Introduction to Launch Vehicle Analysis and Design

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Lecture – 10
Curvilinear Motion Concept

Hello and welcome. The material that we have discussed so far has made one fundamental assumption about the nature of the trajectory that it is along a straight line. As we have already seen earlier, this is an approximation which needs to be relaxed and we need to now look at the possibilities of generating a curvilinear trajectory for the ascent mission. So, in this lecture now we will look at one such option of generating a curvilinear trajectory. So, let us begin.

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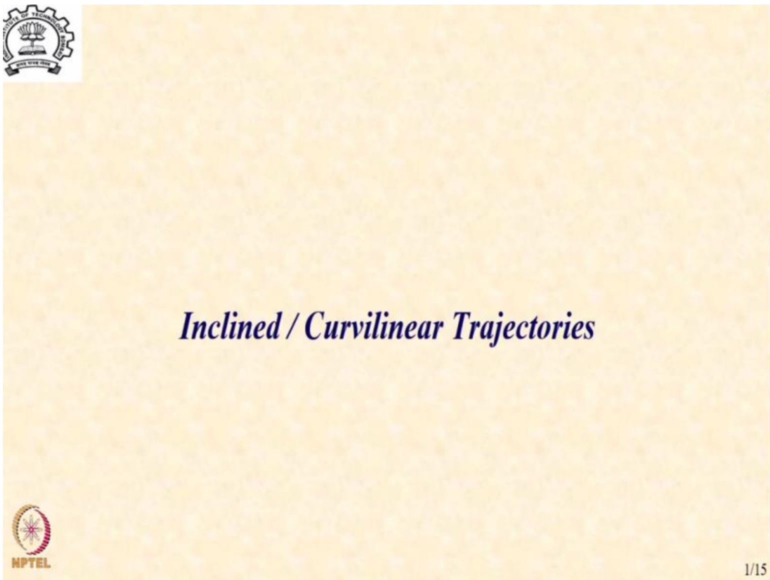

So, we are going to now talk about inclined or curvilinear trajectories.

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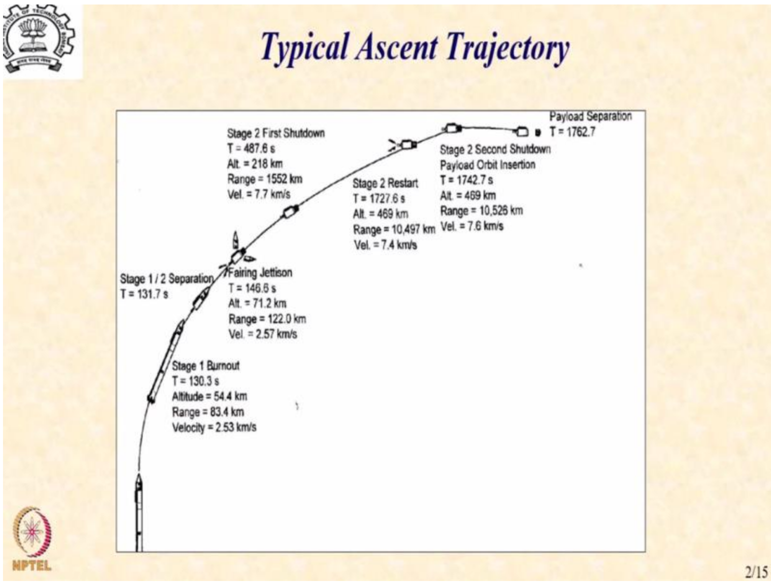

So, let me recall this particular picture that we had seen earlier for a typical ascent mission starting from the lift off until the terminal point. As you can see, the flight path immediately after the lift off starts curving in such a manner that towards the end of the mission the velocity of the residual object is practically parallel to local horizon.

