajectory solution is extremely sensitive to this q_0 value you need to be very careful with the selection of this number as we will see through an example that we will conduct next.

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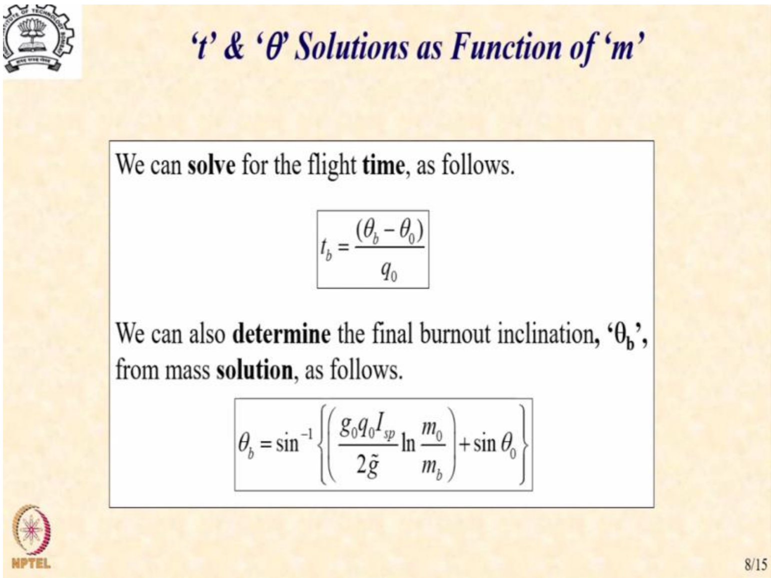

But let us now look at the solution for time and solution of time also as a function of mass. So, because now there is an interplay of variables, we can express different variables in terms of different quantities. So, if we are given the starting and the terminal condition along with q_0 we can directly get the solution for the flight time that is how long the mission will last. On the other hand, if we are given the mass and initial condition it is possible for us to determine the terminal inclination through the mass profile specification.

So, if we know what is the burnout mass and what is the lift off mass that is the mass fraction. If we know the initial angle and if we know the pitch rate, we can directly calculate a terminal angle through this trigonometric expression.

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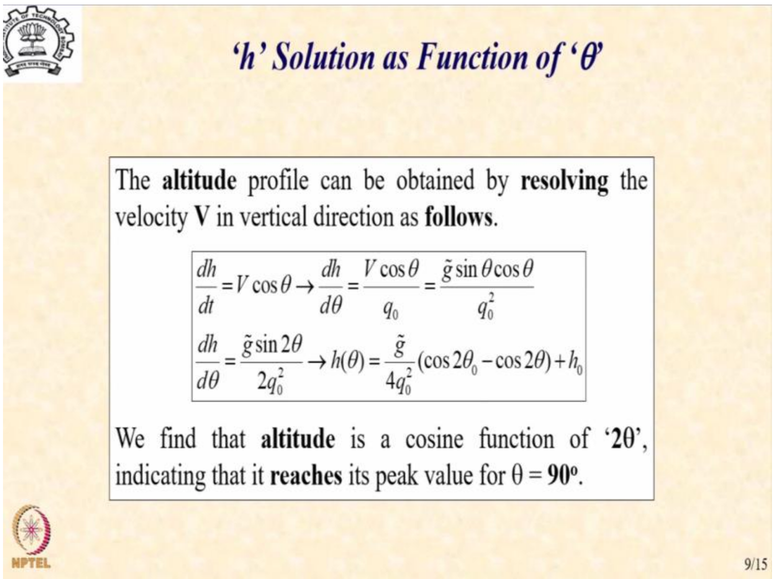

The last two solutions can respond to the trajectory parameters that is the profile in terms of the altitude and the horizontal distance. So, let us first take the altitude equation which is nothing our $\frac{dh}{dt} = V cos\theta$ that is if I take the vertical component of the velocity and integrate it, I should get my altitude. So, again we do the same trick of changing variable from t to θ.