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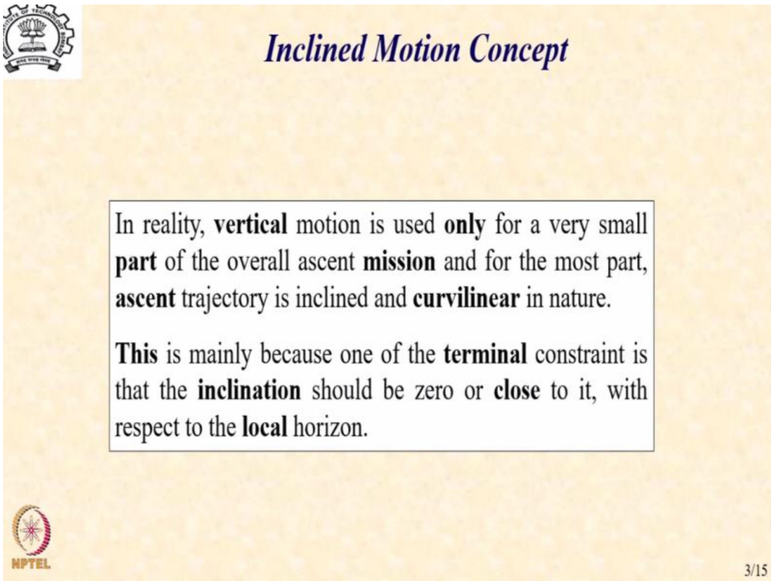

So, we realize that the vertical motion or motion along a radial line is used only for a very small part of the overall mission and for the most part ascent trajectories inclined and curvilinear in nature. Of course, at this point it is worth noting that this is mainly due to one of the terminal constraints that is imposed by the space craft mission that the inclination with respect to the local horizon should be zero or close to it which requires that a vehicle which is moving along a radial line. By the time it completes the mission, should be moving along a local tangent.

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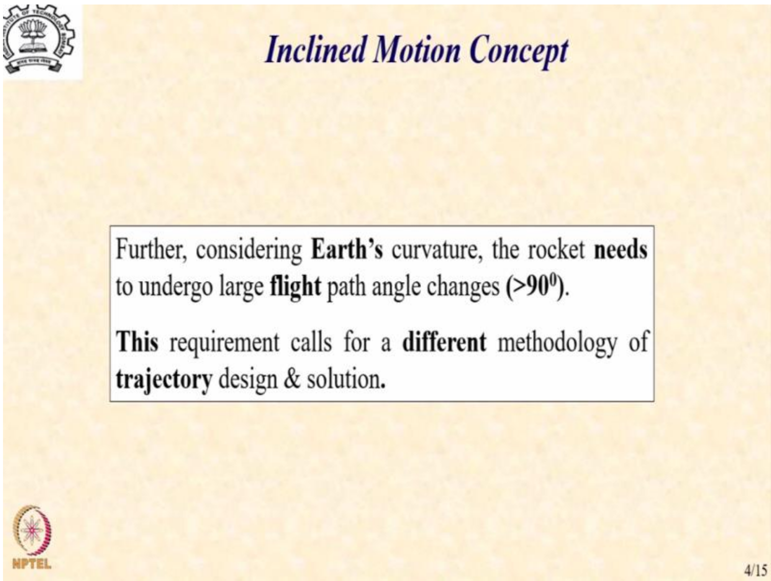

So, obviously you are going to require large amount of curvature while completing the ascent mission. In addition, it is worth nothing that earth itself is no longer a flat surface because you are going to move along a curved path and because of curvature of the earth the local tangent itself changes its inclination with respect to the tangent at the launch point. So, that by the time you reach the terminal point the amount of flight path angle changes that you are going to require are going to be significantly in excess of 90 degree. And this requires a different methodology for trajectory design and solution.

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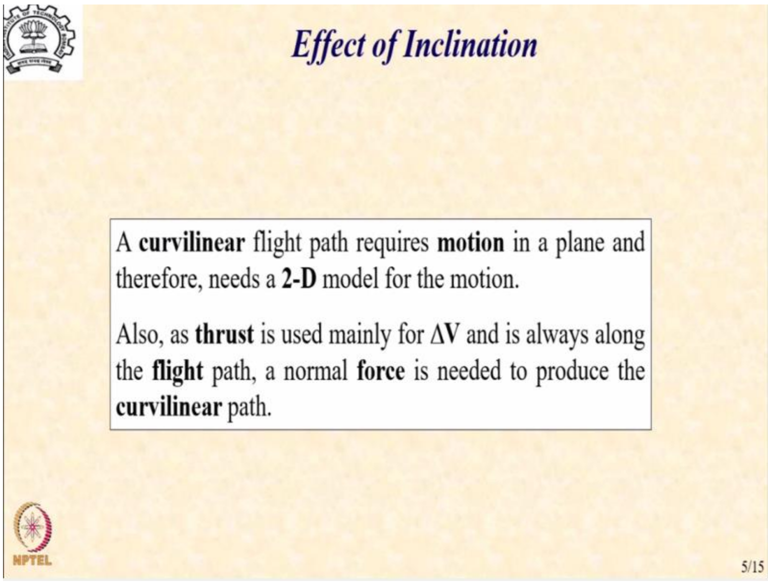

The first thing that happens in this case is that the motion instead of along a straight line is a plane described by the radial line and the tangent. And for that we need a two-dimensional model for the motion. Also, we need to realize the thrust which is the primary force for generating the velocity is generally kept along the flight path so that no amount of propellant is wasted in generating the curvature of the trajectory.

And therefore, we need an alternate, normal force to the velocity vector which will produce the rotation of the velocity vector and hence a curvilinear path.

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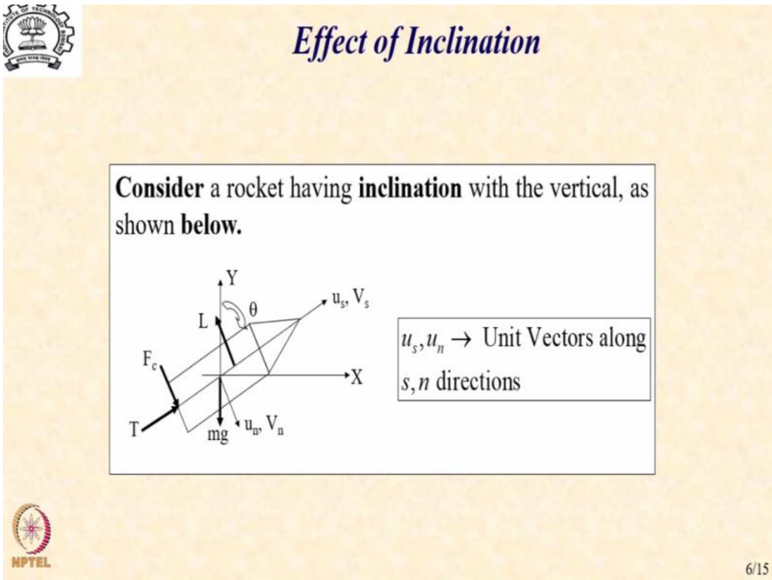

In order to do this, let us create a simple schematic as shown below. So, we have an instantaneous image of a rocket, inclined with respect to a local X axis and has an angle θ with respect to the local Y axis then we have the velocity V_s along with a unit vector u_s along the actual direction of the vehicle and the velocity V_n and the unit vector u_n along the direction normal to the axis or the velocity vector.

The gravity we assume is to be pointing towards the negative wide direction and just to keep the picture fairly general we introduce three additional forces. One the lift which is because of aerodynamics, F_c it controls force for maintaining the trajectory and finally of the thrust which is pushing the vehicle along the flight path. Now, we can easily write down the equations in the Cartesian coordinates X and Y in a planer context.

We can also write the equations in terms of the polar coordinates that is r and θ, but in the present context as you will notice we realize that the trajectory based curvilinear coordinates are the most convenient set of variables that is s and n that we are going to use. This is a coordinate system which is attached to the center of gravity of the vehicle and it is a moving coordinate system which is translating as well as rotating which obviously means that now we need to look at our equations of motion in the context of such a coordinate system and arrive at the applicable relations.