And then we get a differential equation for h which is a simple differential equation of $\frac{\tilde{g}\sin 2\theta}{2q_0^2}$. If we integrate this, I give this for you to verify that integrated expression you get h as a function of θ and as you know θ is a function of time so we can also obtain h as a function of time. Now we find that h is a cosine function of 2θ.

And we know an interesting feature that for $\theta = 90°$ the altitude will reach its maximum value which is a result that we already know from our project time motion. This is something similar to the projectile motion that we have seen in the basic rigid body mechanics.

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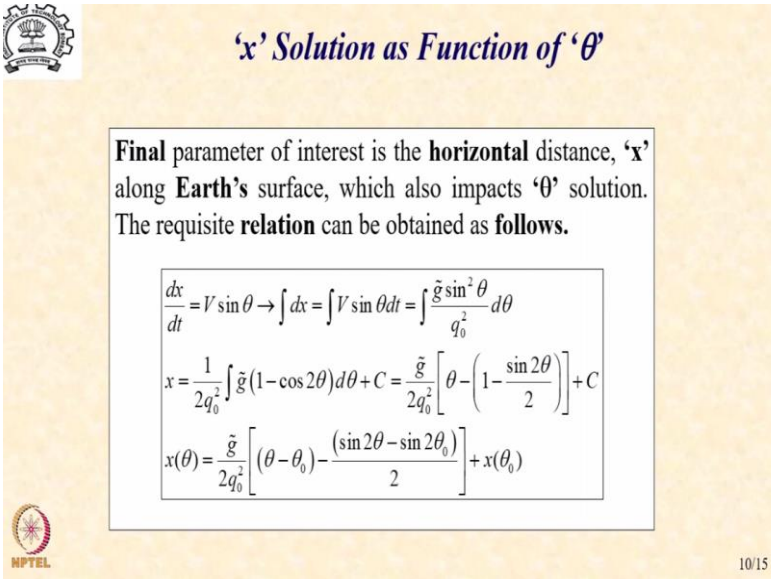

Let us now look at the horizontal distance solution as the solution of the differential equation $\frac{dx}{dt} = Vsin\theta$. So, again we do the same change of variable of dt into $d\theta$ and arrive at integrand that is $\frac{\tilde{g}\sin^2\theta}{q_0^2}$. This is a simple trigonometric integration and now we get a solution for x as containing two terms $(\theta - \theta_0)$ and $(\sin 2\theta - \sin 2\theta_0)$.

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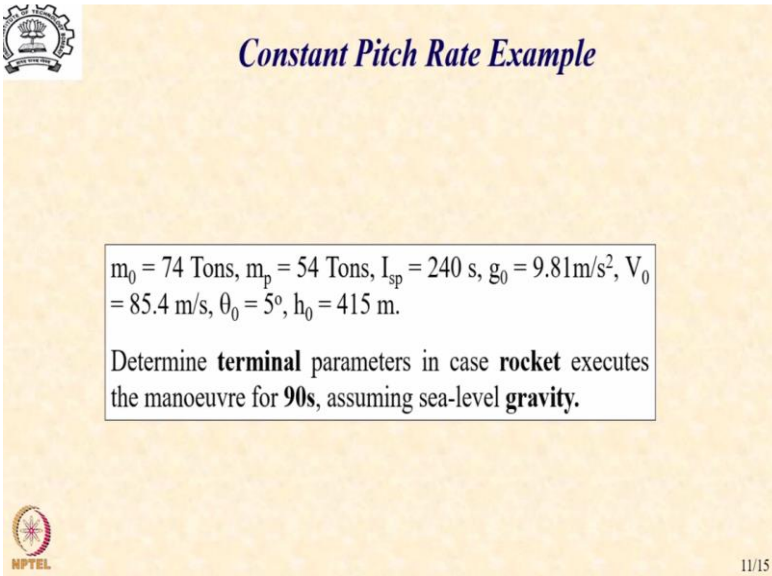

Let us now see all these relations through an example that we have been following from our first idealized solution case, but the fact remains now that we cannot really start our gravity turn with velocity and $\theta = 0°$. So, assume that the vertical lift off has already occurred for some time and at the end of that vertical lift off a small pitch kick has been given which has resulted in an inclination from vertical of about 5°.