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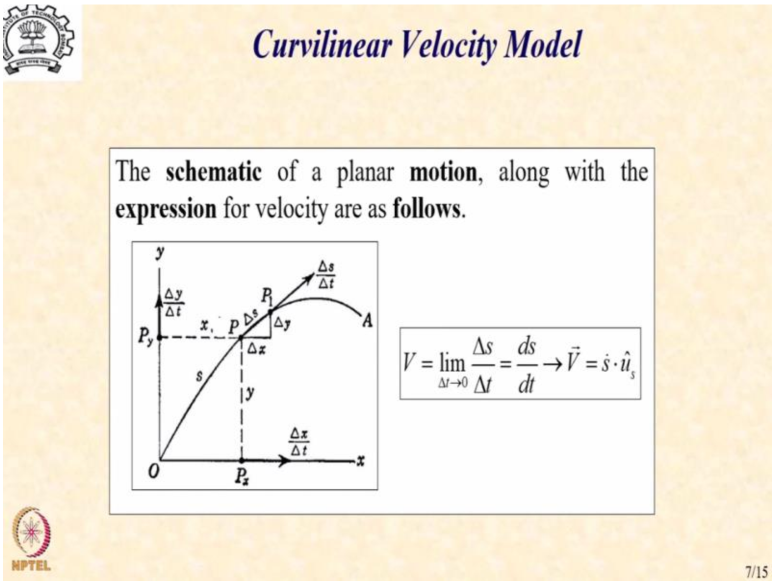

So, let us first look at the expression for the velocity in the context of the curvilinear coordinate system s and n. So, let us assume a two-dimensional system x and y and a curvilinear trajectory A. Let us consider a point P on the trajectory and other point P_1 which is at a distance of Δs along the trajectory as Δs is assumed to be very small infinitesimal quantity P and P_1 can be assumed to be close to each other.

So, that a line joining P and P_1 will be equivalent to a local tangent at P and that tangent if I divide that with Δt that is the time interval between the P and P_1 what we get is a local value of derivative of the trajectory segment Δs or what we call the local velocity which is along the tangent. Mathematically, we can now write the velocity as limit of $\frac{\Delta s}{\Delta t}$ as $\Delta t \to 0$ which is nothing, but $\frac{ds}{dt}$ the derivative of s.

And our velocity vector along the u_s direction is the derivative $\dot{s} \times u_s$. So, you realize that the statement of velocity is fairly simplified if we use the trajectory-based coordinate system u_s.

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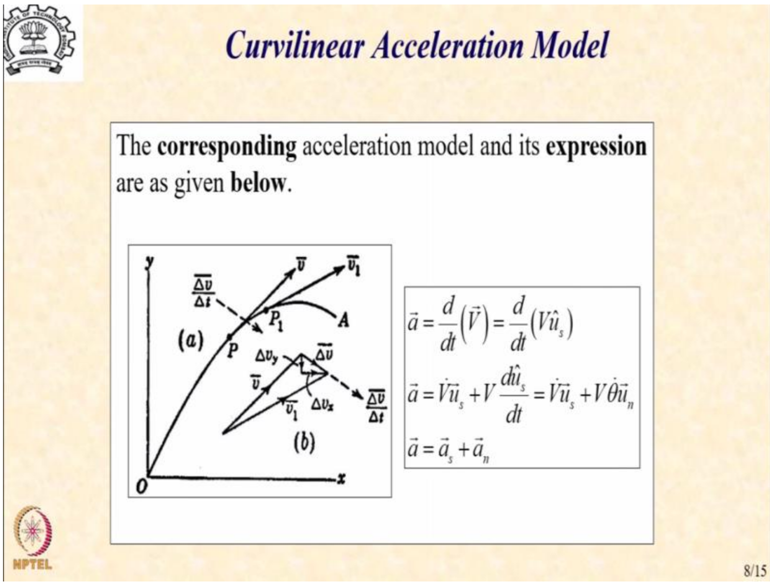

Next, let us look at the acceleration model for the same motion scenario. So, we now consider the same picture and derived the expression for acceleration as follows. So, again we have a same picture, but this time we assume that at point P the local tangent direction is $\vec{v}$. And at P_1 the local tangent direction is $\overrightarrow{v_1}$. We assume that there is an infinitesimal difference between these two local directions that is going from point P to P_1 the velocity vector has rotated by a small angle.

Now the only way this rotation can be achieved is to introduce a velocity component which is normal to $\vec{v}$ at P so that at P_1 the velocity vector will become $\overrightarrow{v_1}$ and this can be done through a simple velocity triangle that is shown below. So, there are two velocity components Δv_x and Δv_y together they generate $\overrightarrow{\Delta v}$ which is the change in the velocity and if we take its ratio with Δt the time taken that becomes our acceleration vector at that particular point.

The mathematical relations are as follows. So, we write acceleration as the derivative of $\vec{V}$ vector which is the velocity. Now as velocity is a vector, we are talking about a vector derivative. So, vector we can write as the scalar part of the velocity $\vec{V}$ and the unit vector and all of us are familiar with the chain rule of differentiation so that we can write the acceleration vector as two terms the $\left(\dot{V}\overrightarrow{u_s} + V\dot{\theta}\overrightarrow{u_n}\right)$.

Now as unit vector is changing its direction the rate of change of unit vector is now related to a rotational rate $\dot{\theta}$ and the local velocity $\vec{V}$. So, this is a locally circular motion concept so that the normal velocity is normal to the tangent direction and $\dot{V}$ represents that particular normal velocity and together with $V\dot{\theta}$ we get normal acceleration.

So, now we have two acceleration components $\overrightarrow{a_s}$ which is $\dot{V}$ that is along the velocity direction and we have another acceleration $\overrightarrow{a_n}$ which is normal to the velocity direction. So, we realize that in the context of the trajectory coordinates s and n the vehicle now has two accelerations, one along the trajectory and one normal to the trajectory.

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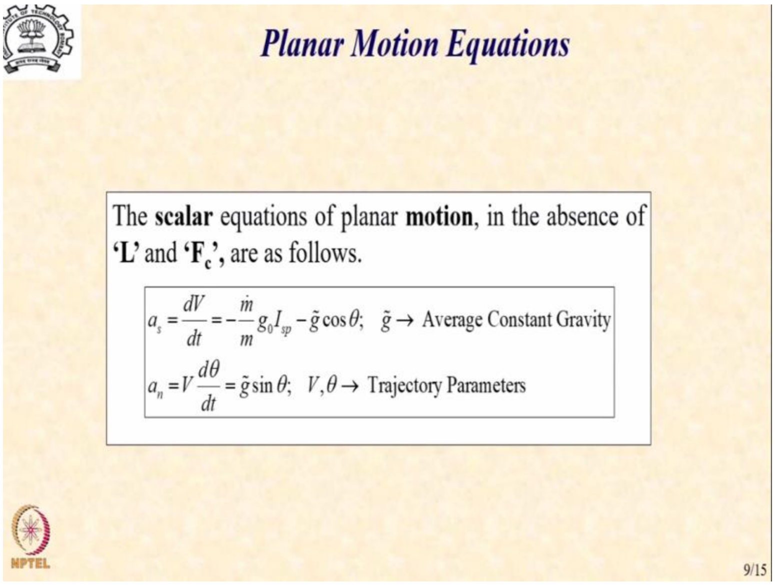

Of course, as we have mentioned earlier, we will assume that the lift is 0 because the vehicle is flying with zero angle of attack and that we do not need an F_c for rotating the velocity vector. So, we assume both these to be zero. And with that we write down now the equations of equilibrium. So, the first equation of equilibrium that is as acceleration along the trajectory direction is nothing, but $\frac{dV}{dt}$ the scalar component of acceleration.

And this is nothing, but the difference between the thrust and a component of gravity in the direction opposite to thrust that is $\tilde{g}cos\theta$. In addition to this, now we have a normal acceleration an which is $V\frac{d\theta}{dt}$ and this should be equal to the normal component of a gravity in the s and n coordinate system that is $\tilde{g}sin\theta$. Here V and θ are treated as trajectory parameters.