And by this time the vehicle has achieved a velocity of 85.4 m/s. Of course, in the process it has consumed 6 tons of propellant and it has also acquired an altitude of about 415 m. Let us now assume that from this point forward we are going to conduct the gravity turn manoeuvre and let us try and find out what would be the terminal parameters if we conduct a constant pitch rate manoeuvre for the next 90s assuming the sea level gravity.

So, in this case I have specified the time, I have specified the initial angle and I have specified the velocity which are the only parameters that I can specify all other parameters now we can calculate using the expression. So, let us see how those parameters can be calculated and what their values are.

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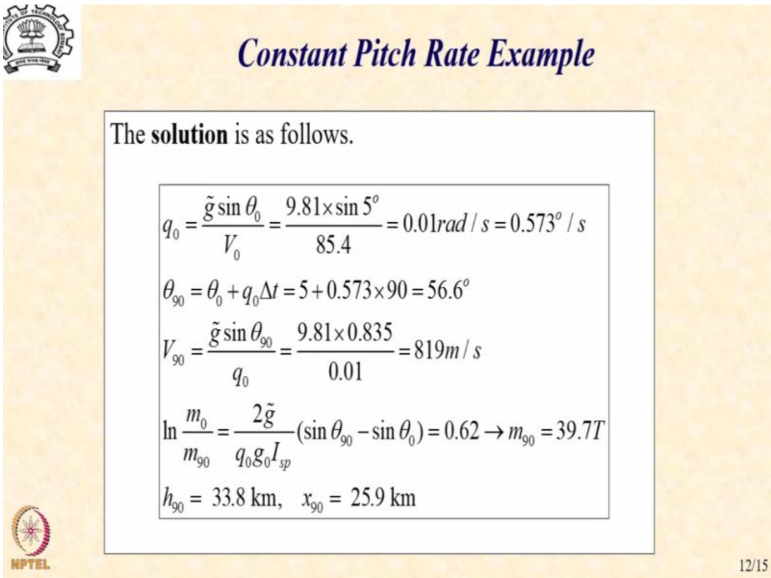

So, q_0 which is $\frac{\tilde{g}\sin\theta_0}{V_0}$ comes out to be 0.01 rad/s which can be converted to degrees per second as 0.573 deg/s. The moment I know the q_0 I already know θ_0. So, I know θ at 90s that is $\theta_0 + q_0\Delta t$ which is 90 and if I do this, I get an angle of 56.6°. So, this particular trajectory when executed will result in the inclination from vertical as 56.6° at the end of 90s.

We can also use the velocity expression that we have derived for obtaining the velocity at the end of 90s which uses the θ at $t = 90$s of 56.6° and gives a value of 890 m/s. So, that is the velocity at that point. Now with θ_{90} known and q_0 known, we can solve for the mass fraction $\frac{m_0}{m_{90}}$ which comes out to be 0.62 and through that we find that mass at 90s will be 39.7 tons.

So, please note we have started from 74 tons and at the end of 90s, 40 tons of mass is left. So, which means that we have approximately consumed 34 tons of propellant during this process which effectively means that we still about 20 tons of propellant left and during this time we have acquired an altitude of around 34 km and have travelled about 26 km over the surface of the earth which is not a very large distance. So, possibly our flat earth approximation is still valid.