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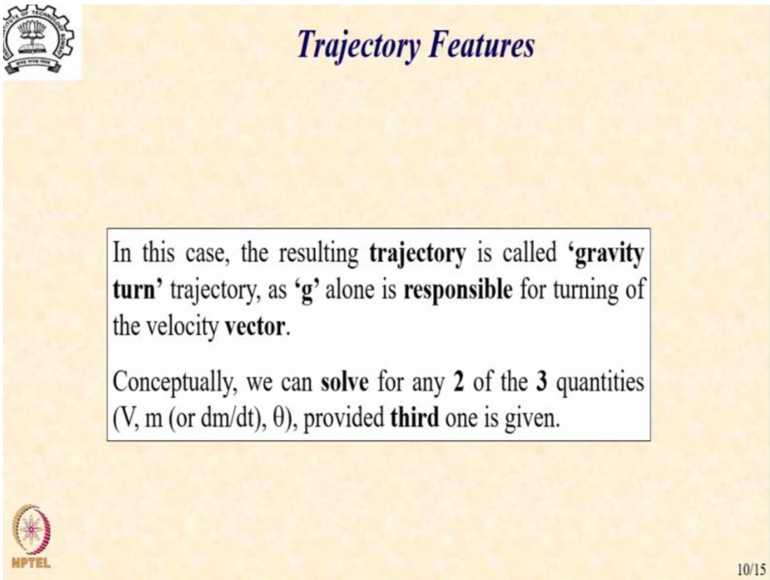

Now as we see from these two equations the velocity solution is now function of the thrust of course, but angle θ. So, it depends now on the angle θ. You can readily see that if I put $\theta = 0°$, which corresponds to the motion along a radial line. The normal equilibrium disappears and we recover our basic rectilinear motion equation, but more importantly what we realize from the second equation that the solution for θ which is nothing.

But the angle by which the velocity vector rotates as it moves along the trajectory is controlled by the gravitational term $gsin\theta$ which means that gravity acceleration alone is now responsible for the turning of the vehicle. Of course, the V is also an important parameter. Here, it is worth nothing that conceptually we now have a set of two coupled nonlinear differential equations for actually three unknowns the velocity, the mass or the $\frac{dm}{dt}$ and the flight path angle θ, but we have only two equations and three unknowns.

So, from our basic understanding of solution of such equations we note that if I specify any one of these, I can solve for the other two which means now I have a requirement that I must specify either velocity or burn profile or flight path angle profile.

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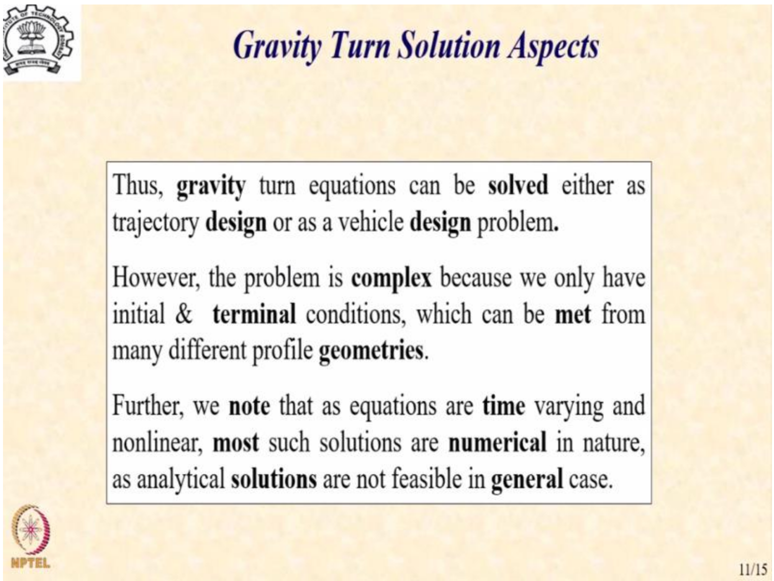

And this gives us an important insight into the solution itself that as much as that these equations can be solved either as a trajectory design problem where I specify your burn rate and solve for V and θ or as a vehicle design problem where I specify a trajectory either in terms of θ or in terms of V and solve explicitly for either m or $\frac{dm}{dt}$ which will enable such a flight path.

However, it is worth noting that the problem is complex because we only have an initial point which is the launch point and terminal conditions which we are going to talk about later which can be met from many different profile geometries. So, obviously there are multiple solutions possible which means depending upon how I specify my m or $\dot{m}$. I am going to get various combinations of V and θ which will be valid trajectories.

And similarly for multiple values of θ or V I will be able to solve for a large number of burn rates or m which will be solutions. Further, we also note that these equations are time varying and nonlinear. So, most of our solutions are going to be numerical in nature as close form analytical solutions are not going to be feasible in a general case.

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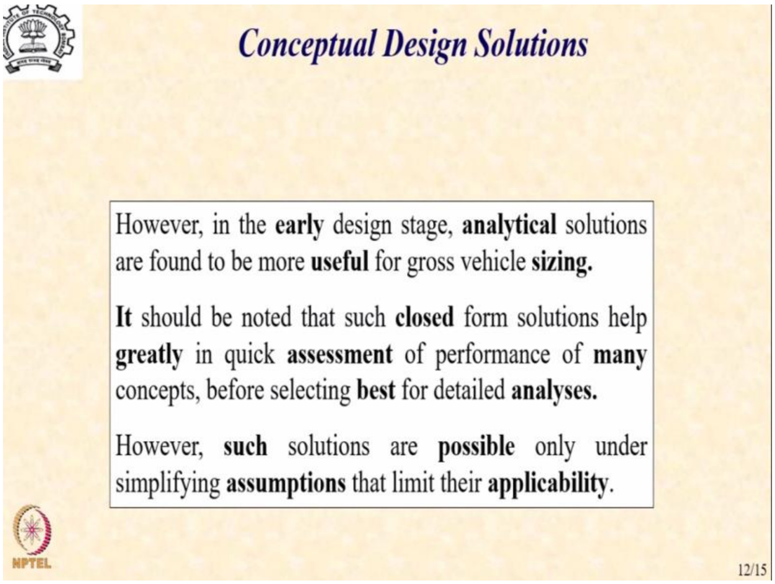

However, as we have been mentioning from the beginning in the early design stage particularly when we are looking at vehicle sizing. Closed form analytical solutions are found to be very useful and we also note that in case we can get such closed form solutions we can get a quick assessment of a large number of concepts and can carry out a trade of study to choose a few of the better performing concepts for more detailed investigations.

Of course, you should realize that such things are going to be possible only under simplifying assumptions that limit their applicability. So, that is the flip side of simplified solutions. They provide quick assessment, but then there is a certain amount of approximation or constraint that we include for arriving at such solutions.

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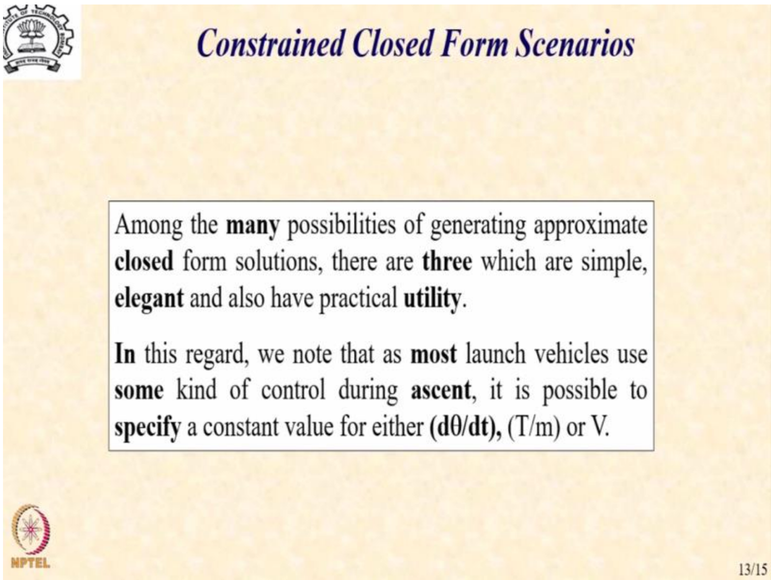

So, what are those kinds of solutions that one can think of? There are many possibilities that one can think of which could generate closed form solutions, but among such possibilities there are three in which are simple elegant and also have practical utility. In this regard, it is worth nothing that most launch vehicles will use some kind of control mechanism during its ascent mission.

And therefore, it is possible to assume that we would have a control to either maintain one of the three variables that is either $\frac{d\theta}{dt}$ or $\frac{T}{m}$ or V. We will find that by keeping one of these as constants it is possible for us to arrive at closed form solution which have lot of practical utility.