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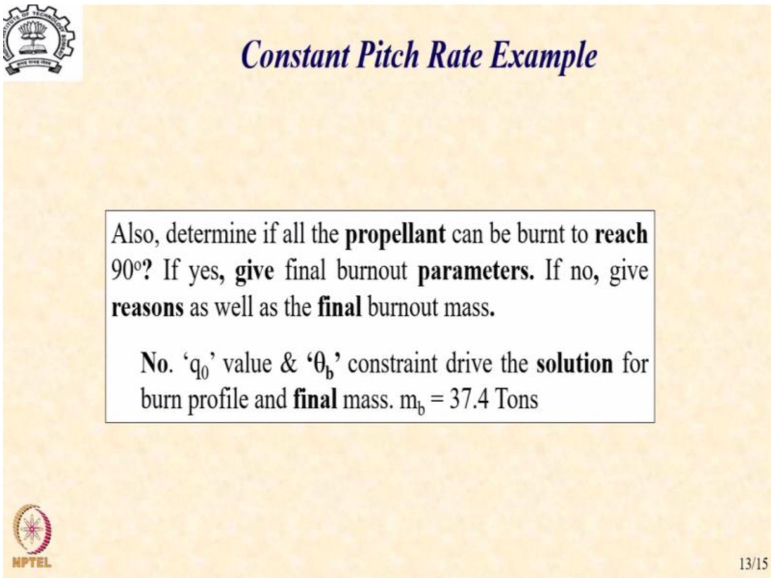

Now, let us explore this problem a little further. As I had mentioned there is still about 20 tons of propellant left and we have not reached 90° which in many cases would be a requirement from terminal condition. So, let us hypothetically say that we would like to reach 90° and we want to find out if by burning the remaining 20 tons propellant we can reach 90°.

Obviously, we know that the time will not be 90s, but it will take a little longer time. But we are not sure, we do not know whether it will happen or not. So, if it happens let us try and find out what are those burnout parameters which we are going to get, but if it does not happen then let us find out the reason as well as to say that whether it reaches 90° what would be the final burnout mass.

I will leave you to work with the formulae that we have derived now that we have seen in the previous example of formulae. I suggest that you become little bit familiar by practicing them. And you will find that it does not happen. In fact, you see a very, very surprising result that from 56.6° that is for 90s to 90° it just requires an additional 2 tons of propellant from 39.7

tons to 37.4 tons by burning additional little more than 2 tons of propellant it has covered an angle of practically 34°.

So, by the time it reaches 90° you still have about 17 tons of propellant left which obviously means that we will not be in a position to consume all the propellant by the time we reach 90°. So, this brings us to the conclusion that there is some disconnect between the rocket mass configuration, the propellant and some of the trajectory parameters that we have assumed.

So, please note an important parameter that we had assumed was θ_0 because that is something which was to be given as a pitch kick and it is something which is under our control and it was somewhat given in an arbitrary fashion there was no reason that θ_0 would be applicable in the present case. So, now let us invert the problem and try and find out that θ_0 which will make this particular mission feasible from a design perspective.

That it will burn all the propellants, it will reach an angle of 90° and let us try and find out what is going to be the θ_0 that will make it happen and that is going to be a design solution. So, that is the θ_0 that you are going to give at the pitch kick point. And with that you should be able to execute the trajectory as desired.

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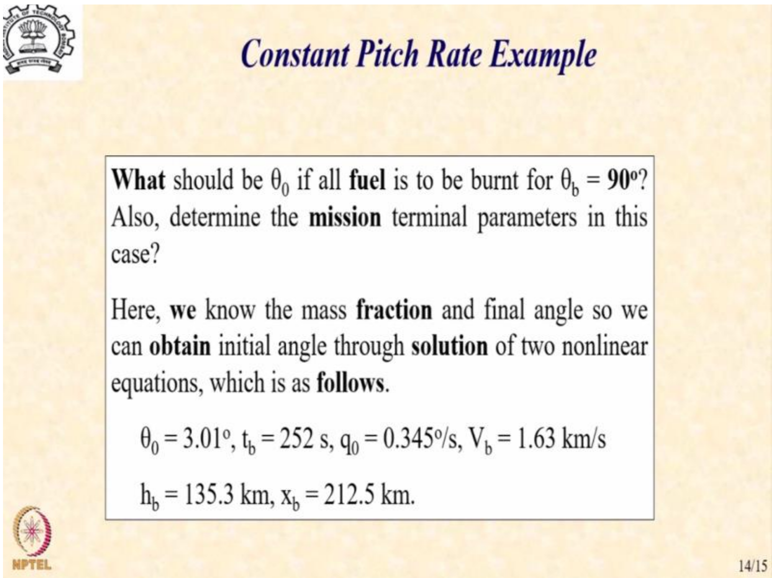