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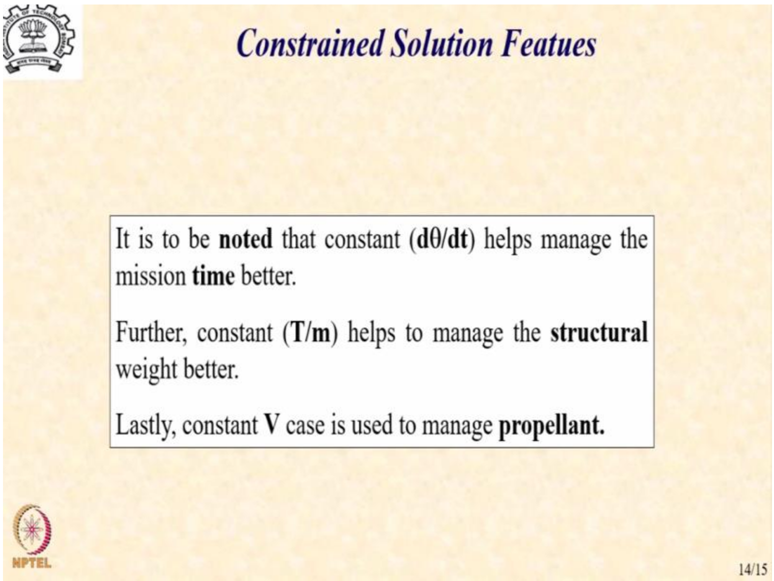

While we will look at these in more detail in the next lecture. It is worth noting here that constant $\frac{d\theta}{dt}$ generally helps us to manage the mission time better, we will see how it happens. Similarly, when we specify constant $\frac{T}{m}$ that helps us to manage the structural mass better. As $\frac{T}{m}$ represents an indicator of the forward acceleration that the vehicle is undergoing and lastly the constant velocity case commonly helps us to manage the overall propellant mass better.

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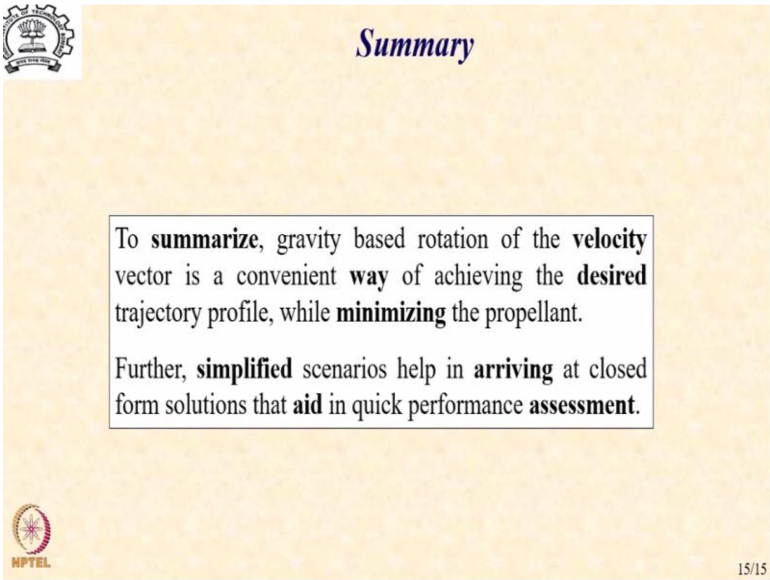

So, to summarize the gravity-based rotation of the velocity vector is a convenient way of achieving the desired trajectory profile by minimizing the propellant. Further, we have also established that simplified scenarios can help in arriving at closed form solutions that aid in quick performance assessment. So, in this lecture segment we have seen the basic idea of a trajectory which is curvilinear in nature and is driven only be the gravitational force so that we can manage our propellant better.

And we have also seen that use of curvilinear coordinate systems s and n helps in writing down the equations in a fairly compact and simplified manner. We have also noted that using simplifying assumptions we can obtain closed form solutions for couple of cases which will help us to generate the trajectories quickly and with practical value. We will do this in our next lecture. So, bye see you in the next lecture and thank you.