So, this is the exercise that we now conduct. Of course, we know the final mass fraction because we want to consume all the propellant. So, we know that the mass fraction is going to be 4 that is $\frac{m_0}{m_b}$ is going to be 4 and final angle also we have specified as $90°$. So, now we are required to determine the θ_0 for which the solution is going to be possible.

And that can be obtained through simultaneous solution of two nonlinear equations one corresponding to the mass fraction and other one corresponding to the final angle. So, if we do this in fact, I suggest that you do this exercise yourself you will find that for $\theta_0 = 3.01°$ such a mission would be feasible, but it will take 252s to complete and in this case where q_0 is going to be 0.345 deg/s instead of 0.547 deg/s that is what we had taken earlier.

So, which means we are reducing q_0 and as we have seen a reduced q_0 automatically results in a higher burnout velocity. So, now we have instead of 816 m/s we have, we have double the velocity that is $1.63\ km/s$ also the altitude which was around 35 km has gone to 135 km and horizontal distance which was around 25 km has become 212 km.

So, you realize that just by a small tweak of the initial angle from $5°$ to $3°$ there is a great change in the trajectory with the same propellant mass and now you realize the value and the potential of the gravity turn trajectory through a constant pitch rate manoeuvre. We realize that the possibilities of generating large number of trajectories that we may desire from the same rocket just by suitably tweaking your q_0 and ensuring that q_0 is maintained throughout the trajectory.

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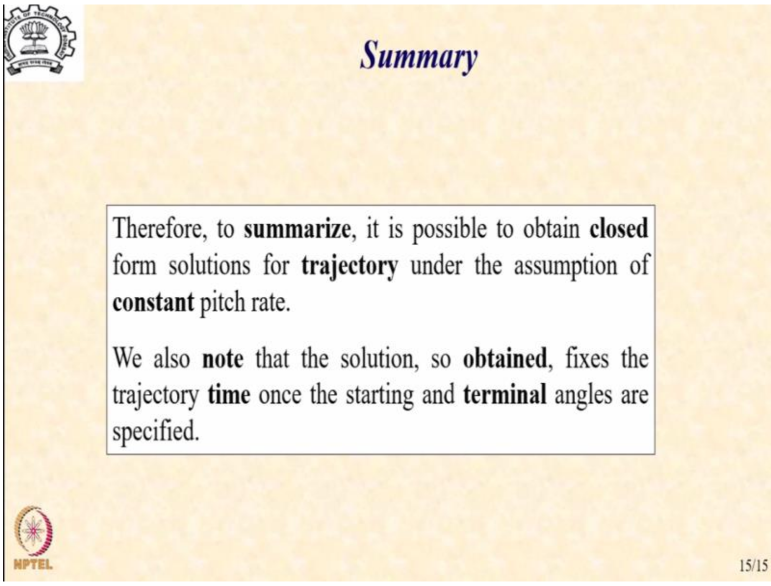

Therefore, to summarize it is possible to obtain closed form solutions for trajectory under the assumption of constant pitch rate. We also note that the solution so obtained fixes the trajectory time once the starting and terminal angles are specified. So, I hope you have realized that gravity turn manoeuvre is an extremely powerful tool for designing the ascent mission trajectories.

What we have also not explicitly stated, but can be seen from the various ratio that we have obtained that for given terminal conditions, it is possible to also determine the rocket configuration that would achieve a particular trajectory configuration. So, you realize that a simple constant pitch rate assumption has provided us with an extremely powerful tool. In the next lecture, we will look at the other two tools that is the constant T/m and a constant velocity. So, bye see you in the next lecture and thank you.